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MULTIVARIATE VORONOVSKAYA TYPE ASYMPTOTIC EXPANSIONS FOR NORMALIZED BELL AND SQUASHING TYPE NEURAL NETWORK OPERATORS

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ABSTRACT. Here we introduce the multivariate normalized bell and squashing type neural network operators of one hidden layer. We derive multivariate Voronovskaya type asymptotic expansions for the error of approximation of these operators to the unit operator.

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1. Background

In [6] the authors presented for the fist time approximation of functions by specific completely described neural network operators. However their approach was only qualitative. The author in [1], [2] continued the work of [6] by presenting for the first time quantitative approximation by determining the rate of convergence and involving the modulus of continuity of the function under approximation. In this work we engage very flexible neural network operators for the first time that derive by normalization of operators of [6], so we are able to produce asymptotic expansions of Voronovkaya type regarding the approximation of these operators to the unit operator.

We use the following (see [6]).

Definition 1.1. A function $b : \mathbb{R} \to \mathbb{R}$ is said to be bell-shaped if b belongs to L^1 and its integral is nonzero, if it is nondecreasing on $(-\infty, a)$ and nonincreasing on $[a, +\infty)$, where a belongs to $\mathbb{R}$. In particular $b(x)$ is a nonnegative number and at a, b takes a global maximum; it is the center of the bell-shaped function. A bell-shaped function is said to be centered if its center is zero.

Definition 1.2 (see [6])**.** A function $b : \mathbb{R}^d \to \mathbb{R}$ $(d \geq 1)$ is said to be a d-dimensional bell-shaped function if it is integrable and its integral is not zero, and for all $i =$

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$1, \ldots, d$,

$$t \to b(x_1, \ldots, t, \ldots, x_d)$$

is a centered bell-shaped function, where $\overrightarrow{x} := (x_1, \ldots, x_d) \in \mathbb{R}^d$ arbitrary.

Example 1.3 (from [6]). Let b be a centered bell-shaped function over $\mathbb{R}$, then $(x_1, \ldots, x_d) \to b(x_1) \cdots b(x_d)$ is a d-dimensional bell-shaped function.

Assumption 1.4. Here $b(\overrightarrow{x})$ is of compact support $\mathcal{B} := \prod_{i=1}^{d} [-T_i, T_i]$, $T_i > 0$ and it may have jump discontinuities there.

Let $f \in C(\mathbb{R}^d)$.

In this article we find a multivariate Voronovskaya type asymptotic expansion for the multivariate normalized bell type neural network operators,

$$M_n(f)(\overrightarrow{x}) :=$$

$$\frac{\sum_{k_1=-n^2}^{n^2} \cdots \sum_{k_d=-n^2}^{n^2} f\left(\frac{k_1}{n}, \ldots, \frac{k_d}{n}\right) b\left(n^{1-\beta}\left(x_1 - \frac{k_1}{n}\right), \ldots, n^{1-\beta}\left(x_d - \frac{k_d}{n}\right)\right)}{\sum_{k_1=-n^2}^{n^2} \cdots \sum_{k_d=-n^2}^{n^2} b\left(n^{1-\beta}\left(x_1 - \frac{k_1}{n}\right), \ldots, n^{1-\beta}\left(x_d - \frac{k_d}{n}\right)\right)}, \tag{1}$$

where $0 < \beta < 1$ and $\overrightarrow{x} := (x_1, \ldots, x_d) \in \mathbb{R}^d$, $n \in \mathbb{N}$. Clearly M_n is a positive linear operator.

The terms in the ratio of multiple sums (1) can be nonzero iff simultaneously

$$\left| n^{1-\beta}\left(x_i - \frac{k_i}{n}\right) \right| \leq T_i, \text{ all } i = 1, \ldots, d$$

i.e., $\left|x_i - \frac{k_i}{n}\right| \leq \frac{T_i}{n^{1-\beta}}$, all $i = 1, \ldots, d$, iff

$$nx_i - T_i n^{\beta} \leq k_i \leq nx_i + T_i n^{\beta}, \text{ all } i = 1, \ldots, d. \tag{2}$$

To have the order

$$-n^2 \leq nx_i - T_i n^{\beta} \leq k_i \leq nx_i + T_i n^{\beta} \leq n^2, \tag{3}$$

we need $n \geq T_i + |x_i|$, all $i = 1, \ldots, d$. So (3) is true when we consider

$$n \geq \max_{i \in \{1, \ldots, d\}} (T_i + |x_i|). \tag{4}$$

When $\overrightarrow{x} \in \mathcal{B}$ in order to have (3) it is enough to suppose that $n \geq 2T^*$, where $T^* := \max\{T_1, \ldots, T_d\} > 0$. Take

$$\widetilde{I}_i := \left[nx_i - T_i n^{\beta}, nx_i + T_i n^{\beta}\right], \quad i = 1, \ldots, d, \quad n \in \mathbb{N}.$$

The length of $\widetilde{I}_i$ is $2T_i n^{\beta}$. By Proposition 2.1, p. 61 of [3], we obtain that the cardinality of $k_i \in \mathbb{Z}$ that belong to $\widetilde{I}_i := card(k_i) \geq \max\left(2T_i n^{\beta} - 1, 0\right)$, any $i \in \{1, \ldots, d\}$. In order to have $card(k_i) \geq 1$ we need $2T_i n^{\beta} - 1 \geq 1$ iff $n \geq T_i^{-\frac{1}{\beta}}$, any $i \in \{1, \ldots, d\}$.

Therefore, a sufficient condition for causing the order (3) along with the interval $\widetilde{I}_i$ to contain at least one integer for all $i = 1, \ldots, d$ is that

$$n \geq \max_{i \in \{1,\ldots,d\}} \left\{ T_i + |x_i|, T_i^{-\frac{1}{\beta}} \right\}. \tag{5}$$

Clearly as $n \to +\infty$ we get that $card(k_i) \to +\infty$, all $i = 1, \ldots, d$. Also notice that $card(k_i)$ equals to the cardinality of integers in $\left[\lceil nx_i - T_i n^{\beta} \rceil, \left[nx_i + T_i n^{\beta}\right]\right]$ for all $i = 1, \ldots, d$.

Here we denote by $\lceil \cdot \rceil$ the ceiling of the number, and by $[\cdot]$ we denote the integral part.

From now on in this article we assume (5). Therefore

$$(M_n(f))(\overrightarrow{x}) = \tag{6}$$

$$\frac{\sum_{k_1=\lceil nx_1 - T_1 n^{\beta}\rceil}^{[nx_1+T_1n^{\beta}]} \cdots \sum_{k_d=\lceil nx_d - T_d n^{\beta}\rceil}^{[nx_d+T_dn^{\beta}]} f\left(\frac{k_1}{n}, \ldots, \frac{k_d}{n}\right) b\left(n^{1-\beta}\left(x_1 - \frac{k_1}{n}\right), \ldots, n^{1-\beta}\left(x_d - \frac{k_d}{n}\right)\right)}{\sum_{k_1=\lceil nx_1 - T_1 n^{\beta}\rceil}^{[nx_1+T_1n^{\beta}]} \cdots \sum_{k_d=\lceil nx_d - T_d n^{\beta}\rceil}^{[nx_d+T_dn^{\beta}]} b\left(n^{1-\beta}\left(x_1 - \frac{k_1}{n}\right), \ldots, n^{1-\beta}\left(x_d - \frac{k_d}{n}\right)\right)}$$

all $\overrightarrow{x} := (x_1, \ldots, x_d) \in \mathbb{R}^d$.

In brief we write

$$(M_n(f))(\overrightarrow{x}) = \frac{\sum_{\overrightarrow{k}=\lceil n\overrightarrow{x} - \overrightarrow{T} n^{\beta}\rceil}^{[n\overrightarrow{x}+\overrightarrow{T}n^{\beta}]} f\left(\frac{\overrightarrow{k}}{n}\right) b\left(n^{1-\beta}\left(\overrightarrow{x} - \frac{\overrightarrow{k}}{n}\right)\right)}{\sum_{\overrightarrow{k}=\lceil n\overrightarrow{x} - \overrightarrow{T} n^{\beta}\rceil}^{[n\overrightarrow{x}+\overrightarrow{T}n^{\beta}]} b\left(n^{1-\beta}\left(\overrightarrow{x} - \frac{\overrightarrow{k}}{n}\right)\right)}, \tag{7}$$

all $\overrightarrow{x} \in \mathbb{R}^d$.

Denote by $\|\cdot\|_{\infty}$ the maximum norm on $\mathbb{R}^d$, $d \geq 1$. So if $\left|n^{1-\beta}\left(x_i - \frac{k_i}{n}\right)\right| \leq T_i$, all $i = 1, \ldots, d$, we find that

$$\left\| \overrightarrow{x} - \frac{\overrightarrow{k}}{n} \right\|_{\infty} \leq \frac{T^*}{n^{1-\beta}}, \tag{8}$$

where $\overrightarrow{k} := (k_1, \ldots, k_d)$.

We also need

Definition 1.5. Let the nonnegative function $S : \mathbb{R}^d \to \mathbb{R}$, $d \geq 1$, S has compact support $\mathcal{B} := \prod_{i=1}^{d} [-T_i, T_i]$, $T_i > 0$ and is nondecreasing for each coordinate. S can be continuous only on either $\prod_{i=1}^{d}(-\infty, T_i]$ or $\mathcal{B}$ and can have jump discontinuities. We call S the multivariate "squashing function" (see also [6]).

Example 1.6. Let $\widehat{S}$ as above when $d = 1$. Then

$$S(\overrightarrow{x}) := \widehat{S}(x_1) \cdots \widehat{S}(x_d), \ \overrightarrow{x} := (x_1, \ldots, x_d) \in \mathbb{R}^d,$$

is a multivariate "squashing function".

Let $f \in C\left(\mathbb{R}^d\right)$.

For $\overrightarrow{x} \in \mathbb{R}^d$ we define also the "multivariate normalized squashing type neural network operators",

$$L_n(f)(\overrightarrow{x}) :=$$

$$\frac{\sum_{k_1=-n^2}^{n^2} \cdots \sum_{k_d=-n^2}^{n^2} f\left(\frac{k_1}{n}, \ldots, \frac{k_d}{n}\right) S\left(n^{1-\beta}\left(x_1 - \frac{k_1}{n}\right), \ldots, n^{1-\beta}\left(x_d - \frac{k_d}{n}\right)\right)}{\sum_{k_1=-n^2}^{n^2} \cdots \sum_{k_d=-n^2}^{n^2} S\left(n^{1-\beta}\left(x_1 - \frac{k_1}{n}\right), \ldots, n^{1-\beta}\left(x_d - \frac{k_d}{n}\right)\right)}. \tag{9}$$

We also here find a multivariate Voronovskaya type asymptotic expansion for $(L_n(f))(\overrightarrow{x})$.

Here again $0 < \beta < 1$ and $n \in \mathbb{N}$:

$$n \geq \max_{i \in \{1, \ldots, d\}} \left\{T_i + |x_i|, T_i^{-\frac{1}{\beta}}\right\},$$

and L_n is a positive linear operator. It is clear that

$$(L_n(f))(\overrightarrow{x}) = \frac{\sum_{\overrightarrow{k}=\lceil n\overrightarrow{x} - \overrightarrow{T} n^\beta \rceil}^{[n\overrightarrow{x} + \overrightarrow{T} n^\beta]} f\left(\frac{\overrightarrow{k}}{n}\right) S\left(n^{1-\beta}\left(\overrightarrow{x} - \frac{\overrightarrow{k}}{n}\right)\right)}{\sum_{\overrightarrow{k}=\lceil n\overrightarrow{x} - \overrightarrow{T} n^\beta \rceil}^{[n\overrightarrow{x} + \overrightarrow{T} n^\beta]} S\left(n^{1-\beta}\left(\overrightarrow{x} - \frac{\overrightarrow{k}}{n}\right)\right)}. \tag{10}$$

For related articles on neural networks approximation, see [1], [2], [3] and [5]. For neural networks in general, see [7], [8] and [9].

Next we follow [4, pp. 284–286].

About Multivariate Taylor formula and estimates

Let $\mathbb{R}^d$; $d \geq 2$; $z := (z_1, \ldots, z_d)$, $x_0 := (x_{01}, \ldots, x_{0d}) \in \mathbb{R}^d$. We consider the space of functions $AC^N\left(\mathbb{R}^d\right)$ with $f : \mathbb{R}^d \to \mathbb{R}$ be such that all partial derivatives of order $(N-1)$ are coordinatewise absolutely continuous functions on compacta, $N \in \mathbb{N}$. Also $f \in C^{N-1}\left(\mathbb{R}^d\right)$. Each N^{th} order partial derivative is denoted by $f_\alpha := \frac{\partial^\alpha f}{\partial x^\alpha}$, where $\alpha := (\alpha_1, \ldots, \alpha_d)$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, \ldots, d$ and $|\alpha| := \sum_{i=1}^d \alpha_i = N$. Consider $g_z(t) := f(x_0 + t(z - x_0))$, $t \geq 0$. Then

$$g_z^{(j)}(t) = \left[\left(\sum_{i=1}^d (z_i - x_{0i}) \frac{\partial}{\partial x_i}\right)^j f\right](x_{01} + t(z_1 - x_{01}), \ldots, x_{0d} + t(z_N - x_{0d})), \tag{11}$$

for all $j = 0, 1, 2, \ldots, N$.

Example 1.7. Let $d = N = 2$. Then

$$g_z(t) = f(x_{01} + t(z_1 - x_{01}), x_{02} + t(z_2 - x_{02})), \quad t \in \mathbb{R},$$

and

$$g_z'(t) = (z_1 - x_{01}) \frac{\partial f}{\partial x_1}(x_0 + t(z - x_0)) + (z_2 - x_{02}) \frac{\partial f}{\partial x_2}(x_0 + t(z - x_0)). \tag{12}$$

Setting

$$(*) = (x_{01} + t(z_1 - x_{01}), x_{02} + t(z_2 - x_{02})) = (x_0 + t(z - x_0)),$$

we get

$$g_z''(t) = (z_1 - x_{01})^2 \frac{\partial f^2}{\partial x_1^2}(*) + (z_1 - x_{01})(z_2 - x_{02}) \frac{\partial f^2}{\partial x_2 \partial x_1}(*) +$$

$$(z_1 - x_{01})(z_2 - x_{02}) \frac{\partial f^2}{\partial x_1 \partial x_2}(*) + (z_2 - x_{02})^2 \frac{\partial f^2}{\partial x_2^2}(*) . \tag{13}$$

Similarly, we have the general case of $d, N \in \mathbb{N}$ for $g_z^{(N)}(t)$.

We mention the following multivariate Taylor theorem.

Theorem 1.8. *Under the above assumptions we have*

$$f(z_1, \ldots, z_d) = g_z(1) = \sum_{j=0}^{N-1} \frac{g_z^{(j)}(0)}{j!} + R_N(z, 0), \tag{14}$$

where

$$R_N(z, 0) := \int_0^1 \left(\int_0^{t_1} \cdots \left(\int_0^{t_{N-1}} g_z^{(N)}(t_N)\, dt_N \right) \cdots \right) dt_1, \tag{15}$$

or

$$R_N(z, 0) = \frac{1}{(N-1)!} \int_0^1 (1-\theta)^{N-1} g_z^{(N)}(\theta)\, d\theta. \tag{16}$$

Notice that $g_z(0) = f(x_0)$.

We make

Remark 1.9. Assume here that

$$\|f_\alpha\|_{\infty, \mathbb{R}^d, N}^{\max} := \max_{|\alpha|=N} \|f_\alpha\|_{\infty, \mathbb{R}^d} < \infty.$$

Then

$$\left\|g_z^{(N)}\right\|_{\infty,[0,1]} = \left\| \left[\left(\sum_{i=1}^d (z_i - x_{0i}) \frac{\partial}{\partial x_i} \right)^N f \right] (x_0 + t(z - x_0)) \right\|_{\infty,[0,1]} \leq \tag{17}$$

$$\left(\sum_{i=1}^d |z_i - x_{0i}| \right)^N \|f_\alpha\|_{\infty, \mathbb{R}^d, N}^{\max},$$

that is

$$\left\|g_z^{(N)}\right\|_{\infty,[0,1]} \leq \left(\|z - x_0\|_{l_1} \right)^N \|f_\alpha\|_{\infty, \mathbb{R}^d, N}^{\max} < \infty. \tag{18}$$

Hence we get by (16) that

$$|R_N(z, 0)| \leq \frac{\left\|g_z^{(N)}\right\|_{\infty,[0,1]}}{N!} < \infty. \tag{19}$$

And it holds

$$|R_N(z, 0)| \leq \frac{\left(\|z - x_0\|_{l_1} \right)^N}{N!} \|f_\alpha\|_{\infty, \mathbb{R}^d, N}^{\max}, \tag{20}$$

$\forall\ z, x_0 \in \mathbb{R}^d$.

Inequality (20) will be an important tool in proving our main results.

2. **Main Results**

We present our first main result.

Theorem 2.1. *Let* $f \in AC^N\left(\mathbb{R}^d\right)$, $d \in \mathbb{N} - \{1\}$, $N \in \mathbb{N}$, *with* $\|f_\alpha\|_{\infty,\mathbb{R}^d,N}^{\max} < \infty$. *Here* $n \geq \max_{i \in \{1,\ldots,d\}} \left\{ T_i + |x_i|, T_i^{-\frac{1}{\beta}} \right\}$, *where* $\overrightarrow{x} \in \mathbb{R}^d$, $0 < \beta < 1$, $n \in \mathbb{N}$, $T_i > 0$. *Then*

$$(M_n(f))(\overrightarrow{x}) - f(\overrightarrow{x}) =$$

$$\sum_{j=1}^{N-1} \left(\sum_{|\alpha|=j} \left(\frac{f_\alpha(\overrightarrow{x})}{\prod_{i=1}^d \alpha_i!} \right) M_n \left(\prod_{i=1}^d (\cdot - x_i)^{\alpha_i}, \overrightarrow{x} \right) \right) + o\left(\frac{1}{n^{(N-\varepsilon)(1-\beta)}} \right), \tag{21}$$

where $0 < \varepsilon \leq N$.

If $N = 1$, *the sum in* (21) *collapses.*

The last (21) *implies that*

$$n^{(N-\varepsilon)(1-\beta)} \left[(M_n(f))(\overrightarrow{x}) - f(\overrightarrow{x}) - \right. \tag{22}$$

$$\left. \sum_{j=1}^{N-1} \left(\sum_{|\alpha|=j} \left(\frac{f_\alpha(\overrightarrow{x})}{\prod_{i=1}^d \alpha_i!} \right) M_n \left(\prod_{i=1}^d (\cdot - x_i)^{\alpha_i}, \overrightarrow{x} \right) \right) \right] \to 0, \text{ as } n \to \infty,$$

$0 < \varepsilon \leq N$.

When $N = 1$, *or* $f_\alpha(\overrightarrow{x}) = 0$, *all* $\alpha : |\alpha| = j = 1, \ldots, N-1$, *then we derive*

$$n^{(N-\varepsilon)(1-\beta)} \left[(M_n(f))(\overrightarrow{x}) - f(\overrightarrow{x}) \right] \to 0,$$

as $n \to \infty$, $0 < \varepsilon \leq N$.

Proof. Put

$$g_{\frac{\overrightarrow{k}}{n}}(t) := f\left(\overrightarrow{x} + t\left(\frac{\overrightarrow{k}}{n} - \overrightarrow{x} \right) \right), \quad 0 \leq t \leq 1.$$

Then

$$g_{\frac{\overrightarrow{k}}{n}}^{(j)}(t) =$$

$$\left[\left(\sum_{i=1}^d \left(\frac{k_i}{n} - x_i \right) \frac{\partial}{\partial x_i} \right)^j f \right] \left(x_1 + t\left(\frac{k_1}{n} - x_1 \right), \ldots, x_d + t\left(\frac{k_d}{n} - x_d \right) \right), \tag{23}$$

and $g_{\frac{\overrightarrow{k}}{n}}(0) = f(\overrightarrow{x})$. By Taylor's formula (14), (16) we obtain

$$f\left(\frac{k_1}{n}, \ldots, \frac{k_d}{n} \right) = g_{\frac{\overrightarrow{k}}{n}}(1) = \sum_{j=0}^{N-1} \frac{g_{\frac{\overrightarrow{k}}{n}}^{(j)}(0)}{j!} + R_N\left(\frac{\overrightarrow{k}}{n}, 0 \right), \tag{24}$$

where

$$R_N\left(\frac{\overrightarrow{k}}{n},0\right)=\frac{1}{(N-1)!}\int_0^1(1-\theta)^{N-1}g_{\frac{\overrightarrow{k}}{n}}^{(N)}(\theta)\,d\theta. \tag{25}$$

More precisely we can rewrite

$$f\left(\frac{\overrightarrow{k}}{n}\right)-f\left(\overrightarrow{x}\right)=$$

$$\sum_{j=1}^{N-1}\sum_{\substack{\alpha:=(\alpha_1,\dots,\alpha_d),\alpha_i\in\mathbb{Z}^+,\\ i=1,\dots,d,|\alpha|:=\sum_{i=1}^d\alpha_i=j}}\left(\frac{1}{\prod_{i=1}^d\alpha_i!}\right)\left(\prod_{i=1}^d\left(\frac{k_i}{n}-x_i\right)^{\alpha_i}\right)f_\alpha\left(\overrightarrow{x}\right)+R_N\left(\frac{\overrightarrow{k}}{n},0\right), \tag{26}$$

where

$$R_N\left(\frac{\overrightarrow{k}}{n},0\right)=N\int_0^1(1-\theta)^{N-1}\sum_{\substack{\alpha:=(\alpha_1,\dots,\alpha_d),\alpha_i\in\mathbb{Z}^+,\\ i=1,\dots,d,|\alpha|:=\sum_{i=1}^d\alpha_i=N}}\left(\frac{1}{\prod_{i=1}^d\alpha_i!}\right)\cdot$$

$$\left(\prod_{i=1}^d\left(\frac{k_i}{n}-x_i\right)^{\alpha_i}\right)f_\alpha\left(\overrightarrow{x}+\theta\left(\frac{\overrightarrow{k}}{n}-\overrightarrow{x}\right)\right)d\theta. \tag{27}$$

By (20) we get

$$\left|R_N\left(\frac{\overrightarrow{k}}{n},0\right)\right|\leq\frac{\left(\left\|\frac{\overrightarrow{k}}{n}-\overrightarrow{x}\right\|_{l_1}\right)^N}{N!}\left\|f_\alpha\right\|_{\infty,\mathbb{R}^d,N}^{\max}. \tag{28}$$

So, since here it holds

$$\left\|\overrightarrow{x}-\frac{\overrightarrow{k}}{n}\right\|_\infty\leq\frac{T^*}{n^{1-\beta}},$$

then

$$\left\|\overrightarrow{x}-\frac{\overrightarrow{k}}{n}\right\|_{l_1}\leq\frac{dT^*}{n^{1-\beta}},$$

and

$$\left|R_N\left(\frac{\overrightarrow{k}}{n},0\right)\right|\leq\frac{d^NT^{*N}}{n^{N(1-\beta)}N!}\left\|f_\alpha\right\|_{\infty,\mathbb{R}^d,N}^{\max}, \tag{29}$$

for all $\overrightarrow{k}\in\left\{\left\lceil n\overrightarrow{x}-\overrightarrow{T}n^\beta\right\rceil,\dots,\left[n\overrightarrow{x}+\overrightarrow{T}n^\beta\right]\right\}$.

Call

$$V\left(\overrightarrow{x}\right):=\sum_{\overrightarrow{k}=\lceil n\overrightarrow{x}-\overrightarrow{T}n^\beta\rceil}^{[n\overrightarrow{x}+\overrightarrow{T}n^\beta]}b\left(n^{1-\beta}\left(\overrightarrow{x}-\frac{\overrightarrow{k}}{n}\right)\right). \tag{30}$$

We observe for

$$U_n(\overrightarrow{x}) := \frac{\sum_{\overrightarrow{k}=\lceil n\overrightarrow{x}-\overrightarrow{T}n^{\beta}\rceil}^{[n\overrightarrow{x}+\overrightarrow{T}n^{\beta}]} R_N\left(\frac{\overrightarrow{k}}{n},0\right) b\left(n^{1-\beta}\left(\overrightarrow{x}-\frac{\overrightarrow{k}}{n}\right)\right)}{V(\overrightarrow{x})}, \tag{31}$$

that

$$|U_n(\overrightarrow{x})| \overset{\text{(by (29))}}{\leq} \frac{d^N T^{*N}}{n^{N(1-\beta)} N!} \|f_\alpha\|_{\infty,\mathbb{R}^d,N}^{\max}. \tag{32}$$

That is

$$|U_n(\overrightarrow{x})| = O\left(\frac{1}{n^{N(1-\beta)}}\right), \tag{33}$$

and

$$|U_n(\overrightarrow{x})| = o(1). \tag{34}$$

And, letting $0 < \varepsilon \leq N$, we derive

$$\frac{|U_n(\overrightarrow{x})|}{\left(\frac{1}{n^{(N-\varepsilon)(1-\beta)}}\right)} \leq \left(\frac{d^N T^{*N} \|f_\alpha\|_{\infty,\mathbb{R}^d,N}^{\max}}{N!}\right) \frac{1}{n^{\varepsilon(1-\beta)}} \to 0, \tag{35}$$

as $n \to \infty$.

I.e.

$$|U_n(\overrightarrow{x})| = o\left(\frac{1}{n^{(N-\varepsilon)(1-\beta)}}\right). \tag{36}$$

By (26) we get

$$\frac{\sum_{\overrightarrow{k}=\lceil n\overrightarrow{x}-\overrightarrow{T}n^{\beta}\rceil}^{[n\overrightarrow{x}+\overrightarrow{T}n^{\beta}]} f\left(\frac{\overrightarrow{k}}{n}\right) b\left(n^{1-\beta}\left(\overrightarrow{x}-\frac{\overrightarrow{k}}{n}\right)\right)}{V(\overrightarrow{x})} - f(\overrightarrow{x}) =$$

$$\sum_{j=1}^{N-1} \sum_{\substack{\alpha:=(\alpha_1,\ldots,\alpha_d),\alpha_i\in\mathbb{Z}^+,\\ i=1,\ldots,d,|\alpha|:=\sum_{i=1}^d \alpha_i=j}} \left(\frac{f_\alpha(\overrightarrow{x})}{\prod_{i=1}^d \alpha_i!}\right). \tag{37}$$

$$\frac{\left(\sum_{\overrightarrow{k}=\lceil n\overrightarrow{x}-\overrightarrow{T}n^{\beta}\rceil}^{[n\overrightarrow{x}+\overrightarrow{T}n^{\beta}]} \left(\prod_{i=1}^d \left(\frac{k_i}{n}-x_i\right)^{\alpha_i}\right)\right) b\left(n^{1-\beta}\left(\overrightarrow{x}-\frac{\overrightarrow{k}}{n}\right)\right)}{V(\overrightarrow{x})} + U_n(\overrightarrow{x}).$$

The last says

$$(M_n(f))(\overrightarrow{x}) - f(\overrightarrow{x}) -$$

$$\sum_{j=1}^{N-1}\left(\sum_{|\alpha|=j}\left(\frac{f_\alpha(\overrightarrow{x})}{\prod_{i=1}^d \alpha_i!}\right) M_n\left(\prod_{i=1}^d (\cdot - x_i)^{\alpha_i}, \overrightarrow{x}\right)\right) = U_n(\overrightarrow{x}). \tag{38}$$

The proof of the theorem is complete. □

We present our second main result

Theorem 2.2. *Let* $f \in AC^N(\mathbb{R}^d)$, $d \in \mathbb{N} - \{1\}$, $N \in \mathbb{N}$, *with* $\|f_\alpha\|_{\infty,\mathbb{R}^d,N}^{\max} < \infty$. *Here* $n \geq \max_{i \in \{1,\ldots,d\}} \left\{ T_i + |x_i|, T_i^{-\frac{1}{\beta}} \right\}$, *where* $\overrightarrow{x} \in \mathbb{R}^d$, $0 < \beta < 1$, $n \in \mathbb{N}$, $T_i > 0$. *Then*

$$(L_n(f))(\overrightarrow{x}) - f(\overrightarrow{x}) =$$

$$\sum_{j=1}^{N-1} \left(\sum_{|\alpha|=j} \left(\frac{f_\alpha(\overrightarrow{x})}{\prod_{i=1}^d \alpha_i!} \right) L_n \left(\prod_{i=1}^d (\cdot - x_i)^{\alpha_i}, \overrightarrow{x} \right) \right) + o\left(\frac{1}{n^{(N-\varepsilon)(1-\beta)}} \right), \tag{39}$$

where $0 < \varepsilon \leq N$.

If $N = 1$, *the sum in* (39) *collapses.*

The last (39) *implies that*

$$n^{(N-\varepsilon)(1-\beta)} \Bigg[(L_n(f))(\overrightarrow{x}) - f(\overrightarrow{x}) - \tag{40}$$

$$\sum_{j=1}^{N-1} \left(\sum_{|\alpha|=j} \left(\frac{f_\alpha(\overrightarrow{x})}{\prod_{i=1}^d \alpha_i!} \right) L_n \left(\prod_{i=1}^d (\cdot - x_i)^{\alpha_i}, \overrightarrow{x} \right) \right) \Bigg] \to 0, \text{ as } n \to \infty,$$

$0 < \varepsilon \leq N$.

When $N = 1$, *or* $f_\alpha(\overrightarrow{x}) = 0$, *all* $\alpha : |\alpha| = j = 1, \ldots, N-1$, *then we derive that*

$$n^{(N-\varepsilon)(1-\beta)} \left[(L_n(f))(\overrightarrow{x}) - f(\overrightarrow{x}) \right] \to 0,$$

as $n \to \infty$, $0 < \varepsilon \leq N$.

Proof. As similar to Theorem 2.1 is omitted. □

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