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DISCRETE DELAYS AND PIECEWISE CONSTANT ARGUMENT IN NEURONICS

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ABSTRACT. Sufficient conditions are established for the convergence of solutions of difference equation

$$x_{n+1} = a\ \tanh[x_n - b_{n-1} - c]\,, \quad n = 0, , 1, 2, 3, \cdot,$$

discritized from the mother version of continuous case

$$\frac{dx(t)}{dt} = -x(t) + a \tanh[x(t) - bx(t-\tau) - c], \quad t > 0,$$

with a, b, c and τ are all positive, that models the dynamics of a single effective neuron with dynamical threshold with delay. A similar approach can also be applied to establish the stability conditions for a differential equation with piecewise constant arguments as in the following

$$\frac{dx(t)}{dt} = -ax(t) + \tanh[bx(t) + cx([t])], \quad t > 0, \quad t \neq 0, 1, 2, \cdots .$$

The stability analysis of the generalized time delay model

$$\frac{dx(t)}{dt} = -ax(t) + \tanh\left\{bx(t) + \sum_{j=0}^{k} c_j x([t-j])\right\}, \quad t > 0, \quad t \neq 0, 1, 2, \cdots,$$

is also discussed using the lemma due to Cooke and Huang 1991.

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1. DELAY INDEPENDENT STABILITY

The literature of neural network models has been intensively established in the last two decades. As one of the most interested topics of studying the dynamics of a neural networks system, parametrical-induced, dynamical stability draws a vast majority of attention in the recent years of researches. For examples, in their recent study, Akhmet & Yilmaz 2009 considered the Hopfield type neural networks system with piecewise constant argument of generalized type without delays. They showed the global asymptotic stability of the system by using Gronwall-Bellman inequality. Abbas 2009 investigated the k-pseudo almost periodic sequence solutions of a discrete time neural network. They studied the existence, uniqueness and the exponential attractively of the solution. Coombes & Laing 2009 also discussed the stability of

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the Wilson-Cowan neural network. They focused to study on the impact of discrete delays in neural population models without spatial extent. On the other hand, Gyori & Hartung 2003 discussed the stability of a single neuron model with delay. At where, they showed a new sufficient conditions for global asymptotic stability of constant equilibriums by checking the existence of the limit of the solution. While Gou & Xue 2009 studied the dynamic of periodic solutions and exponents stability of a particular class of networks with time-varying delays. Gopalsamy 2008 investigated the stability the combined dynamics in a kind of networks with synaptic adaptation and neuronic activation time delays. Studying the dynamical stability of neural networks or a single neuronal model with time delays has been intensively studied in the current trend of research activities in recent years. However, current investigations of the dynamics of such neural networks is predominantly concerned with the autonomous system with the deviated argument of the activation functions being a discrete constant or a continuous time dependence. Literatures of either continuous or discrete time neural networks dealing with piecewise continuous argument with delays in the activation function appears to be scares.

To fill the little gap, as a starting point, we shall show in this paper the stability of a single neuron model of the Hopfield-type equation with piecewise continuous argument with delay, which will be of the form

$$\frac{dx(t)}{dt} = -ax(t) + \tanh\left\{bx(t) + \sum_{j=0}^{k} c_j x([t-j])\right\}, \quad t > 0, \quad t \neq 0, 1, 2, \cdots ,$$

In the literature on neural network dynamics, there are numerous investigations concerned with the stability and local Hopf-bifurcations of Hopfield-type equations with discrete delays (for instance see Babcock & Westerverlt 1986a,b. Most of these results do not consider the persistence of bifurcated periodic solution far from the bifurcation point in the parameter space. For the purpose of completeness, we firstly consider the dynamics of a single neuron modelled by the delay differential equation

$$\frac{dx(t)}{dt} = -x(t) + a\tanh[x(t) - bx(t-\tau) - c], \quad t > 0 \tag{1.1}$$

where a, b, c and τ are positive numbers; the equation (1.1) is usually supplemented with an initial condition of the form

$$x(s) = \phi(s), \quad s \in [-\tau, 0] \tag{1.2}$$

where ϕ is assumed to be continuous, $\phi : [-\tau, 0] \mapsto \mathbb{R}$. If we consider the dynamics of (1.1) for $t > \tau$, then we can introduce a change of variables such that

$$y(t) \equiv x(t) - bx(t-\tau) - c \tag{1.3}$$

and obtain from (1.1) that y is governed by

$$\frac{dy(t)}{dt} = -[y(t) + c] + a\tanh[y(t)] - ab\tanh[y(t-\tau)], \quad t > 0 \tag{1.4}$$

In the following we study the dynamics of (1.4) and with a translation of time, we can assume that (1.4) is valid for $t > 0$. We assume that

$$a > 0, \quad 0 < a(1-b) < 1 \tag{1.5}$$

It is now elementary to note that (1.4) has a unique equilibrium solution y^* satisfying

$$y^* + c = a(1-b)\tanh(y^*). \tag{1.6}$$

Our first result below provides sufficient conditions for the asymptotic stability of y^* of (1.4). It was done by Gopalsamy & Leung 1997. We state here for the purpose of completeness.

Theorem 1.1. *Suppose that the parameters a and b satisfy*

$$a > 0, \quad 0 < a(1-b) < 1 \quad \text{and} \quad a(1+b) < 1. \tag{1.7}$$

Then all solutions of (1.4) satisfy

$$\lim_{t\to\infty} y(t) = y^*. \tag{1.8}$$

Proof. One may consider a Lyapunov-like functional $V(y)(t)$ defined by

$$V(y)(t) = |y(t) - y^*| + ab\int_{t-\tau}^{t} |y(s) - y^*|\, ds, \quad t > 0 \tag{1.9}$$

to establish the bounded condition

$$\int_0^\infty |y(s) - y^*|\, ds < \infty. \tag{1.10}$$

Which will imply, by Barbalatt's lemma (see Gopalsamy 1992), that

$$\lim_{t\to\infty} y(t) = y^*$$

We refer the readers to the detail of proof from Gopalsamy & Leung 1997. □

2. A DISCRETE MODEL WITH THRESHOLDS

It is believed that the dynamical characteristics of the discretized model is not necessarily similar to that of the mother version of the continuous model. It is of great interest to investigate the stability for the corresponding discrete model equation. Let us consider the following discrete analogue of (1.1) in the form

$$x_{n+1} = a\tanh[x_n - bx_{n-1} - c], \quad n = 0, 1, 2, 3, \cdots. \tag{2.1}$$

We let

$$y_n \equiv x_n - bx_{n-1} - c \tag{2.2}$$

in (2.1) and obtain

$$y_{n+1} = a\tanh(y_n) - ab\tanh(y_{n-1}) - c. \tag{2.3}$$

An equilibrium y^* of (2.3) is a solution of

$$y^* + c = a(1-b)\tanh y^*. \tag{2.4}$$

If we assume $a(l-b) < 1$, then there exists a unique equilibrium y^* satisfying (2.4) for each $c \in (-\infty, \infty)$. The following result provides sufficient conditions for all solutions of (2.3) to approach y^* as $n \to \infty$.

Theorem 2.1. *Let $a \in (0, \infty)$, $b, c \in (-\infty, \infty)$ and suppose that*

$$0 < a(1-b) < 1; \quad a(1+b) < 1. \tag{2.5}$$

Then every solution of (2.3) *satisfies*

$$y_n \to y^* \quad as \quad n \to \infty. \tag{2.6}$$

Proof. Consider a Lyapunov-type function V_n defined by

$$V_n = |y_n - y^*| + ab|y_{n-1} - y^*|, \quad n = 0, 1, 2, 3, \cdots. \tag{2.7}$$

It is found that

$$\begin{aligned} V_{n+1} &= |y_{n+1} - y^*| + ab|y_n - y^*| \\ &\leq a|\tanh(y_n) - \tanh(y^*)| + ab|\tanh(y_{n_1}) - \tanh(y^*)| + ab|y_n - y^*| \\ &\leq a|y_n - y^*| + ab|y_{n-1} - y^*| + ab|y_n - y^*| \\ &= a(1+b)|y_n - y^*| + ab|y_{n-1} - y^*| \end{aligned} \tag{2.8}$$

and hence

$$\begin{aligned} V_{n+l} - V_n &\leq -\{1 - a(1+b)\}|y_n - y^*|; \\ V_n - V_{n-1} &\leq -\{1 - a(1+b)\}|y_{n-1} - y^*|; \\ \cdots &\quad \cdots\cdots \\ V_2 - V_1 &\leq -\{1 - a(1+b)\}|y_1 - y^*|; \end{aligned} \tag{2.9}$$

adding the respective sides of (2.9),

$$V_{n+1} + \{1 - a(1+b)\}\sum_{j=1}^{n} |y_j - y^*| \leq V_1. \tag{2.10}$$

It follows that the series

$$\sum_{j=1}^{\infty} |y_j - y^*|$$

converges and hence $\lim_{n\to\infty} |y_n - y^*| = 0$. □

Corollary 2.2. *Under the assumptions of Theorem* 2.1*, all solutions of* (2.1) *converge to x^* as $n \to \infty$, where x^* denotes the unique solution of*

$$x = a\tanh[x(1-b) - c]. \tag{2.11}$$

Proof. It follows from (2.1) that $x_n \to a\tanh(y^*)$ as $n \to \infty$. If we let $lim_{n\to\infty}x_n = \alpha$, then we have from (2.1) and (2.2) that

$$\begin{cases} a = a\tanh y^* \\ y^* = \alpha - b\alpha - c \end{cases} \tag{2.12}$$

leading to

$$\alpha = a\tanh[\alpha(1-b) - c];$$

by the uniqueness of the solutions of (2.11) the result follows. □

We consider system (2.1) with $a = 0.5$, $b = 1.85$ and $c = -0.25$, which satisfy the conditions in (2.5);

$$x_{n+1} = 0.5\tanh[x_n - 1.85x_{n-1} + 0.25], \quad n = 0, 1, 2, 3, \cdots, \tag{2.13}$$

which can be transformed to a two dimensional system

$$\begin{cases} x_{n+1} &= 0.5\tanh[x_n - 1.85y_n + 0.25], \\ y_{n+1} &= x_n \end{cases}, \qquad n = 0, 1, 2, 3, \cdots. \tag{2.14}$$

The following computer simulations show that solutions of (2.13) and (2.14), with $x_0 = 0.92$, converge to nontrivial equilibrium respectively:
Bifurcation analysis depending on parameters of model (2.1) can also be discussed. But we do not do it here until the next stage where we will consider a discretized system of a more general model with continuously distributed delays.

3. AN EQUATION WITH PIECEWISE CONSTANT ARGUMENT

There exists an extensive literature on the dynamical characteristic of neural networks of a large number of interacting neurons and small netlets of one or two neurons (for relevant literature we refer to Gopalsamy & He 1994a,b, Gopalsamy & Leung 1996). Experimental evidence related to complex and chaotic dynamics in neural systems has been reported by a few authors (Babloyantz & Salazar 1985, Babloyantz & Destexhe 1986, Holden et. al 1982, Skarda & Freeman 1987). For the references of oscillation and periodicity in various model systems, one may refer the work by Mohamad & Gopalsamy 2002, or Guo & Xue 2009 and the work in a single neuron model by El-Morshedy & Gopalsamy 2000. In theoretical models of neural systems, the emphasis has been on the stability of fixed points or cycles. However a few models of neural system have been proposed which indicate chaotic behaviour and these models rely on complex architectures, complicated equations or stochastic elements. It is the purpose of this section to consider the dynamics of a single effective neuron and investigate a variety of dynamical characteristics. We consider a continuous time formulation with a gain occurring at a discrete sequence. It is the opinion of the author that a model of this type is new to the literature and

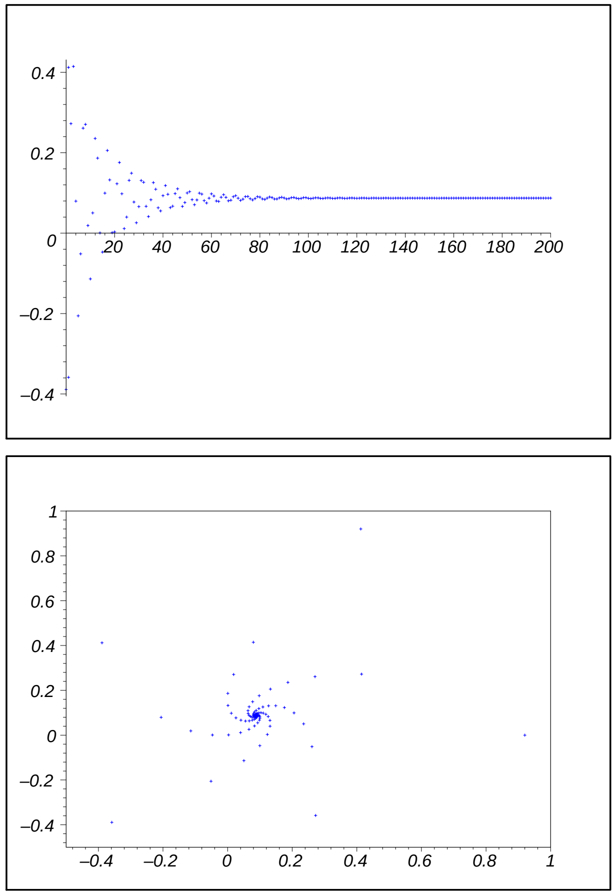


FIGURE 1. (a) Solution of (2.13) for $x_0 = 0.92$. (b) Phase plot of (2.14); $x_0 = 0.92$.

has a wide ranging spectrum of dynamical characteristics. For further review on the models with piecewise constant arguments, one can refer the works by Akhmet and Yilmaz 2009 and the related literatures.

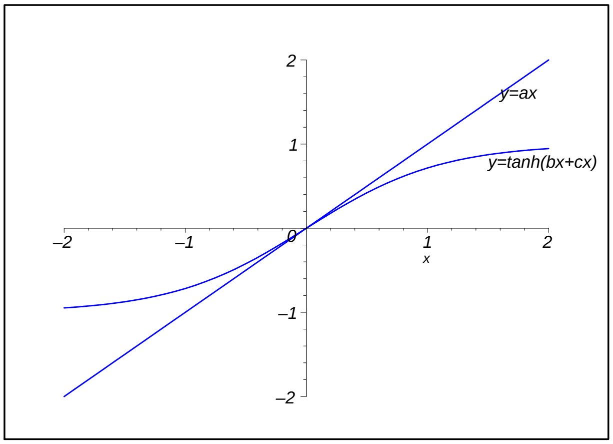
Consider the dynamics of a single neuron modelled by the differential equation

$$\frac{dx(t)}{dt} = -ax(t) + \tanh[bx(t) + cx([t])], \quad t > 0, \quad t \neq 0, 1, 2, \cdots, \tag{3.1}$$

in which a, b, c denote real numbers and $a > 0$ and $[t]$ denotes the greatest integer in t. From the properties of the $\tanh(\cdot)$ function, it will follow that if

$$a > |b| + |c|, \tag{3.2}$$

then there exists an unique equilibrium solution of (3.1) coinciding with the trivial solution; for instance we have the following situation:
In the following we show that when a, b, c satisfy (3.2), all solutions of (3.1) converge as $t \to \infty$ to the trivial equilibrium solution. We define a functional $\sigma(\cdot)$ as follows:



if $y : (a, b) \to (-\infty, \infty)$ is differentiable at t then

$$\sigma(y)(t) = \begin{cases} 1 & \text{if } y(t) > 0 \text{ or } y(t) = 0 \text{ and } \left.\dfrac{dy}{dt}\right|_t > 0, \\ 0 & \text{if } y(t) = 0 \text{ and } \left.\dfrac{dy}{dt}\right|_t = 0, \\ -1 & \text{if } y(t) < 0 \text{ or } y(t) < 0 \text{ and } \left.\dfrac{dy}{dt}\right|_t < 0. \end{cases} \tag{3.3}$$

It will follow from (3.3) that

$$y(t)\sigma(y)(t) = |y(t)| \text{ and } \frac{d^+|y(t)|}{dt} = \sigma(y)(t)\frac{dy(t)}{dt} \tag{3.4}$$

where $\frac{d^+}{dt}$ denotes the right derivative. Now consider (3.1) on an interval of the form $[n, n+1)$ where n denotes a nonnegative integer. Then we have from (3.1),

$$\frac{dx(t)}{dt} = -ax(t) + \tanh[bx(t) + cx(n)], \quad t \in [n, n+1), \tag{3.5}$$

and hence from (3.4) and (3.5) we obtain

$$\begin{aligned} \frac{d^+|x(t)|}{dt} &= -ax(t)\sigma(x)(t) + \sigma(x)(t)\tanh[bx(t) + cx(n)] \\ &\le -a|x(t)| + |b||x(t)| + |c||x(n)| \\ &= -(a - |b|)|x(t)| + |c|x(n), \quad t \in [n, n+1) \end{aligned} \tag{3.6}$$

Integrating both sides of (3.6),

$$\begin{aligned} |x(t)| &\le |x(n)|e^{-(a-|b|)(t-n)} + \frac{|c||x(n)|}{a - |b|}\left\{1 - e^{-(a-|b|)(t-n)}\right\} \\ &\le |x(n)|\left[e^{-(a-|b|)} + \frac{|c|}{a-|b|}(1 - e^{-(a-|b|)})\right] \\ &= \alpha|x(n)|, \quad t \in [n, n+1) \end{aligned} \tag{3.7}$$

where

$$\alpha = e^{-(a-|b|)} + \frac{|c|}{a - |b|}(1 - e^{-(a-|b|)}). \tag{3.8}$$

One can verify that when (3.2) holds, that $0 < \alpha < 1$; now letting $t \to (n+1)$ in (3.7), we obtain

$$|x(n+1)| \leq \alpha |x(n)|$$

and hence derive that

$$|x(n+1)| \leq \alpha^{n+1}|x(0)|, \quad n = 0, 1, 2, \cdots . \tag{3.9}$$

Since $0 < \alpha < 1$, it follows from (3.9) that

$$\lim_{n\to\infty} |x(n)| = 0; \tag{3.10}$$

we can now conclude from (3.7) and (3.10) that

$$\lim_{t\to\infty} |x(t)| = 0; \tag{3.11}$$

which implies that the trivial solution of (3.1) is globally attractive when (3.2) holds.

Although the previous discussion implies that all solutions of (3.1) remain bounded, the following analysis will be useful for systems more general than (3.1). We note that all solutions of (3.1) satisfy

$$-ax - 1 \leq \frac{dx(t)}{dt} \leq -ax + 1, \quad t \neq 0, 1, 2, \cdots . \tag{3.12}$$

It follows from (3.12) that if $x(t)$ denotes any solution of (3.1), then we have

$$|x(t)| \leq \max\{x(0), \frac{1}{a}\}, \quad t \geq 0. \tag{3.13}$$

Thus (3.13) provides an explicit a priori estimate for solutions of equations of the type (3.1).

We proceed to consider the following generalization of (3.1) in the form

$$\frac{dx(t)}{dt} = -ax(t) + \tanh\left[bx(t) + \sum_{j=0}^{k} c_j x([t-j])\right]; \quad t \neq 0, 1, 2, \cdots , \tag{3.14}$$

The technique used to show the attractively of the trivial equilibrium of (3.1) is not easily adaptable for the discussion of (3.14) and we need a modification of the method. We use the next Lemma due to Cooke & Huang 1991 whose proof is included for completeness.

Lemma 3.1. *Let y be continuous with a piecewise continuous derivative on $[0,\infty)$; suppose y is bounded on $[0,\infty)$. Let*

$$\limsup_{t\to\infty} y(t) = y^* \quad \textit{and} \quad \liminf_{t\to\infty} y(t) = y_*. \tag{3.15}$$

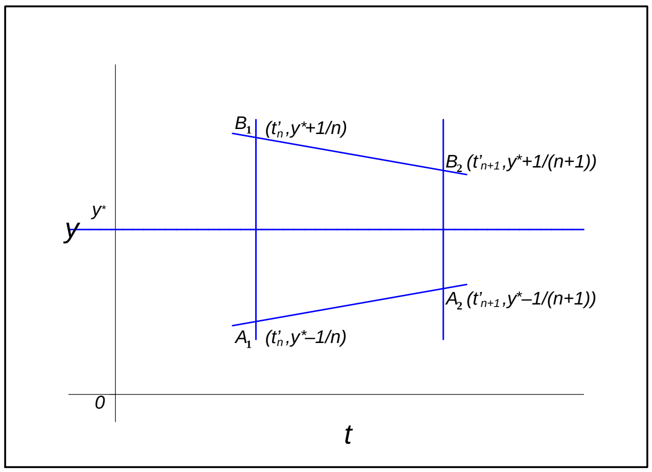
Then there exist sequences $\{t_n\}$ and $\{s_n\}$ such that $t_n \to \infty$, $s_n \to \infty$ as $n \to \infty$ for which

$$\left\{\begin{array}{lll} |y(t_n) - y^*| \le \dfrac{1}{n}, & \left.\dfrac{dy}{dt}\right|_{t_n} \ge -\dfrac{2}{n}, & n = 1, 2, 3, \cdots, \\ |y(s_n) - y_*| \le \dfrac{1}{n}, & \left.\dfrac{dy}{dt}\right|_{s_n} \le +\dfrac{2}{n}, & n = 1, 2, 3, \cdots. \end{array}\right. \tag{3.16}$$

Proof. By the definition of limit superior, there is a sequence $\{t'_n - y^*\}$ such that

$$|t'_{n+1} - t'_n| \ge 1, \qquad |y(t'_n) - y^*| < \frac{1}{n}. \tag{3.17}$$

Let D denote the set of points of the plane bounded by the quadrilateral $A_1A_2B_2B_1$ displayed in the figure below:



There are two cases to consider: (i) the graph of $y(t)$, $t \in [t'_n, t'_{n+1}]$ lies above the segment B_1B_2 or lies below the segment A_1A_2; (ii) the graph of $y(t)$ lies inside $A_1A_2B_2B_1$, for some $t \in (t'_n, t'_{n+1})$.
In case (i), the graph of y leaves D and has to return to D since

$$|y(t'_{n+1}) - y^*| < \frac{1}{n+1}. \tag{3.18}$$

Hence there exists a $t_n \in (t'_n, t'_{n+1})$ such that

$$|y(t_n) - y^*| \le \frac{1}{n} \quad \text{and} \quad \left.\frac{dy}{dt}\right|_{t_n} \ge 0. \tag{3.19}$$

Consider now the case (ii) above; we have

$$|y(t_n) - y^*| \le \frac{1}{n} \quad \text{for some } t_n \in (t'_n, t'_{n+1}). \tag{3.20}$$

We know that y is continuous with a piecewise continuous derivative and a finite jump discontinuity. Therefore

$$\left|\int_{t'_n}^{t'_{n+1}} \dot{y}(s)\, ds\right| = |y(t'_{n+1}) - y(t'_n)| \le \frac{2}{n}. \tag{3.21}$$

We claim that there exists a $t_n \in (t_n', t_{n+1}')$ such that the assertion holds. Suppose the assertion is false, then $\frac{dy(t)}{dt} < -\frac{2}{n}$ for almost all $t_n \in (t_n', t_{n+1}')$. It will then follow that

$$\int_{t_n'}^{t_{n+1}'} \dot{y}(s)\, ds < -\frac{2}{n}(t_{n+1}' - t_n') < -\frac{2}{n}. \tag{3.22}$$

which implies that $y(t_{n+1}') - y(t_n') < -\frac{2}{n}$ and this contradicts (3.22). Thus the first assertion holds. □

The existence of $\{s_n\}$ and the validity of (3.16) can be proved similarly. We can now prove the following:

Theorem 3.2. *Consider the scalar system*

$$\frac{dx(t)}{dt} = -ax(t) + \tanh\left\{bx(t) + \sum_{j=0}^{k} c_j x([t-j])\right\}; \quad t > 0, t \neq 0, 1, 2, \cdots \tag{3.23}$$

in which a, b, c_j $(j = 0, 1, 2, \cdots, k)$ *denote real numbers satisfying*

$$a > 0 \quad \textit{and} \quad a > |b| + \sum_{j=0}^{k} |c_j|, \tag{3.24}$$

Then the trivial solution of (3.23) *is globally attractive in the sense that an arbitrary solution* $x(t)$ *of* (3.23), *satisfies*

$$\lim_{t\to\infty} x(t) = 0; \tag{3.25}$$

Proof. We know that solutions of (3.23) remain bounded for $t \in [0, \infty)$. Also solutions of (3.23) satisfy

$$\frac{d^+}{dt}|x(t)| \leq -a|x(t)| + |b||x(t)| + \sum_{j=0}^{k} |c|x([t-j]); \quad t \neq 0, 1, 2, \cdots. \tag{3.26}$$

In order to prove (3.25) it is sufficient to show that

$$\lim_{t\to\infty} x(t) = 0. \tag{3.27}$$

Suppose (3.27) does not hold; there exists therefore $\rho > 0$ such that

$$\limsup_{t\to\infty} |x(t)| = \rho > 0. \tag{3.28}$$

Hence there exists a sequence $\{t_n\} \to \infty$ as $n \to \infty$ such that

$$|x(t_n)| \to \rho \quad \text{as} \quad n \to \infty; \tag{3.29}$$

it follows also from Lemma 2.1 that for the sequence $\{t_n\}$ we have

$$-\frac{2}{n} \leq \left.\frac{d^+|x(t)|}{dt}\right|_{t_n} \leq -a|x(t_n)| + |b||x(t_n)| + \sum_{j=0}^{k} |c||x([t_n - j])|. \tag{3.30}$$

Passing to the limit as $n \to \infty$ in (3.30) we obtain

$$\begin{aligned} 0 &\leq -(a - |b|)\rho + \textstyle\sum_{j=0}^{k} |c_j|\rho \\ &= -\left\{a - (|b| + \textstyle\sum_{j=0}^{k} |c_j|\right\} \rho < 0 \end{aligned}$$

which is impossible due to (3.24). Hence $\rho = 0$ from which (3.25) follows. This completes the proof. □

The result of Lemma 3.1 can be used for the derivation of sufficient conditions for the attractively of equilibria of dynamical systems modelled by equations involving both delayed and advanced arguments. The proof of the next result is similar to that of the Theorem above and hence we formulate the result without its proof.

Corollary 3.3. *Let $a, b, c_j, d_j, l_j, m_j, \tau_j, \sigma_j$ be real numbers and let p_1, p_2, p_3, p_4 be positive integers; suppose that*

$$a > |b| + \sum_{j=1}^{p_1} |c_j| + \sum_{j=1}^{p_2} |d_j| + \sum_{j=1}^{p_3} |l_j| + \sum_{j=1}^{p_4} |m_j|. \tag{3.31}$$

Then all nontrivial solutions of the scalar system

$$\begin{aligned} \frac{dx(t)}{dt} &= -ax(t) + \tanh\left\{ bx(t) + \sum_{j=1}^{p_1} c_j x(t - \tau_j) + \sum_{j=1}^{p_2} d_j x(t + \sigma_j) \right. \\ &\left. + \sum_{j=1}^{p_3} l_j x([t - j]) + \sum_{j=1}^{p_4} m_j x([t + j]) \right\}; \quad t \neq 0, 1, 2, \cdots; t > 0 \end{aligned} \tag{3.32}$$

converge to the trivial solution such that

$$\lim_{t \to \infty} x(t) = 0. \tag{3.33}$$

4. CONVERGENCE IN NEURAL NETWORKS

In this section we consider the dynamics of a system of interconnected neurons modelled by the following coupled system of equations:

$$\begin{aligned} \frac{dx_i(t)}{dt} &= -a_i x_i(t) + \tanh\left\{ b_i x_i(t) + \sum_{j=1}^{n} c_{ij} x_i([t]) \right\}; \\ & i = 1, 2, 3, \cdots, n; \quad t > 0 \text{ and } t \neq 0, 1, 2, \cdots. \end{aligned} \tag{4.1}$$

The system (4.1) has the trivial solution as one of its equilibria. We are interested in the derivation of sufficient conditions for the attractivity of the trivial solution. We assume that the coefficients in (4.1) satisfy

$$\min_{1 \leq i \leq n} (a_i - [b_i]) > \max_{1 \leq i \leq n} \sum_{j=1}^{n} |c_{ji}|. \tag{4.2}$$

On any interval of the form $[k, k+1)$ where k is a nonnegative integer, solutions of (4.1) are continuously differentiable and on such intervals we have

$$\frac{d^+}{dt}|x_i(t)| \le -a_i|x_i(t)| + |b_i||x_i(t)| + \sum_{j=1}^{n} |c_{ij}||x_i(k)|; \tag{4.3}$$
$$i = 1, 2, 3, \cdots, n; \quad t \in [k, k+1).$$

We obtain from (4.3) by vertical addition that

$$\begin{aligned} \frac{d^+}{dt}\left[\sum_{i=1}^{n} |x_i(t)|\right] &\le \sum_{i=1}^{n}\left[-a_i|x_i(t)| + |b_i||x_i(t)| + \sum_{j=1}^{n}|c_{ij}||x_i(k)|\right] \\ &\le \sum_{i=1}^{n}\left\{-(a_i - |b_i|)|x_i(t)| + \sum_{j=1}^{n}|c_{ji}||x_i(k)|\right\} \\ &\le \left(-\min_{1\le i\le n}\{a_i - [b_i]\}\right)\sum_{i=1}^{n}|x_i(t)| \\ &\quad + \left(\max_{1\le i\le n}\sum_{j=1}^{n}|c_{ji}|\right)\sum_{i=1}^{n}|x_i(k)| \\ &\quad t \in [k, k+1), k \in \mathbb{N}. \end{aligned} \tag{4.4}$$

As in the case of (3.1), solutions of (4.1) remain bounded. Suppose then that

$$\limsup_{t\to\infty}\sum_{i=1}^{n}|x_i(t)| = \rho; \tag{4.5}$$

if $\rho = 0$ then there is nothing to prove since it will follow from (4.5) and

$$0 \le \liminf_{t\to\infty}|x_i(t)| \le \lim_{t\to\infty}|x_i(t)| \le \limsup_{t\to\infty}|x_i(t)| = 0$$

that $\lim_{t\to\infty}|x_i(t)|$ exists and equals zero. Suppose then that $\rho > 0$. Then once again we derive from (4.4) that

$$\begin{aligned} -\frac{2}{n} \le \frac{d^+}{dt}\left[\sum_{i=1}^{n}|x_i(t)|\right]_{t_n} &\le \left(-\min_{1\le i\le n}\{a_i - [b_i]\}\right)\sum_{i=1}^{n}|x_i(t_n)| \\ &\quad + \left(\max_{1\le i\le n}\sum_{j=1}^{n}|c_{ji}|\right)\sum_{i=1}^{n}|x_i(k)|, \quad t_n \in [k, k+1). \end{aligned} \tag{4.6}$$

Passing to the limit as $t_n \to \infty$, we have from (4.6) that

$$0 \le -\left[\min_{1\le i\le n}(a_i - [b_i]) - \max_{1\le i\le n}\sum_{j=1}^{n}|c_{ji}|\right]\rho < 0$$

and this is impossible. Hence we conclude that $\rho = 0$ implying that

$$\lim_{t\to\infty}\sum_{i=1}^{n}|x_i(t)| = 0$$

which implies that the trivial solution of (4.1) is globally attractive.

In the following we plot the solution of the system (4.1) for $j = 2$ with appropriate parameters satisfying condition (4.2) and the initial condition $(x(0), y(0)) = (1.05, 1.85)$. We see that the solution converges to the trivial solution.

$$\begin{cases} \dfrac{dx(t)}{dt} = -1.25x(t) + \tanh\{0.1x(t) - 0.1x([t]) + 0.1y([t])\} \\ \dfrac{dy(t)}{dt} = -y(t) + \tanh\{0.2y(t) + 0.1x([t]) - 0.1y([t])\} \end{cases} \tag{4.7}$$

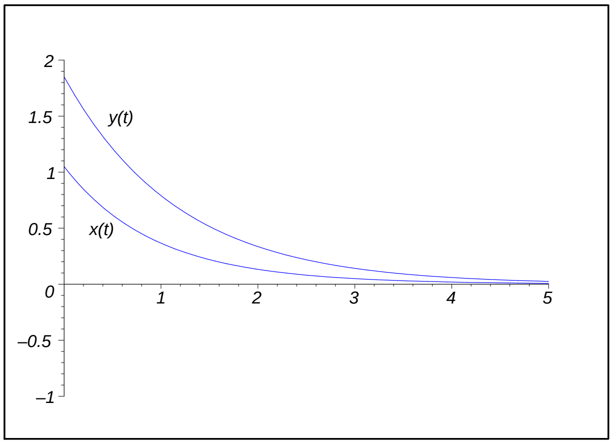


FIGURE 2. Solution $\{x(t), y(t)\}$ of (4.7) converges.

5. CONCLUSION

In this piece of work, we focus on the conditions on parameters to ensure the stability of the equilibrium. The variation of parameters may onset the stability to bifurcation. Such bifurcation may be a Hopf type (see for examples Gopalsamy & Leung 1996 and Guo & Xiu 2009 and even chaotic behaviours.

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