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ANALYSIS OF FITTED SPLINE IN COMPRESSION FOR CONVECTION DIFFUSION PROBLEMS WITH TWO SMALL PARAMETERS

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ABSTRACT. In this paper, we derive an exponentially fitted difference scheme using spline in compression technique for singularly perturbed ordinary differential equation with two small parameters affecting the convection and diffusion terms. The solution of the problem exhibits the boundary layer on the left hand side of the domain. Bounds on the derivatives of the solution are derived. A first order monotone numerical method is constructed. Numerical results are presented to authenticate the theoretical results and to establish the efficiency of the method.

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1. INTRODUCTION

Consider the singularly perturbed convection diffusion problem

$$Ly \equiv -\epsilon y'' - \mu a(x)y' = f(x) \qquad \text{on} \quad \Omega = (0,1) \tag{1.1}$$

$$y(0) = y_0; \quad y(1) = y_1; \quad y_0, y_1 \in \mathbb{R} \tag{1.2}$$

with two small parameters $0 < \epsilon \ll 1$ and $0 < \mu \ll 1$. The functions $a(x)$ and $f(x)$ are assumed to be sufficiently smooth with $a(x) \geq \alpha > 0$; $\alpha \in \mathbb{R}$ for $x \in [0,1]$.

When the parameter $\mu = 1$, the problem is well studied as one parameter convection diffusion singularly perturbed problem. Only few people have discussed two-parameter singular perturbation problem This class of problems occur in dc motor theory, biology, chemical reactor theory and lubrication theory [4], [12], [13], [14].

The asymptotic nature of the solution of two parameter problem is studied extensively by O'Malley, [5], [1], [2], [3].The solution not only depends on the parameters ϵ and μ but also on the ratio ϵ/μ^2. Not much work has been done numerically. In [17] second order parameter uniform methods on a uniform mesh was constructed. In [20], [21] the authors showed that standard upwind finite difference operator on two different choices of Shishkin mesh is first order parameter uniform. In [22] authors

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constructed a second order parameter uniform method on Shishkin mesh. An almost first order convergent method using upwind difference scheme on Shishkin mesh was obtained in [19]. In [16] the author developed non standard finite difference method to tackle two parameter problems.

Spline techniques for numerical solution of singularly perturbed two point boundary value problems have been extensively used. Flaherty and Mathon [23] used tension splines, Sakai and Usmani [24] have used exponential splines to solve one parameter singulary perturbed problems.

There are two ways to obtain small truncation error inside the boundary layer. The first is to use a non-uniform mesh (eg. Shishkin mesh) in the region with more mesh points in the boundary layer region than the rest of the region. The second approach is fitted operator method which is so developed that it reflects the behavior of the solution in the boundary layer region.

In this paper we use spline in compression [6] with a fitting factor, an extension of the approach given in [15], for solving two parameter problems for the case $\epsilon/\mu^2 \to 0$ as $\mu \to 0$. The fitting factor is obtained using exponential fitting. A monotone tridiagonal difference scheme is obtained and the method is first order accurate.

The paper is organized as follows. Section 2 has results on the bounds of the derivatives of the solution of the SPP (1.1)–(1.2). In Section 3 the numerical method is discussed and the fitting factor is obtained for the parameter ϵ. Convergence results are discussed in Section 4. Section 5 has some numerical results to testify the claims made in Section 4. The paper concludes with a discussion.

Notation: For any given function $g(x) \in C^k(\overline{\Omega})$ (k is a non negative integer), $\|g\|$ is a global maximum norm over the domain $\overline{\Omega}$ given by

$$\|g\| = \max_{x\in\overline{\Omega}} |g(x)|$$

Throughout the paper C denotes a generic positive constant independent of ϵ, μ, n.

2. PROPERTIES OF EXACT SOLUTION

Lemma 2.1 (Continuous Maximum principle)**.** *Assume that* $\Phi(x)$ *be any sufficiently smooth function satisfying* $\Phi(0) \geq 0$ *and* $\Phi(1) \geq 0$. *Then* $L\Phi(x) \geq 0$ *for all* $x \in (0,1)$ *implies that* $\Phi(x) \geq 0 \ \forall \ x \in [0,1]$

Proof. Let $x^* \in [0,1]$ be such that $\Phi(x^*) < 0$ and

$$\Phi(x^*) = \min_{x\in[0,1]} \Phi(x)$$

So $x^* \notin \{0,1\}$ and $\Phi'(x^*) = 0$ and $\Phi''(x^*) \geq 0$

$$\implies L\Phi(x)|_{x=x^*} \leq 0, \quad \text{which is a contradiction.}$$

Thus, $\Phi(x) \geq 0 \;\forall\; x \in [0, 1]$ □

Lemma 2.2. *Let $u(x)$ be a solution of (1.1)–(1.2), then*

$$\|u\| \leq C(1 + 1/\mu)$$

Proof. Consider two barrier functions defined by

$$\Psi^{\pm}(x) = \max\{y_0, y_1\} + \frac{1}{\alpha\mu}\|f\|(1 - x) \pm u(x)$$

then, $\Psi^{\pm}(0) \geq 0$ and $\Psi^{\pm}(1) \geq 0$

$$\begin{aligned} L(\Psi^{\pm}(x)) &= -\epsilon(\Psi^{\pm}(x))'' - \mu a(x)(\Psi^{\pm}(x))' \\ &= \frac{a(x)}{\alpha}\|f\| \pm f(x) \\ &\geq \|f\| \pm f(x) \geq 0 \qquad (\text{ since } a(x) \geq \alpha) \end{aligned}$$

Hence by maximum principle,

$$\begin{aligned} \Psi^{\pm}(x) &\geq 0 \\ \Longrightarrow \quad |u(x)| &\leq \max\{y_0, y_1\} + \frac{1}{\alpha\mu}\|f\|(1 - x) \\ &\leq \max\{y_0, y_1\} + \frac{1}{\alpha\mu}\|f\| \\ &\leq C(1 + 1/\mu) \end{aligned}$$

□

Theorem 2.3. *Assuming that $a(x)$ and $f(x)$ are sufficiently smooth. Then the solution $u(x)$ of the boundary value problem (1.1)–(1.2) satisfies*

$$|u^{(k)}(x)| \leq C\left(\frac{1}{\mu^{k+1}} + \frac{1}{\epsilon^k}e^{\frac{-\mu\alpha}{\epsilon}x}\right) \qquad \forall \;\; x \in [0, 1] \tag{2.1}$$

for all $0 \leq k \leq 3$.

Proof. We follow the method of proof adopted in [11] and prove this by induction. By previous lemma

$$\|u\| \leq C(1 + 1/\mu) \qquad \forall \;\; x \in [0, 1]$$

We first find the differential equation satisfied by the derivatives of $u(x)$ by differentiating k times the original equation (1.1) to obtain the bounds on derivatives of $u(x)$

$$Lu^{(k)} = f_k \tag{2.2}$$

where,

$$f_k = f_0 = f, \qquad \text{for } k = 0.$$

and

$$f_k = f^{(k)} + \mu \sum_{s=0}^{k-1} \binom{k}{s} a^{(k-s)} u^{(s+1)} \qquad \forall \ 1 \le k \le 3$$

The inhomogeneous term f_k of the equation satisfied by $u^{(k)}$ depends on the derivatives of u up to order k, the coefficient a and its derivatives along with the k^{th} order derivative of f.

Assume that for all j; $0 \le j \le k$, the following estimates hold

$$|u^{(j)}(x)| \le C\left(\frac{1}{\mu^{j+1}} + \frac{1}{\epsilon^j} e^{\frac{-\mu\alpha}{\epsilon}x}\right) \qquad \forall \ x \in [0,1] \tag{2.3}$$

Thus, the above assumption (2.3) gives

$$Lu^{(k)} = f_k$$

where

$$|u^{(k)}(x)| \le C\left(\frac{1}{\mu^{k+1}} + \frac{1}{\epsilon^k} e^{\frac{-\mu\alpha}{\epsilon}x}\right)$$

and

$$|f^{(k)}(x)| \le C\left(\frac{1}{\mu^{k+1}} + \frac{1}{\epsilon^k} e^{\frac{-\mu\alpha}{\epsilon}x}\right)$$

In particular

$$|u^{(k)}(0)| \ \le \ C\left(\frac{1}{\mu^{k+1}} + \frac{1}{\epsilon^k}\right) \tag{2.4}$$

$$|u^{(k)}(1)| \ \le \ C\left(\frac{1}{\mu^{k+1}} + \frac{1}{\epsilon^k} e^{\frac{-\mu\alpha}{\epsilon}}\right) \le C\left(\frac{1}{\mu^{k+1}} + \frac{1}{\mu\epsilon^{k-1}}\right) \tag{2.5}$$

(as, $\frac{\mu}{\epsilon} e^{\frac{-\mu\alpha}{\epsilon}} \le C$)

Define

$$\theta_k(x) = -\frac{1}{\epsilon}\int_0^x e^{-(A(x)-A(t))} dt$$

where $A(x) = \frac{\mu}{\epsilon}\int_0^x a(s)ds$. Then

$$u_p^{(k)}(x) := \int_0^x \theta_k(t)dt$$

is a particular solution of (2.2). So, the general solution of (2.2) could be written as

$$u^{(k)} = u_p^{(k)} + u_h^{(k)}$$

where, homogeneous solution $u_h^{(k)}$ satisfies

$$Lu_h^{(k)} = 0, \qquad u_h^{(0)} = u^{(k)}(0), \qquad u_h^{(1)} = u^{(k)}(1) - u_p^{(k)}(1)$$

Introducing the function

$$\varphi(x) = \frac{\int_0^x e^{-A(t)}dt}{\int_0^1 e^{-A(t)}dt}$$

also,

$$L\varphi(x) = 0 \ ; \quad \varphi(0) = 0 \ ; \quad 0 \le \varphi(x) \le 1$$

Now, $u_h^{(k)}$ can be written as

$$u_h^{(k)} = u^{(k)}(0)(1-\varphi(x)) + (u^{(k)}(1) - u_p^{(k)}(1))\varphi(x) \tag{2.6}$$

This leads to

$$u^{(k+1)} = u_p^{(k+1)} + u_h^{(k+1)} = \theta_k + (u^{(k)}(1) - u_p^{(k)}(1) - u^{(k)}(0))\varphi'(x)$$

As

$$\varphi'(x) = \frac{e^{-A(x)}}{\int_0^1 e^{-A(t)}dt}$$

the bounds on $a(x)$ leads to

$$|\varphi'(x)| \le C\frac{\mu}{\epsilon}e^{\frac{-\mu\alpha}{\epsilon}x} \tag{2.7}$$

Also, the lower bound on $a(x)$ and bounds on f_k gives

$$|\theta_k(x)| \le C\left(\frac{1}{\alpha\mu^{k+2}} + \frac{1}{\epsilon^{k+1}}e^{\frac{-\mu\alpha}{\epsilon}x}\right) \tag{2.8}$$

Since, $u_p^{(k)}(1) = \int_0^1 \theta_k(t)dt$, it gives

$$|u_p^{(k)}(1)| \le C\left(\frac{1}{\mu^{k+2}} + \frac{1}{\mu\epsilon^{k+1}}\right) \tag{2.9}$$

From (2.6)

$$|u^{(k+1)}| \le |\theta_k| + (|u^{(k)}(1)| + |u_p^{(k)}(1)| + |u^{(k)}(0)|)|\varphi'(x)|$$

Hence, by using the fact that $\epsilon/\mu^2 \to 0$ as $\mu \to 0$ and the equations (2.4), (2.5), (2.7), (2.8), (2.9), we obtain

$$|u^{(k+1)}(x)| \le C\left(\frac{1}{\mu^{k+2}} + \frac{1}{\epsilon^{k+1}}e^{\frac{-\mu\alpha}{\epsilon}x}\right)$$

Hence the proof. □

3. DESCRIPTION OF THE METHOD

Let $0 = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = 1$ be the mesh points. For $x \in [x_{j-1}, x_j]$, we define $\alpha_j = (a_{j-1} + a_j)/2$ and $\gamma_j = (f_{j-1} + f_j)/2$. We define the fitting factor problem associated with (1.1) by

$$Ly \equiv -\sigma(x,\epsilon,\mu)y'' - \mu a(x)y' = f(x) \qquad \text{on} \quad \Omega = (0,1) \tag{3.1}$$

$$y(0) = y_0; \quad y(1) = y_1; \quad y_0, y_1 \in \mathbb{R} \tag{3.2}$$

where $\sigma(x,\epsilon,\mu)$ is an exponential fitting factor to be determined later.

The approximate solution of this problem (1.1) is sought in the form of the function $S(x)$, which on each interval $[x_{j-1}, x_j]$, (denoted by $S_j(x)$) satisfies the following relations:

(i) the differential equation

$$-\sigma_j S_j''(x) - \mu\alpha_j S_j(x) = \gamma_j \tag{3.3}$$

(ii) the interpolation condition

$$S_j(x_{j-1}) = u_{j-1}, \qquad S_j(x_j) = u_j \tag{3.4}$$

(iii) the continuity condition

$$S_j'(x_j^+) = S_j'(x_j^-) \tag{3.5}$$

(iv) the consistency condition

$$\frac{p_j}{2} = \tan\frac{p_j}{2}, \qquad p_j = h\mu\frac{(a_{j-1}+a_j)}{2\sigma_j} = h\mu\frac{\alpha_j}{\sigma_j} \tag{3.6}$$

where,

$$x \in [x_{j-1}, x_j], \qquad x_j = jh \qquad j = 0(1)n, \qquad h = 1/n$$

Solving (3.3) with the help of (3.4), we obtain

$$\begin{aligned} S_j(x) &= \frac{1}{F_j}\left[D_j \exp\left(\frac{-\mu\alpha_j x_{j-1}}{\sigma_j}\right) - E_j \exp\left(\frac{-\mu\alpha_j x_j}{\sigma_j}\right)\right] \\ &\quad + \frac{E_j - D_j}{F_j}\exp\left(\frac{-\mu\alpha_j x}{\sigma_j}\right) + \frac{\gamma_j}{\mu\alpha_j}\left(-x + \frac{\sigma_j}{\mu\alpha_j}\right) \end{aligned} \tag{3.7}$$

where,

$$\begin{aligned} D_j &= u_j + \frac{\gamma_j}{\mu\alpha_j}\left(x_j - \frac{\epsilon}{\mu\alpha_j}\right) \\ E_j &= u_{j-1} + \frac{\gamma_j}{\mu\alpha_j}\left(x_{j-1} - \frac{\epsilon}{\mu\alpha_j}\right) \\ \text{and, } F_j &= \exp(-\mu x_{j-1}\alpha_j/\epsilon) - \exp(-\mu x_j \alpha_j/\epsilon) \end{aligned}$$

3.1. **Derivation of the scheme.** Since $S(x) \in C^2[0,1]$, hence we have

$$S_j'(x_j) = S_{j+1}'(x_j) \tag{3.8}$$

Using (3.7) and (3.8), we obtain

$$\begin{aligned} \left(\frac{-p_j}{e^{p_j}-1}\right)u_{j-1} &+ \left(\frac{p_j}{e^{p_j}-1} + \frac{p_{j+1}}{1-e^{-p_{j+1}}}\right)u_j \\ &+ \left(\frac{p_{j+1}}{1-e^{-p_{j+1}}}\right)u_{j+1} = \left(1 - \frac{p_j}{e^{p_j}-1}\right)\frac{h}{2\mu\alpha_j}f_{j-1} \\ &+ \left(\frac{p_{j+1}}{1-e^{-p_{j+1}}} - 1\right)\frac{h}{2\mu\alpha_{j+1}}f_{j+1} \\ &+ \frac{h}{2\mu}\left[\frac{1}{\alpha_j}\left(1 - \frac{p_j}{e^{p_j}-1}\right) + \frac{1}{\alpha_{j+1}}\left(\frac{p_{j+1}}{1-e^{-p_{j+1}}} - 1\right)\right]f_j \end{aligned} \tag{3.9}$$

Now by using (3.6), we obtain the following difference scheme,

$$Ru_j = Qf_j \qquad j = 1, 2, \ldots, n-1, \tag{3.10}$$

where,

$$
\begin{aligned}
Ru_j &= r_j^- u_{j-1} + r_j^c u_j + r_j^+ u_{j+1} \\
Qf_j &= q_j^- f_{j-1} + q_j^c f_j + q_j^+ f_{j+1}
\end{aligned}
$$

where, $u_0 = y_0$ and $u_1 = y_1$ and

$$r_j^- = -1 + p_j/2; \quad r_j^+ = -(1 + p_{j+1}/2); \quad r_j^c = -(r_j^- + r_j^+)$$

and

$$q_j^- = \frac{h^2}{4\sigma_j}; \quad q_j^+ = \frac{h^2}{4\sigma_{j+1}}; \quad q_j^c = q_j^+ + q_j^-$$

where σ_j is to be determined.

Remark 3.1. The consistency condition (3.6) can be replaced by $(1,1)$ order Padé Approximate for $\exp(x)$ in (3.9) and rewriting all other terms in form of $\exp(x)$. The resulting scheme is identical to (3.10)

3.2. **Determination of fitting factor.** If $a(x) \equiv a =$ constant in (1.1), then the asymptotic behavior of the solution [1] for $\epsilon/\mu^2 \to 0$ as $\mu \to 0$ is

$$y(x) = \exp(-\mu x \alpha/\epsilon) \tag{3.11}$$

To find the fitting factor we require the truncation error for the boundary layer function to be zero when $a(x) \equiv a$

$$(T)_i y = R(y_i) - Qf_i = R(y_i) - Q(Ly)_i$$

With y as in (3.11), $Q(Ly)_i = 0$. So,

$$(T)_i y = 0 \implies R(y_i) = 0$$

which gives the fitting factor as

$$\sigma = \frac{h\mu\alpha}{2} \coth\left(\frac{\mu h \alpha}{2\epsilon}\right), \qquad \text{when} \;\; a(x) \equiv a = \text{constant} \tag{3.12}$$

and

$$\sigma_j = \frac{h\mu\alpha_j}{2} \coth\left(\frac{\mu h \alpha_j}{2\epsilon}\right), \qquad \text{when} \;\; a(x) \neq \text{constant} \tag{3.13}$$

4. ANALYSIS OF THE NUMERICAL METHOD

The resulting system of equation generated by (3.10) for $j = 1, 2, \ldots, n-1$, after incorporating the boundary conditions and taking them to right hand side, is denoted by the matrix A.

Lemma 4.1. *For all $\epsilon, \mu > 0$ and all $h = 1/n$, the matrix A is monotone.*

Proof. Clearly, A is a tridiagonal matrix. Hence, A is irreducible if its co-diagonals contains non zero elements only. The co-diagonals contain $1 \pm \tanh(\mu h/2\epsilon\alpha_j)$. As $0 \leq \tanh(x) < 1 \; \forall \; x \geq 0$, the co-diagonal elements are always non-zero. Hence A is irreducible.

Now consider a row of A, but not the first or the last row. Then the sum of two off-diagonal elements equals the modulus of the diagonal element. Only in the first and the last row the modulus of the diagonal element dominates the off-diagonal element. Thus A is irreducibly diagonally dominant. In addition the off-diagonal elements are positive and the diagonal elements are negative. Thus A is non-singular and $A^{-1} > 0$. This implies A is an M-Matrix and thus monotone. □

Also by [10], the numerical scheme satisfies discrete maximum principle.

4.1. **Proof of uniform convergence.** For error analysis we use the comparison function method developed by Berger et al. [7] and Kellogg and Tsan [8]. By a comparison function we mean a function ϕ such that $L\phi_i > 0$ and $\phi_{\pm n} > 0$, where L is a differential operator and n is a positive integer. These functions are used together with the maximum principle to convert the bounds on truncation error to bounds on discretization error. We use the following two lemma.

Lemma 4.2 (Discrete maximum principle)**.** *Let $\{u_j\}$ be a set of values at the grid points $\{x_j\}$, satisfying $u_0 \leq 0$, $u_1 \leq 0$ and $Ru_j \geq 0$, $j = 1(1)n-1$ then $u_j \leq 0$, $j = 1(1)n$.*

Lemma 4.3. *If $K_1(h,\epsilon,\mu) \geq 0$; $K_2(h,\epsilon,\mu) \geq 0$ are such that*

$$R(K_1(h,\epsilon,\mu)\;\phi_j + K_2(h,\epsilon,\mu)\;\psi_j) \geq R(\pm e_j) = \pm\tau_j(y)$$

for each $j = 1, 2, \ldots, n-1$, then the discrete maximum principle implies that

$$|e_j| \leq K_1(h,\epsilon,\mu)\;|\phi_j| + K_2(h,\epsilon,\mu)\;|\psi_j|$$

where, $|e_j| = |u_j - y(x_j)|$, for each j and ϕ and ψ are two comparison function.

Theorem 4.4. *Let $y(x)$ be the solution of (1.1) and $\{u_j\}$; $j = 0, 1, \ldots, n$, be a set of values of the approximate solution to $y(x)$, obtained by using (3.10). Then there are positive constants β and M (independent of h, ϵ and μ) such that the following estimate holds:*

$$|u_j - y(x_j)| \leq Ch\left(\frac{1}{\mu^3} + \frac{1}{\mu\epsilon}e^{\frac{-\mu\beta}{\epsilon}x_j}\right)$$

Proof. By (3.12) we have,

$$|\sigma_j - \epsilon| \;=\; \left|\frac{h\mu\alpha_j}{2}\coth(\frac{h\mu\alpha_j}{2\epsilon}) - \epsilon\right|$$

Using the expansion for $x\coth(x) = 1 + \frac{x^2}{3} + O(x^4)$ and the consistency condition for this case requires $h < \epsilon/\alpha_j$, we obtain

$$|\sigma_j - \epsilon| \le Ch \tag{4.1}$$

We use two comparison function $\phi = 2 - x$ and $\psi = e^{-\beta\mu x/\epsilon}$, where β will be taken as the minimum of all the constants appearing in the proof. Using (4.1), we obtain

$$R\phi_j \ge Ch^2\mu/\epsilon\ , \quad R\psi_j \ge C(e^{-\beta\mu h/\epsilon})^j h^2\mu^2/\epsilon^2 \tag{4.2}$$

Now we estimate the truncation error of the scheme (3.10)

$$\begin{aligned} \tau_j(y) &= Ry_j - Q(Ly)_j \\ &= T_0 y_j + T_1 y_j' + T_2 y_j'' + \text{remainder terms} \end{aligned}$$

where

$$\begin{aligned} T_0 &= r_j^- + r_j^c + r_j^+ \\ T_1 &= h(r_j^+ - r_j^-) - \mu(q_j^- a_{j-1} + q_j^c a_j + q_j^+ a_{j+1}) \\ T_2 &= \frac{h^2}{2}(r_j^+ - r_j^-) + \epsilon(q_j^- + q_j^c + q_j^+) - h\mu(q_j^- a_{j-1} - q_j^+ a_{j+1}) \end{aligned}$$

Using (3.10), we obtain $T_0 = 0$, $T_1 = 0$ and

$$T_2 = \frac{h^2}{2}\left\{\left(-1 + \frac{h\mu\alpha_j}{2\sigma_j}\right) + \left(-1 - \frac{h\mu\alpha_{j+1}}{2\sigma_{j+1}}\right)\right\} + \epsilon\frac{h^2}{2}\left(\frac{1}{\sigma_j} + \frac{1}{\sigma_{j+1}}\right) - \mu\frac{h^3}{4}\left(\frac{\alpha_{j-1}}{\sigma_j} - \frac{\alpha_{j+1}}{\sigma_{j+1}}\right)$$

now, using (4.1), we get,

$$|T_2| \le C\mu\frac{h^3}{\epsilon}$$

(2.3) gives,

$$|y^{(2)}| \le C\left(\frac{1}{\mu^3} + \frac{1}{\epsilon^2}e^{\frac{-\mu\alpha}{\epsilon}x}\right)$$

Using Theorem 2.3, we get,

$$|T_2 y_j^{(2)}| \le Ch^3\left(\frac{1}{\epsilon\mu^2} + \frac{\mu}{\epsilon^3}e^{\frac{-\mu\alpha}{\epsilon}x}\right)$$

So,

$$|\tau_j(y)| \le Ch^3\left(\frac{1}{\epsilon\mu^2} + \frac{\mu}{\epsilon^3}e^{\frac{-\mu\alpha}{\epsilon}x}\right)$$

By choosing $K_1 = \frac{h}{\mu^3}$ and $K_2 = \frac{h}{\mu\epsilon}$, the Lemma 4.3 is satisfied, we get,

$$|e_j| = |y(x_j) - u_j| \le K_1|\phi_j| + K_2|\psi_j|$$

which on simplification gives,

$$|y(x_j) - u_j| \le Ch\left(\frac{1}{\mu^3} + \frac{1}{\mu\epsilon}e^{\frac{-\mu\beta}{\epsilon}x_j}\right)$$

Hence the theorem. □

5. NUMERICAL RESULTS

This section presents numerical examples to show the applicability of the method. The maximum nodal errors and the order of convergence is calculated using the exact solution (when available) or by using the double mesh principle (when exact solution is not known).

The numerical rate of convergence (if exact solution is known) is calculated as follows. Let $u_j \equiv$ approximate solution for given value of ϵ, μ and n and $u(x_j) \equiv$ exact solution for given value of ϵ, μ and n. Maximum error at all mesh points:

$$E_{\epsilon,\mu,n} = \max_j |u(x_j) - u_j|$$

The numerical rate of convergence is given as

$$r_n = \log_2(E_{\epsilon,\mu,n}/E_{\epsilon,\mu,2n})$$

If exact solution is not known then the rate of convergence is calculated as follows:

$$U_n = \max_j |u_j^{h/2^n} - u_{2j}^{h/2^{n+1}}|$$

where, $u_j^{h/2^n}$ denotes the value of u_j for the mesh length $h/2^n$.

$$r_n = \log_2(U_n/U_{n+1})$$

where, r_n denotes the numerical rate of convergence.

Example 1: $-\epsilon y'' - \mu y' = 2x$; $x \in (0,1)$ with $y(0) = 0$; $y(1) = 0$ the exact solution is

$$y(x) = -\frac{x^2}{\mu} + \frac{(2\epsilon - \mu)}{\mu^2(\exp(-\mu/\epsilon) - 1)}(1 - \exp(-x\mu/\epsilon)) + \frac{2\epsilon x}{\mu^2}$$

TABLE 1. Computed order of convergence for $\mu = 10^{(-1)}$ for Example 1

ϵ	$n = 16$	$n = 32$	$n = 64$	$n = 128$	$n = 256$	$n = 512$
$10^{(-6)}$	0.9531	0.9778	0.9904	0.9980	1.0046	1.0139
$10^{(-7)}$	0.9527	0.9770	0.9989	0.9947	0.9979	1.0001
$10^{(-8)}$	0.9527	0.9769	0.9986	0.9944	0.9972	.9987
$10^{(-9)}$	0.9527	0.9769	0.9986	0.9943	0.9972	.9986
$10^{(-10)}$	0.9527	0.9769	0.9986	0.9943	0.9972	.9986
$10^{(-11)}$	0.9527	0.9769	0.9986	0.9943	0.9972	.9986
$10^{(-12)}$	0.9527	0.9769	0.9986	0.9943	0.9972	.9986

Example 2: $-\epsilon y'' - \mu y' = e^x$; $x \in (0,1)$ with $y(0) = 0$; $y(1) = 1$ the exact solution is

$$y(x) = -\frac{e^x}{\epsilon + \mu} + \frac{(\epsilon + \mu + e - 1)}{(\epsilon + \mu)(\exp(-\mu/\epsilon) - 1)} \exp(-x\mu/\epsilon) - \frac{(\epsilon + \mu + e - e^{-\mu/\epsilon})}{(\epsilon + \mu)(\exp(-\mu/\epsilon) - 1)}$$

TABLE 2. Computed order of convergence for $\mu = 10^{(-2)}$ for Example 1

ϵ	$n = 16$	$n = 32$	$n = 64$	$n = 128$	$n = 256$	$n = 512$
$10^{(-6)}$	0.9573	0.9862	1.0074	1.0327	1.0077	1.0173
$10^{(-8)}$	0.9527	0.9770	0.9888	0.9947	0.9979	1.0001
$10^{(-10)}$	0.9527	0.9769	0.9886	0.9943	0.9972	0.9986
$10^{(-12)}$	0.9527	0.9769	0.9886	0.9943	0.9972	0.9986

TABLE 3. Computed order of convergence for $\mu = 10^{(-4)}$ for Example 1

ϵ	$n = 16$	$n = 32$	$n = 64$	$n = 128$	$n = 256$	$n = 512$
$10^{(-10)}$	0.9527	0.9770	0.9889	0.9447	0.9979	1.0001
$10^{(-12)}$	0.9769	0.9769	0.9886	0.9943	0.9972	0.9986

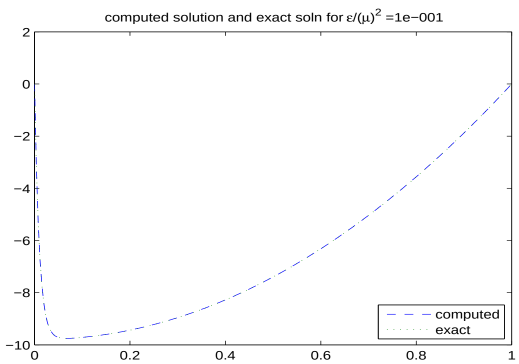

FIGURE 1. Exact and Computed solution of Example 1 for $\epsilon = 10^{-3}, \mu = 10^{-1}$

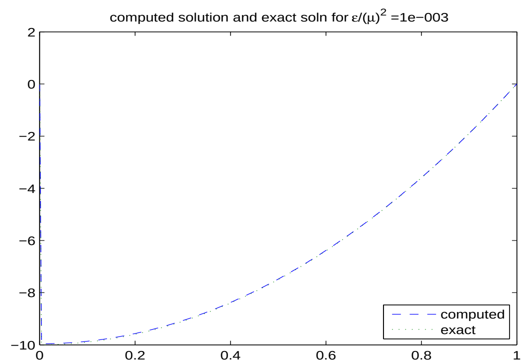

FIGURE 2. Exact and Computed solution of Example 1 for $\epsilon = 10^{-5}, \mu = 10^{-1}$

TABLE 4. Computed order of convergence for $\mu = 10^{(-1)}$ for Example 2

ϵ	$n = 16$	$n = 32$	$n = 64$	$n = 128$	$n = 256$	$n = 512$
$10^{(-6)}$	0.9646	0.9835	0.9933	0.9994	1.0053	1.0139
$10^{(-7)}$	0.9642	0.9827	0.9916	0.9961	0.9986	1.0004
$10^{(-8)}$	0.9641	0.9826	0.9914	0.9958	0.9979	0.9990
$10^{(-9)}$	0.9641	0.9826	0.9914	0.9958	0.9979	0.9989
$10^{(-10)}$	0.9641	0.9826	0.9914	0.9958	0.9979	0.9989
$10^{(-11)}$	0.9641	0.9826	0.9914	0.9958	0.9979	0.9989
$10^{(-12)}$	0.9641	0.9826	0.9914	0.9958	0.9979	0.9990

Example 3: $-\epsilon y'' - \mu(1+x)y' = (1+x)^2$; $x \in (0,1)$ with $y(0) = 0$; $y(1) = 0$ the exact solution is not known so we use double mesh principle to calculate the order of convergence.

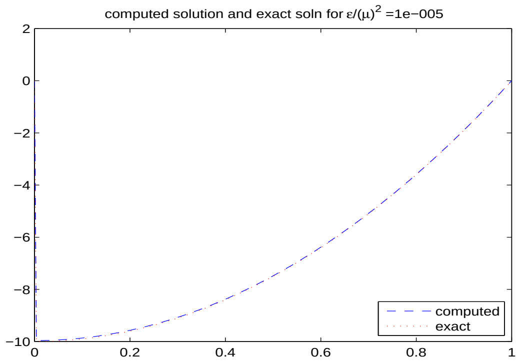

FIGURE 3. Exact and Computed solution of Example 1 for $\epsilon = 10^{-7}, \mu = 10^{-1}$

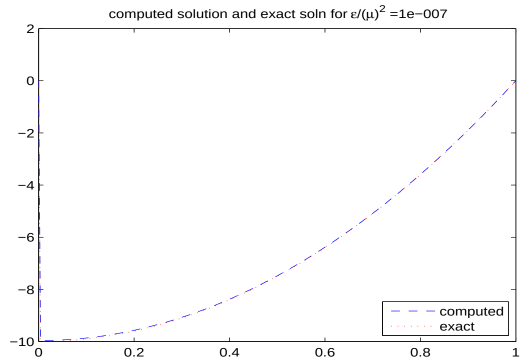

FIGURE 4. Exact and Computed solution of Example 1 for $\epsilon = 10^{-9}, \mu = 10^{-1}$

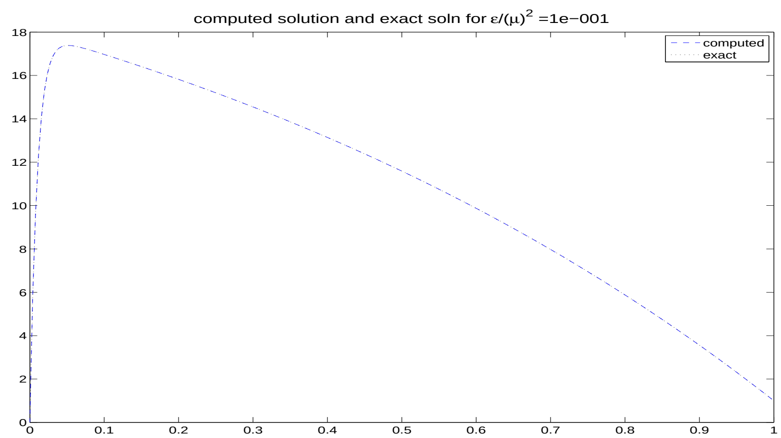

FIGURE 5. Exact and Computed solution of Example 2 for $\epsilon = 10^{-3}, \mu = 10^{-1}$

TABLE 5. Computed order of convergence for $\mu = 10^{(-2)}$ for Example 2

ϵ	$n = 16$	$n = 32$	$n = 64$	$n = 128$	$n = 256$	$n = 512$
$10^{(-6)}$	0.9919	0.9919	1.0102	1.0341	1.0077	1.0135
$10^{(-8)}$	0.9641	0.9827	0.9916	0.9961	0.9986	1.0004
$10^{(-10)}$	0.9641	0.9826	0.9914	0.9957	0.9979	0.9990
$10^{(-12)}$	0.9641	0.9826	0.9914	0.9958	0.9979	0.9990

6. DISCUSSION

In this paper we presented an approximate method based on exponentially fitted spline in compression for solving two parameter singularly perturbed two point boundary value problem. The spline in compression was used for their ease in use

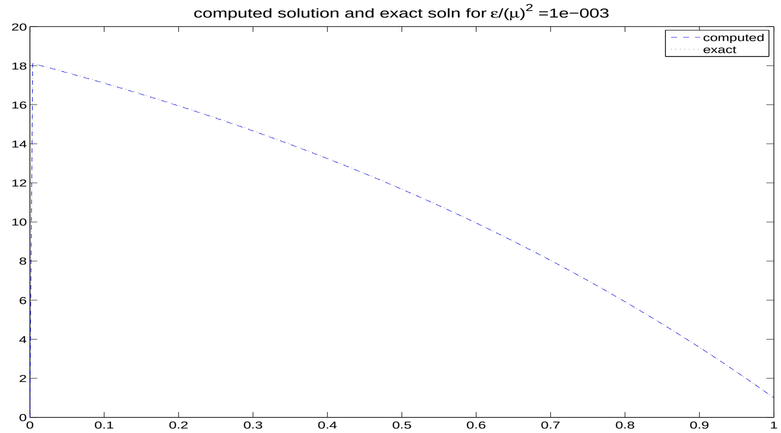

FIGURE 6. Exact and Computed solution of Example 2 for $\epsilon = 10^{-5}, \mu = 10^{-1}$

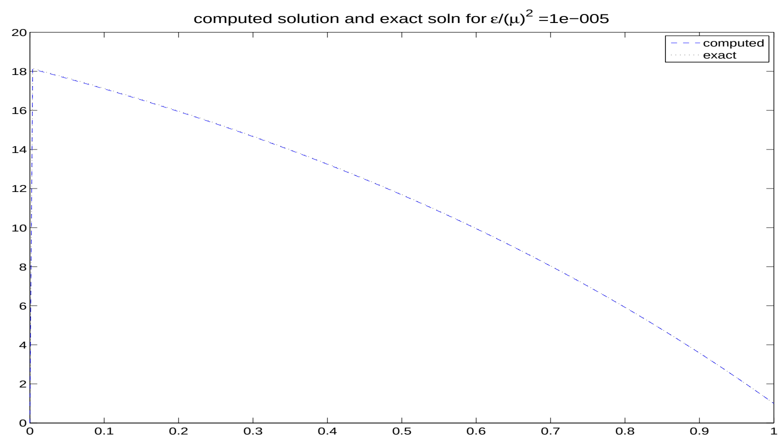

FIGURE 7. Exact and Computed solution of Example 2 for $\epsilon = 10^{-7}, \mu = 10^{-1}$

TABLE 6. Computed order of convergence for $\mu = 10^{(-4)}$ for Example 2

ϵ	$n = 16$	$n = 32$	$n = 64$	$n = 128$	$n = 256$	$n = 512$
$10^{(-10)}$	0.9641	0.9827	0.9916	0.9961	0.9986	1.0004
$10^{(-12)}$	0.9641	0.9826	0.9914	0.9958	0.9979	0.9990

and easy computer implementation. The difference scheme obtained was monotone. The method also leads to tridiagonal matrix as opposed to full matrices using polynomials, trigonometric functions as approximates. Test examples have been given to show the efficiency of the proposed method. To substantiate the suitability of the proposed method, graphs have been plotted for Examples 1 and 2 for different values of the parameter ϵ and μ. As the mesh is refined for a fixed value of the parameter ϵ and μ the computed rate of convergence increases to 1.

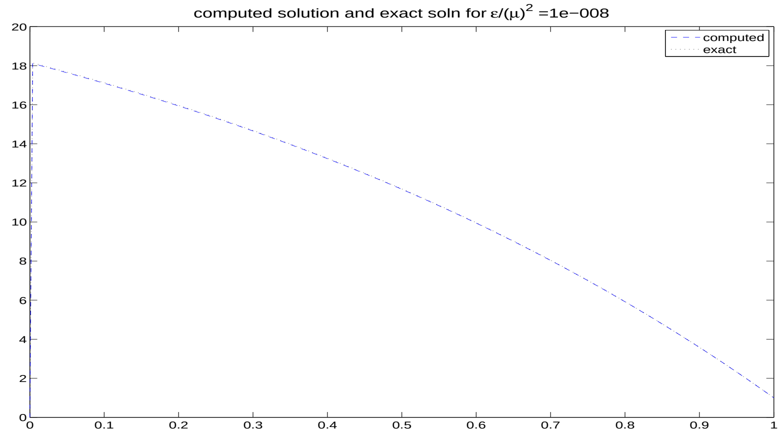

FIGURE 8. Exact and Computed solution of Example 2 for $\epsilon = 10^{-10}, \mu = 10^{-1}$

TABLE 7. Computed order of convergence for $\mu = 10^{(-1)}$ for Example 3

ϵ	$n = 16$	$n = 32$	$n = 64$	$n = 128$	$n = 256$	$n = 512$
$10^{(-6)}$	0.9287	0.9651	0.9827	0.9914	0.9957	0.9978
$10^{(-7)}$	0.9287	0.9651	0.9827	0.9914	0.9957	0.9978
$10^{(-8)}$	0.9287	0.9651	0.9827	0.9914	0.9957	0.9978
$10^{(-9)}$	0.9287	0.9651	0.9827	0.9914	0.9957	0.9978
$10^{(-10)}$	0.9287	0.9651	0.9827	0.9914	0.9957	0.9978
$10^{(-11)}$	0.9287	0.9651	0.9827	0.9914	0.9957	0.9978
$10^{(-12)}$	0.9287	0.9651	0.9827	0.9914	0.9957	0.9978

TABLE 8. Computed order of convergence for $\mu = 10^{(-2)}$ for Example 3

ϵ	$n = 16$	$n = 32$	$n = 64$	$n = 128$	$n = 256$	$n = 512$
$10^{(-6)}$	0.9287	0.9651	0.9827	0.9914	0.9957	1.0023
$10^{(-8)}$	0.9287	0.9651	0.9827	0.9914	0.9957	0.9978
$10^{(-10)}$	0.9287	0.9651	0.9827	0.9914	0.9957	0.9978
$10^{(-12)}$	0.9287	0.9651	0.9827	0.9914	0.9957	0.9978

TABLE 9. Computed order of convergence for $\mu = 10^{(-4)}$ for Example 3

ϵ	$n = 16$	$n = 32$	$n = 64$	$n = 128$	$n = 256$	$n = 512$
$10^{(-10)}$	0.9287	0.9651	0.9827	0.9914	0.9957	0.9979
$10^{(-12)}$	0.9287	0.9651	0.9827	0.9914	0.9957	0.9979

The numerical results established the claim of the first order accurate method calculated for various values of the parameter ϵ and μ with the ratio $\epsilon/\mu^2 \to 0$. All calculations are done in MATLAB.

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