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FITTED COLLOCATION METHOD FOR CONVECTION-DIFFUSION PROBLEMS WITH TWO SMALL PARAMETERS

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ABSTRACT. In this paper a numerical method is constructed for a singularly perturbed ordinary differential equation with two small parameters affecting the convection and diffusion terms. Depending on the size of the parameters the solution of the problem may exhibit boundary layers at both end points of the domain. We use B-spline collocation method, which leads to a tridiagonal linear system. Fitted artificial viscosity parameter is designed on a uniform mesh, which permits its extension to the case of adaptive meshes, which may be used to improve the solution. The accuracy of the proposed method is demonstrated by test problems. The numerical result is found in good agreement with exact solution.

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1. INTRODUCTION

The boundary value problems for ordinary differential equations in which one or more small positive parameter(s) multiplying the derivative(s), arise in the field of physics and applied mathematics. The problems with one small positive parameter multiplying to highest derivative has been considered by many authors [1, 2, 3, 4, 5, 6]. In this paper, second order two point boundary value problems with two small parameters multiplying to the highest and second highest derivative has been considered. These kind of problem arises in transport phenomena in chemistry, biology, chemical reactor theory, lubrication theory and dc motor theory [12, 8, 13, 17, 7].

$$Lu(x) \equiv -\epsilon u''(x) - \mu a(x)u'(x) + b(x)u(x) = f(x), \quad x \in \Omega = (0,1) \tag{1.1}$$

with

$$u(0) = \alpha, \quad u(1) = \beta, \tag{1.2}$$

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with two small parameters $0 < \epsilon \ll 1$ and $0 < \mu \ll 1$, such that $\epsilon/\mu^2 \to 0$ as $\mu \to 0$. The functions $a(x)$, $b(x)$, and $f(x)$ are assumed to be sufficiently smooth with $a(x) \geq a^\star > 0$ and $b(x) \geq b^\star > 0$ for $x \in [0,1]$. When the parameter $\mu = 1$, the problem is the well-studied one-parameter convection-diffusion problem. In this case, a boundary layer of width $O(\epsilon)$ appears in the neighborhood of the point $x = 0$. This problem encompasses the reaction-diffusion problem when $\mu = 0$ and boundary layers of width $O(\sqrt{\epsilon})$ appear at both $x = 0$ and $x = 1$. It is well-known that standard numerical methods are unsuitable for singularly perturbed problems and fail to give accurate solutions. There is a vast literature dealing with numerical methods for convection-diffusion and associated problems [15, 10]. A good number of research papers can be found in the litrature for single parameter convection-diffusion and reaction-diffusion problems but only a very few authors have discussed two-parameter singular perturbation problems [16, 11, 22, 9, 20, 14].

The nature of the two-parameter problem was asymptotically examined by O'Malley [18, 4, 19] in 1967, where the ratio of μ^2 to ϵ was identified as significant. After a decade Shishkin and Titov [9], used an exponentially fitted finite difference scheme on a uniform mesh for solving (1.1) and (1.2), showed that method is convergent. Linßand Roos [22], considered linear two-parameter singularly perturbed convection-diffusion problem and used the simple upwind-difference scheme on Shishkin mesh to establish almost first-order convergence. Roos and Uzelac [11], also considered the two parameter singularly perturbed boundary value problem and used stream-line diffusion finite element method (SDFEM) on a properly chosen Shishkin mesh.

In this paper, the analysis is based on using B-spline collocation method on a uniform mesh to solve the two parameter boundary value problem given by (1.1) and (1.2). Here, we replace the perturbation parameters ε, μ affecting the derivatives by fitting artificial viscosity $\eta_1(x,\varepsilon), \eta_2(x,\varepsilon)$ respectively. The artificial viscosity is then determined using the asymptotic expansion approximation of the solution and the corresponding collocation scheme for the problem. B-spline approximation method for numerical solutions have been researched by various researchers. They are useful basis functions and based on piece polynomials that possess attractive properties: piecewise smooth, compact support, symmetry, rapidly decaying, differentiability, linear combination, which leads to matrices that are easier to diagonalize. The resulting matrices are sparse, but always, banded. The argument consists of establishing a maximum principle, stability estimate and deriving sharp parameter-explicit bounds of solution and its derivatives. These bounds are used in convergence analysis. Three numerical examples are given and comparisons are made with other solutions. Finally, a summary of the main conclusions is given at the end of the paper. Through out the paper we will take C as positive generic constant which can take different values independent of the parameters ε, μ and N (the dimension of the discrete problem).

2. A PRIORI ESTIMATES OF THE CONTINUOUS PROBLEM

The differential operator L satisfies the following maximum principle:

Lemma 2.1 (Maximum principle)**.** *If π is any sufficiently smooth function satisfying $\pi(0) \geq 0$ and $\pi(1) \geq 0$ and let $L\pi(x) \geq 0,\ \forall\, x \in \Omega$ implies that $\pi(x) \geq 0,\ \forall\, x \in \bar{\Omega}$.*

Proof. The proof is by contradiction. If possible suppose that there is a point $x^* \in [0,1]$ such that $\pi(x^*) < 0$ and $\pi(x^*) = \min_{x \in \bar{\Omega}} \pi(x)$. It is clear from the given conditions $x^* \notin \{0,1\}$, therefore $\pi'(x^*) = 0$ and $\pi''(x^*) \geq 0$. Thus we have

$$\begin{aligned} L\pi(x) \mid_{x=x^*} &= -\epsilon\pi''(x) - \mu a(x)\pi'(x) + b(x)\pi(x) \mid_{x=x^*} \\ &= -\epsilon\pi''(x^*) - \mu a(x^*)\pi'(x^*) + b(x^*)\pi(x^*) < 0, \end{aligned}$$

a contradiction. It follows that $\pi(x^*) \geq 0$ and so $\pi(x) \geq 0\ \forall\, x \in \bar{\Omega}$. □

Lemma 2.2 (Stability estimate)**.** *Let u(x) be the solution of the given problem* (1.1)*, then we have*

$$\|u(x)\|_\infty \leq b^{*-1}\|f\| + \max(|u(0)|, |u(1)|\ ).$$

Proof. We introduce two barrier functions $\psi^{\pm}$ defined by

$$\psi^{\pm}(x) = \pm u(x) + b^{*-1}\|f\| + \max(|\alpha|, |\beta|).$$

Then we have

$$\psi^{\pm}(0) = \pm\alpha + b^{*-1}\|f\| + \max(|\alpha|, |\beta|) \geq 0,$$

$$\psi^{\pm}(1) = \pm\beta + b^{*-1}\|f\| + \max(|\alpha|, |\beta|) \geq 0.$$

It follows that

$$\begin{aligned} L\psi^{\pm}(x) &= \pm Lu(x) + b(x)(b^{*-1}\|f\| + \max(|\alpha|, |\beta|)) \\ &= \pm f(x) + b(x)(b^{*-1}\|f\| + \max(|\alpha|, |\beta|)) \\ &\geq (-\|f\| \pm f(x)) + b(x)\max(|\alpha|, |\beta|) \\ &\geq 0,\ \forall\, 0 < x < 1. \end{aligned}$$

Then the maximum principle gives $\psi^{\pm}(x) \geq 0,\ \forall\, x \in [0,1]$ and so the required bound. □

Theorem 2.3. *Suppose $a(x)$, $b(x)$, $f(x)$ are sufficiently smooth and has derivatives at least of order k then the solution $u(x)$ of the BVP* (1.1) *with* (1.2) *satisfies*

$$|\ u^{(i)}(x)\ | \leq C\{1 + \mu^{-i}\exp(-\nu(1-x)/\mu) + \epsilon^{-i}\exp(-a^* x/(\epsilon/\mu))\} \tag{2.1}$$

for all $i \leq k$, where $\frac{b(x)}{a(x)} \geq \nu$.

Proof. The proof for the bounds of the derivatives follows by induction. From Lemma 2.1 and Lemma 2.2 we have

$$|u(x)| \le C \qquad \forall\, x \in [0,1].$$

Differentiating i times both side of the original equation $Lu = f$ we have

$$\widehat{L}u^{(i)} = f_i,$$

where $\widehat{L}u(x) \equiv -\varepsilon u''(x) - \mu a(x)u'(x)$,$f_0 = f$ and f_i depends on u, a, b, f, μ and their derivatives of order up to and including i. Using induction hypotheses the following estimates hold

$$|u^{(i)}(x)| \le C[1 + \mu^{-i}\exp(-\nu(1-x)/\mu) + \epsilon^{-i}\exp(-a^* x/(\epsilon/\mu))],$$

and

$$|f_i(x)| \le C[1 + \mu^{-i}\exp(-\nu(1-x)/\mu) + \epsilon^{-i}\exp(-a^* x/(\epsilon/\mu))], \qquad \forall\, x \in [0,1].$$

Further we define the function

$$\theta_i(x) = \frac{1}{\varepsilon}\int_0^x f_i(t)e^{-(A(x)-A(t))}dt,$$

where $A(x) = \int_0^x [a(t)/(\varepsilon/\mu)]dt$. It is easy to verify that

$$u_p^{(i)}(x) = -\int_0^x \theta_i(t)dt \tag{2.2}$$

is a particular solution of the equation $\widehat{L}u^{(i)} = f_i$. Therefore its general solution can be written as

$$u^{(i)} = u_p^{(i)} + u_h^{(i)},$$

where the homogeneous solution $u_h^{(i)}$ satisfies

$$\widehat{L}u_h^{(i)} = 0, \quad u_h^{(i)}(0) = u^{(i)}(0), \quad u_h^{(i)}(1) = u^{(i)}(1) - u_p^{(i)}(1).$$

Now introducing the function

$$\phi(x) = \frac{\int_0^x e^{-A(t)}dt}{\int_0^1 e^{-A(t)}dt}.$$

It is clear that $\widehat{L}\phi(x) = 0, \phi(0) = 0, \phi(1) = 1$ and $0 \le \phi(x) \le 1$. Then $u_h^{(i)}$ is given by

$$u_h^{(i)}(x) = u^{(i)}(0)(1 - \phi(x)) + (u^{(i)}(1) - u_p^{(i)}(1))\phi(x).$$

Thus above leads to the expression for $u^{(i+1)}$ given by

$$u^{(i+1)}(x) = u_p^{(i+1)} + u_h^{(i+1)} = (u^{(i)}(1) - u^{(i)}(0) - u_p^{(i)}(1))\phi'(x) - \theta_i(x).$$

Using bounds lead to the estimate

$$|\phi'(x)| \le C(\varepsilon/\mu)^{-1}e^{-a^* x/(\varepsilon/\mu)}.$$

Furthermore, putting the estimate for f_i, and the fact that $\epsilon/\mu^2 \to 0$ as $\mu \to 0$, gives

$$|\theta_i(x)| \leq C[1 + \mu^{-(i+1)} \exp(-\nu(1-x)/\mu) + \epsilon^{-(i+1)} \exp(-a^* x/(\epsilon/\mu))]. \tag{2.3}$$

Since $u_p^{(i)}(1) = -\int_0^1 \theta_i(t)dt$, it follows that $|u_p^{(i)}(1)| \leq C\varepsilon^{-i}\mu^{-1}$. But

$$|u^{(i+1)}(x)| \leq |\theta_i(x)| + (|u^{(i)}(0)| + |u^{(i)}(1)| + |u_p^{(i)}(1)|)|\phi'(x)|.$$

Therefore from the equations (2.3) and above estimate for $u_p^{(i)}$ we lead to

$$\begin{aligned}|u^{(i+1)}(x)| \leq C\{1 &+ \mu^{-(i+1)} \exp(-\nu(1-x)/\mu) \\ &+ \epsilon^{-(i+1)} \exp(-a^* x/(\epsilon/\mu))\}, \qquad \forall\, x \in [0,1].\end{aligned}$$

□

3. B-SPLINE COLLOCATION METHOD

In this section, we describe the B-spline collocation method to obtain the approximate solution to the two parameter problem discussed in Section 1 with uniform mesh. We redefine the problem by introducing the artificial viscosity, which will be determined later, as

$$\widetilde{L}u(x) \equiv -\eta_1(x,\varepsilon)u''(x) - \eta_2(x,\varepsilon)a(x)u'(x) + b(x)u(x) = f(x), \quad x \in \Omega = (0,1) \tag{3.1}$$

with

$$u(0) = \alpha, \quad u(1) = \beta,$$

where the coefficients $a(x), b(x)$ and $f(x)$ are sufficiently smooth functions. We subdivide the interval $[0,1]$ and choose uniform mesh points represented by $\pi = \{x_0, x_1, x_2, \ldots, x_N\}$, such that $x_0 = 0$ and $x_N = 1$ and h is the uniform spacing. We define $L_2[0,1]$ a vector space of all the square integrable function on $[0,1]$, and X be the linear subspace of $L_2[0,1]$. Now define for $i = 0, 1, 2, \ldots, N$.

(3.2)

$$B_i(x) = \frac{1}{h^3}\begin{cases} (x - x_{i-2})^3, & x_{i-2} \leq x \leq x_{i-1}, \\ h^3 + 3h^2(x - x_{i-1}) + 3h(x - x_{i-1})^2 - 3(x - x_{i-1})^3, & x_{i-1} \leq x \leq x_i, \\ h^3 + 3h^2(x_{i+1} - x) + 3h(x_{i+1} - x)^2 - 3(x_{i+1} - x)^3, & x_i \leq x \leq x_{i+1}, \\ (x_{i+2} - x)^3, & x_{i+1} \leq x \leq x_{i+2}, \\ 0, & \text{otherwise.} \end{cases}$$

We define four additional knots $x_{-2} < x_{-1} < x_0$ and $x_{N+2} > x_{N+1} > x_N$. From the above equation, we can simply check that each of the function $B_i(x)$ is twice continuously differentiable on the entire real line. Each $B_i(x)$ is also a piecewise cubic at π and $B_i(x) \in X$. Let $\Lambda = \{B_{-1}, B_0, \ldots, B_{N+1}\}$ and let $\Phi_3(\pi)$=span Λ. The function B_i's are linearly independent on $[0,1]$, thus $\Phi_3(\pi)$ is $(N+3)$ dimensional. In fact $\Phi_3(\pi) \subseteq_{subpace} X$. Let $\widetilde{L}$ be a linear operator whose domain is X and whose range is also in X. Now the global approximation $S(x)$ to the function $u(x)$ can be written in terms of the B-splines as

$$S(x) = \sum_{i=-1}^{N+1} \gamma_i B_i(x), \tag{3.3}$$

where γ_i are unknown real coefficients. This form shows the variation of all contributing cubic B-splines over a single element and is useful for working out the solution inside the element. At nodal points, values of $S(x)$ and its derivatives $S'(x)$ and $S''(x)$ can be determined in terms of the element parameters γ_i:

$$S(x_i) = \gamma_{i-1} + 4\gamma_i + \gamma_{i+1},\ hS'(x_i) = 3(\gamma_{i+1} - \gamma_{i-1}),\ h^2 S''(x_i) = 6(\gamma_{i-1} - 2\gamma_i + \gamma_{i+1}),$$

where $'$ and $''$ denotes the first and second derivatives with respect to x, respectively.

Here we have introduced two extra cubic B-splines, B_{-1} and B_{N+1} to satisfy the boundary conditions. Therefore we have

$$\widetilde{L}S(x_i) = f(x_i), \qquad 0 \le i \le N, \tag{3.4}$$

and

$$S(x_0) = \alpha, \qquad S(x_N) = \beta. \tag{3.5}$$

On solving the collocation equations (3.4) and putting the values of B-spline functions and its derivatives at mesh points, we obtain a system of $(N+1)$ linear equations in $(N+3)$ unknowns

$$\begin{aligned}(-6\eta_{1,i} + 3\eta_{2,i}a_i h + b_i h^2)\gamma_{i-1} + (12\eta_{1,i} + 4b_i h^2)\gamma_i \\ + (-6\eta_{1,i} - 3\eta_{2,i}a_i h + b_i h^2)\gamma_{i+1} = f_i h^2, \quad 0 \le i \le N.\end{aligned} \tag{3.6}$$

The given boundary conditions become

$$\gamma_{-1} + 4\gamma_0 + \gamma_1 = \alpha, \tag{3.7}$$

and

$$\gamma_{N-1} + 4\gamma_N + \gamma_{N+1} = \beta. \tag{3.8}$$

Thus the Eqs. (3.6), (3.7) and (3.8) lead to a $(N+3)\times(N+3)$ system with $(N+3)$ unknowns $(\gamma_{-1}, \gamma_0, \gamma_1, \ldots, \gamma_{N+1})$. Now eliminating γ_{-1} from first equation of (3.6) and (3.7) we find

$$(36\eta_{1,0} - 12\eta_{2,0}a_0 h)\gamma_0 + (-6\eta_{2,0}a_0 h)\gamma_1 = h^2 f_0 - \alpha(-6\eta_{1,0} + 3\eta_{2,0}a_0 h + b_0 h^2). \tag{3.9}$$

Similarly, eliminating γ_{N+1} from the last equation of (3.6) and from (3.8), we find

$$(6\eta_{2,N}a_N h)\gamma_{N-1} + (36\eta_{1,N} + 12\eta_{2,N}a_N h)\gamma_N = h^2 f_N - \beta(-6\eta_{1,N} - 3\eta_{2,N}a_N h + b_N h^2). \tag{3.10}$$

Thus by the elimination of γ_{-1} and γ_{N+1}, coupling equations (3.9) and (3.10) with the second through (N-1) equations of (3.6), we lead to a system of $(N+1)$ linear equations in $(N+1)$ unknowns

$$Tx_N = d_N. \tag{3.11}$$

The co-efficient matrix $T = [t_{i,j}]$ is given by

$$\begin{bmatrix} (\widetilde{L}B_{0,0} - 4\widetilde{L}B_{0,-1}) & (\widetilde{L}B_{0,1} - \widetilde{L}B_{0,-1}) & 0 & 0 & \dots & 0 \\ \widetilde{L}B_{1,0} & \widetilde{L}B_{1,1} & \widetilde{L}B_{1,2} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \widetilde{L}B_{i,i-1} & \widetilde{L}B_{i,i} & \widetilde{L}B_{i,i+1} & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & \widetilde{L}B_{N-1,N-2} & \widetilde{L}B_{N-1,N-1} & \widetilde{L}B_{N-1,N} \\ 0 & \dots & 0 & 0 & (\widetilde{L}B_{N,N-1} - \widetilde{L}B_{N,N+1}) & (\widetilde{L}B_{N,N} - 4\widetilde{L}B_{N,N+1}) \end{bmatrix}$$

where $x_N = (\gamma_0, \gamma_1, \dots, \gamma_N)^t$, $\widetilde{L}B_{i,i-1} = -6\eta_{1,i} + 3\eta_{2,i}a_ih + b_ih^2$, $\widetilde{L}B_{i,i} = 12\eta_{1,i} + 4b_ih^2$, $\widetilde{L}B_{i,i+1} = -6\eta_{1,i} - 3\eta_{2,i}a_ih + b_ih^2$, $d_N = (h^2 f_0 - \alpha\widetilde{L}B_{0,-1}, h^2 f_1, \dots, h^2 f_{N-1}, h^2 f_N - \beta\widetilde{L}B_{N,N+1})^t$.

It can be easily seen that the matrix T is strictly diagonally dominant under the condition, $0 < h < (-3a_i\eta_{2,i} + \sqrt{9a_i^2\eta_{2,i}^2 + 24b_i\eta_{1,i}})/2b_i$ or $h > (3a_i\eta_{2,i} + \sqrt{9a_i^2\eta_{2,i}^2 + 24b_i\eta_{1,i}})/2b_i$ $\forall\ i = 0, 1, \dots, N$ and hence nonsingular. Since T is nonsingular, we can solve the system $Tx_N = d_N$ for $\gamma_0, \gamma_1, \dots, \gamma_N$ and substitute into the boundary condition (3.7) and (3.8) to obtain γ_{-1} and γ_{N+1}. Hence the method of collocation using a basis of cubic B-splines has a unique solution $S(x)$ given by (3.3).

4. DESIGN OF THE ARTIFICIAL VISCOSITY

We consider the asymptotic expansion of (1.1) with (1.2) of the form

$$\begin{aligned} u(x,\epsilon,\mu) &= \{u_0 + (\epsilon/\mu)u_1 + (\epsilon/\mu)^2u_2 + O(\epsilon/\mu)^3\} \\ &\quad + \{z_0 + (\epsilon/\mu)z_1 + (\epsilon/\mu)^2z_2 + O(\epsilon/\mu)^3)\} \\ &= (u_0 + z_0) + (\epsilon/\mu)(u_1 + z_1) + (\epsilon/\mu)^2(u_2 + z_2) + O((\epsilon/\mu)^3) \\ &= \widetilde{u}_0 + (\epsilon/\mu)\widetilde{u}_1 + (\epsilon/\mu)^2\widetilde{u}_2 + O((\epsilon/\mu)^3), \end{aligned}$$

where the zeroth order asymptotic expansion $\widetilde{u}_0$ is given by

$$\widetilde{u}_0(x) = u_0 + z_0, \tag{4.1}$$

where u_0 is the solution of the reduced problem of (1.1) with (1.2) given by

$$-\mu a(x)u_0'(x) + b(x)u_0(x) = f(x), \tag{4.2}$$

with

$$u_0(1) = \beta, \tag{4.3}$$

and u_1 is the solution of

$$-\epsilon u_1''(x) - \mu a(x)u_1'(x) + b(x)u_1(x) = \mu u_0''(x), \tag{4.4}$$

with

$$u_1(0) = 0, \quad u_1(1) = -(\epsilon/\mu)^{-1}z_0(1), \tag{4.5}$$

and z_0 is a layer correction given by

$$-(d^2 z_0/d\tau^2) - a(0)(dz_0/d\tau) = 0, \tag{4.6}$$

with

$$z_0(0) = \alpha - u_0(0), \quad z_0(\infty) = 0, \tag{4.7}$$

where $\tau = x/(\epsilon/\mu)$.

Solving Eq. (4.2) with (4.3) we get

$$u_0(x) = \exp\left(\int_x^1 \frac{b(s)}{\mu\, a(s)} d\, s\right)\left[\beta - \int_x^1 \frac{f(s)}{\mu\, a(s)} \exp\left(-\int_x^1 \frac{b(s)}{\mu\, a(s)} d\, s\right) d\, s\right], \tag{4.8}$$

which can also be solved using asymptotic expansion to give

$$u_0(x) = \frac{f(x)}{b(x)} + \left(\beta - \frac{f(1)}{b(1)}\right)\exp\left(\frac{b(1)}{a(1)\mu}(1-x)\right) + O(\mu). \tag{4.9}$$

Again solving (4.6) with (4.7) we get

$$z_0(x) = \{\alpha - u_0(0)\}\exp(-a(0)x/(\epsilon/\mu)). \tag{4.10}$$

Lemma 4.1. *Let $\tilde{u} = u_0 + z_0$ be the zeroth order asymptotic approximation to the solution, where u_0 represents the zeroth order approximate solution of the reduced problem and z_0 the inner region solution. Then, we have*

$$|u(x) - \tilde{u}(x)| \le C\varepsilon/\mu. \tag{4.11}$$

Proof. Since,

$$L(u(x) - \tilde{u}(x)) \;\; = \;\; \varepsilon u_o''(x) + \tau a'(\xi)z_o'(\tau) - b(x)z_o(\tau)$$

where $\tau = x/\varepsilon/\mu$ is the stretched variable and $\xi \in (0,1)$. As $z_0(\tau) = \{\alpha - u_0(0)\}\exp(-a(0)\tau)$, using the fact that $t\exp(-t) \le \exp(-t/2)$, in the above inequality, we get

$$|L(u(x) - \tilde{u}(x))| \le \varepsilon|u_0''(x)| + C\exp((-a^*x)/2(\varepsilon/\mu)).$$

Since $|u_0^i(x)| \le C[1 + \mu^{-i}\exp((-\nu(1-x))/\mu)]$, [Pg 8, [5]], we obtain

$$|L(u(x) - \tilde{u}(x))| \le C\left[\varepsilon + \frac{\varepsilon}{\mu^2}\exp((-\nu(1-x))/\mu) + \exp((-a^*x)/2(\varepsilon/\mu))\right]. \tag{4.12}$$

We define two barrier functions

$$\varrho_{\pm}(x) = \pm(u(x) - \tilde{u}(x)) + C_1\frac{\varepsilon}{\mu}(1 - x/2) + C_2\frac{\varepsilon}{\mu^2}\exp(-a^*x/2(\varepsilon/\mu))$$
$$+ C_3\frac{\varepsilon}{\mu^2}\exp((-\nu(1-x))/2\mu),$$

where C_1, C_2 and C_3 are constants. We have

$$\begin{aligned} L(\varrho_{\pm}(x)) &= -\epsilon\varrho_{\pm}''(x) - \mu a(x)\varrho_{\pm}'(x) + b(x)\varrho_{\pm}(x) \\ &\geq -C[\epsilon + \frac{\epsilon}{\mu^2}\exp((-\nu(1-x))/\mu) + \exp((-a^*x)/2(\epsilon/\mu))] \\ &+ C_1[\epsilon a^*/2 + \frac{\epsilon}{\mu}b^*(1-x/2)] + C_2[(a^*/2)^2 + \frac{\epsilon}{\mu^2}b^*]\exp(-a^*x/2(\epsilon/\mu)) \\ &+ C_3\frac{\epsilon}{\mu^2}[2\nu a^* - \nu^2\frac{\epsilon}{\mu^2}]\exp((-\nu(1-x))/2\mu). \end{aligned}$$

As second, third and fourth terms are positive while first term is negative on right side of the above inequality. Using the fact that $\frac{\varepsilon}{\mu^2}$ is small, we choose the constants C_1, C_2 and C_3 such that the total of the positive terms dominate the negative term. Thus we obtain

$$L(\varrho_{\pm}(x)) \geq 0. \tag{4.13}$$

Also, we have $\varrho_{\pm}(x) \geq 0$ at both ends of the interval $[0, 1]$, then maximum principle implies that

$$\varrho_{\pm}(x) \geq 0,$$

which on simplification gives

$$|u(x) - \tilde{u}(x)| \leq C\varepsilon/\mu. \tag{4.14}$$

Then (4.2) and (4.9) at the computational grid points $x_i, i = (0, 1, \ldots, N)$, in the limiting case as $h \to 0$, gives

$$\lim_{h\to 0}\frac{\eta_{2,i}}{h} = \frac{b(0)}{3a(0)}\frac{\left(\widetilde{\gamma}_{i-1} + 4\widetilde{\gamma}_i + \widetilde{\gamma}_{i+1} - \frac{f(0)}{b(0)}\right)}{\widetilde{\gamma}_{i-1} - \widetilde{\gamma}_{i+1}} \tag{4.15}$$

and

$$\widetilde{\gamma}_{i-1} + 4\widetilde{\gamma}_i + \widetilde{\gamma}_{i+1} = \frac{f(0)}{b(0)} + \left(\beta - \frac{f(1)}{b(1)}\right)\exp\left(\frac{b(1)}{a(1)}(1/\mu - i\rho_2)\right) + O(\mu). \tag{4.16}$$

where $\widetilde{\gamma}_i$'s are unknown real coefficients corresponding to the approximation $\widetilde{S}(x)$ to the function $u_0(x)$, which can be written in terms of the B-splines as $\widetilde{S}(x) = \sum_{i=-1}^{N+1}\widetilde{\gamma}_i B_i(x)$. Similarly, evaluating (3.6) and (4.11) at the grid points $x_i, i = (0, 1, \ldots, N)$ and in the limiting case as $h \to 0$, it gives

$$\lim_{h\to 0}\frac{\eta_{1,i}}{h} = \frac{a(0)\mu}{2}\frac{\gamma_{i-1} - \gamma_{i+1}}{\gamma_{i-1} - 2\gamma_i + \gamma_{i+1}} \tag{4.17}$$

and

$$\gamma_{i-1} + 4\gamma_i + \gamma_{i+1} = \lim_{h\to 0} u_0(ih) + (\alpha - u_0(0)) \exp\big(-a(0)i\rho_1\big) + O(\varepsilon/\mu), \tag{4.18}$$

where $\rho_1 = \frac{h}{(\varepsilon/\mu)}$ and $\rho_2 = \frac{h}{\mu}$. Evaluating the values of $\lim_{h\to 0} \frac{\eta_{1,i}}{h}, \lim_{h\to 0} \frac{\eta_{2,i}}{h}$ at the nodal points x_{i-1}, x_i, x_{i+1} and adding in the proportion $1, 4, 1$ respectively and eliminating $\gamma_i' s, \widetilde{\gamma}_i' s$ using above equations, we get

$$\lim_{h\to 0} \frac{\eta_{1,i}}{h} = \frac{a(0)\mu}{2} \coth\Big(\frac{a(0)\rho_1}{2}\Big),\ \lim_{h\to 0} \frac{\eta_{2,i}}{h} = \frac{b(0)}{3a(0)}\Big[\coth\Big(\frac{b(1)\rho_2}{a(1)}\Big) + 2\text{cosech}\Big(\frac{b(1)\rho_2}{a(1)}\Big)\Big].$$

We define

$$\eta_{1,i} = \eta_1(x_i) = \frac{a_i h\mu}{2} \coth\Big(\frac{a_i\rho_1}{2}\Big) \tag{4.19}$$

and

$$\eta_{2,i} = \eta_2(x_i) = \frac{b_i h}{3a_i}\Big[\coth\Big(\frac{b_i\rho_2}{a_i}\Big) + 2\text{cosech}\Big(\frac{b_i\rho_2}{a_i}\Big)\Big]. \tag{4.20}$$

Now, we have

$$\eta_{1,i} - \varepsilon = \varepsilon\Big[\frac{a_i\rho_1}{2} \coth\big(\frac{a_i\rho_1}{2}\big) - 1\Big].$$

Since $\lim_{x\to 0} x\coth x = 1$, $x\text{coth}x = 1 + x^2/3 + O(x^4)$, we have

$$|x\text{coth}x - 1| \le Cx^2 \quad \text{for } x \le 1.$$

Also, $\lim_{x\to\infty} \coth x = 1$, we have

$$|x\text{coth}x - 1| \le Cx \quad \text{for } x \ge 1.$$

Hence,

$$|x\text{coth}x - 1| \le C\frac{x^2}{1+x} \quad \text{for } x \ge 0,$$

$$\Rightarrow |\eta_{1,i} - \epsilon| \le C\epsilon\frac{\rho_1^2}{1+\rho_1}. \tag{4.21}$$

Similarly, using expansion for $x\text{coth}x$, $x\text{cosech}x$, it can be shown that

$$|\eta_{2,i} - \mu| \le C\mu\frac{\rho_2^4}{1+\rho_2^3}. \tag{4.22}$$

□

5. STABILITY ANALYSIS

In this section, we consider the stability of the B-spline collocation method.

The collocation method gives a system of linear equations. We want to show that the difference between actual solution and computed solution depends continuously on the perturbations in the prescribed coefficients of the linear system (3.11), that is, the method is stable. Let $\widetilde{x}_N$ be the solution of the perturbed system, i.e.,

$$(T+\sigma)\widetilde{x}_N = d_N + \partial, \tag{5.1}$$

where σ and ∂ are small perturbations in T and d_N respectively. Subtracting (5.1) from (3.11) we get

$$x_N - \widetilde{x}_N = (T+\sigma)^{-1}(\sigma x_N - \partial). \tag{5.2}$$

Now we claim that

$$\|(T+\sigma)^{-1}\|_\infty \leq C\|T^{-1}\|_\infty. \tag{5.3}$$

We have

$$\begin{aligned}
\|(T+\sigma)^{-1}\|_\infty &= \|T^{-1}(I+\sigma T^{-1})^{-1}\|_\infty \\
&\leq \|T^{-1}\|_\infty \|(I+\sigma T^{-1})^{-1}\|_\infty \\
&\leq \frac{\|T^{-1}\|_\infty}{1-\|\sigma T^{-1}\|_\infty}, \qquad \text{provided } \|\sigma T^{-1}\|_\infty < 1, \\
&\leq \frac{\|T^{-1}\|_\infty}{1-\|\sigma\|_\infty \|T^{-1}\|_\infty} \\
&\leq \|T^{-1}\|_\infty \left\{ 1 + \frac{\|\sigma\|_\infty \|T^{-1}\|_\infty}{1-\|\sigma\|_\infty \|T^{-1}\|_\infty} \right\} \\
&\leq C\|T^{-1}\|_\infty.
\end{aligned}$$

Also we have

$$\|T^{-1}\|_\infty \leq \left[\min_{0\leq i\leq N} \left(| \, t_{i,i} \, | - \sum_{j\neq i} | \, t_{i,j} \, | \right) \right]^{-1} = C < \infty. \tag{5.4}$$

On combining (5.2) (5.3) and (5.4), we conclude that

$$\|x_N - \widetilde{x}_N\|_\infty \leq C(\|\sigma\|_\infty \|x_N\|_\infty + \|\partial\|_\infty),$$

which shows the stability of the collocation system.

6. PROOF OF THE CONVERGENCE

For the derivation of uniform convergence, we use the following lemma proved in [21].

Lemma 6.1. *The B-splines set $\{B_i\}_{i=-1}^{N+1}$ defined in equation (3.2), satisfy the inequality*

$$\sum_{i=-1}^{N+1} |B_i(x)| \leq 10, \quad x \in \overline{\Omega}. \tag{6.1}$$

Theorem 6.2. *Let $u(x)$ be the sufficiently smooth solution of the given two parameter problem and let $S(x)$ be the cubic B-spline collocation approximate on a uniform mesh. Then the error estimate is given by*

$$\|u(x) - S(x)\| \leq C\frac{h^2}{h + \epsilon/\mu}. \tag{6.2}$$

Proof. Let $Y(x)$ be a unique spline from $\Phi_3(\pi)$ interpolating the solution $u(x)$ of our boundary value problem, then in view of De Boor-Hall error estimates [21] that

$$\|D^i(u(x) - Y(x))\| \leq c_i\|u^{(4)}\|h^{4-i}, \quad i = 0, 1, 2 \tag{6.3}$$

where c_i's are independent of h and N. Let $Y(x) \in \Phi_3(\pi)$, then

$$Y(x) = \sum_{i=-1}^{N+1} \overline{\gamma}_i B_i(x). \tag{6.4}$$

Now collocating conditions are $\widetilde{L}S(x_i) = Lu(x_i) = f(x_i)$. Also let $\widetilde{L}Y(x_i) = \tilde{f}(x_i)\ \forall\ i = 0, 1, \ldots, N$ with boundary conditions $Y(x_0) = \alpha, Y(x_N) = \beta$. It is immediate from estimate (6.3) that

$$\begin{aligned} |\widetilde{L}S(x_i) - \widetilde{L}Y(x_i)| &= |f(x_i) - \tilde{f}(x_i)| \\ &= |Lu(x_i) - \widetilde{L}Y(x_i)| \\ &\leq |\eta_{1,i} - \epsilon||u^{(2)}(x)| + |\eta_{2,i} - \mu||u^{(1)}(x)| \\ &\quad + (|\eta_{1,i}|c_2h^2 + |\eta_{2,i}|\|a\|c_1h^3 + \|b\|c_0h^4)|u^{(4)}(x)|. \end{aligned} \tag{6.5}$$

Using relationship (3.11), $\widetilde{L}(S(x_i) - Y(x_i))$ leads to the linear system,

$$T(x_N - \overline{x}_N) = (d_N - \overline{d}_N), \tag{6.6}$$

where

$$\begin{aligned} x_N - \overline{x}_N &= (\gamma_0 - \overline{\gamma}_0, \gamma_1 - \overline{\gamma}_1, \ldots, \gamma_N - \overline{\gamma}_N)^t, \\ d_N - \overline{d}_N &= (h^2(f(x_0) - \tilde{f}(x_0)), h^2(f(x_1) - \tilde{f}(x_1)), \ldots, h^2(f(x_N) - \tilde{f}(x_N)))^t. \end{aligned}$$

The inequalities (6.5), (4.21), (4.22) and estimate of derivatives of u, together with the argument that $\mu^{-k}\exp(-\nu(1-x)/\mu) \to 0$ as $\mu \to 0$ and $\epsilon^{-i}\exp(-a^*x/(\epsilon/\mu)) \to 0$ as $\epsilon \to 0\ \forall x \in \Omega$ and $k \in \mathbb{I}^+$, we get

$$\|d_N - \overline{d}_N\| \leq C\frac{h^4}{h + \epsilon/\mu}. \tag{6.7}$$

We have seen that for sufficiently small values of h, the coefficient matrix T is strictly diagonally dominant. Also note that except the first row and the last row of T, the off diagonal elements $t_{i,i-1},\ t_{i,i+1},\ 1 \le i \le N-1$, are positive and the main diagonal elements $t_{i,i}$ are negative, therefore

$$|\,t_{i,i}| - (|\,t_{i,i-1}| + |\,t_{i,i+1}|) = 6b_i h^2 > 0, \quad 1 \le i \le N-1.$$

Also, from the first row of T, we have

$$|\,t_{0,0}| - |\,t_{0,1}| = 36\eta_{1,0} - 18a_0\eta_{2,0}h.$$

Again from last row of T, we have

$$|\,t_{N,N}| - |\,t_{N,N-1}| = 36\eta_{1,0} + 6a_N\eta_{2,0}h.$$

The above diagonal dominant property for smaller values of h [23], implies

$$\|T^{-1}\|_\infty \le \frac{C}{h^2}. \tag{6.8}$$

Using the relation (6.6), (6.7) and (6.8), we obtain

$$|\gamma_i - \overline{\gamma}_i| \le C\frac{h^2}{h+\epsilon/\mu}, \quad 0 \le i \le N. \tag{6.9}$$

Now to estimate $|\gamma_{-1} - \overline{\gamma}_{-1}|$ and $|\gamma_{N+1} - \overline{\gamma}_{N+1}|$, we use the boundary conditions (3.9) and (3.10), we have

$$|\gamma_{-1} - \overline{\gamma}_{-1}| \le C\frac{h^2}{h+\epsilon/\mu},\ |\gamma_{N+1} - \overline{\gamma}_{N+1}| \le C\frac{h^2}{h+\epsilon/\mu}. \tag{6.10}$$

The above inequality together with the Lemma 6.1 enables us to estimate $\|S(x) - Y(x)\|$, hence $\|u(x) - S(x)\|$. In particular

$$S(x) - Y(x) = \sum_{i=-1}^{N+1} (\gamma_i - \overline{\gamma}_i)B_i(x),$$

$$\|S(x) - Y(x)\| \le C\frac{h^2}{h+\epsilon/\mu}.$$

Combining the triangle inequality with the above results, we have

$$\|u(x) - S(x)\| \le C\frac{h^2}{h+\epsilon/\mu}.$$

□

7. TEST EXAMPLES AND NUMERICAL RESULTS

We now validate the theoretical results by some numerical results. We consider three test problems.

Example 1. Consider the boundary value problem

$$-\epsilon u''(x) - \mu u'(x) + u(x) = x,$$

with

$$u(0) = 1, \quad u(1) = 0.$$

The exact solution of the problem is given by

$$u(x) = \frac{(1+\mu) + (1-\mu)e^{m_2}}{e^{m_2} - e^{m_1}} e^{m_1 x} + \frac{(1+\mu) + (1-\mu)e^{m_1}}{e^{m_1} - e^{m_2}} e^{m_2 x} + x + \mu,$$

where

$$m_{1,2} = \frac{-\mu \mp \sqrt{\mu^2 + 4\epsilon}}{2\epsilon}.$$

Example 2. Now we consider the boundary value problem

$$-\epsilon u''(x) - 2\mu u'(x) + 4u(x) = 1,$$

with

$$u(0) = 0, \quad u(1) = 1.$$

The exact solution of the problem is given by

$$u(x) = \frac{3 + e^{m_2}}{4(e^{m_1} - e^{m_2})} e^{m_1 x} + \frac{3 + e^{m_1}}{4(e^{m_1} - e^{m_1})} e^{m_2 x} + \frac{1}{4},$$

where

$$m_{1,2} = \frac{-\mu \mp \sqrt{\mu^2 + 4\epsilon}}{\epsilon}.$$

Example 3. Consider the non-homogeneous problem

$$-\epsilon u^{''}(x) - \mu u^{'}(x) + u(x) \ = \cos \pi x, \quad x \in (0,1), \quad u(0) = 0, \quad u(1) = 0.$$

The exact solution is given by

$$u(x) = a \cos \pi x + b \sin \pi x + A e^{m_1 x} + B e^{-m_2(1-x)},$$

where

$$a = \frac{\varepsilon \pi^2 + 1}{\mu^2 \pi^2 + (\varepsilon \pi^2 + 1)^2}, \quad b = \frac{-\mu \pi}{\mu^2 \pi^2 + (\varepsilon \pi^2 + 1)^2},$$

$$A = -a \frac{1 + e^{-m_2}}{1 - e^{m_1 - m_2}}, \quad B = a \frac{1 + e^{m_1}}{1 - e^{m_1 - m_2}},$$

$$m_{1,2} = \frac{-\mu \mp \sqrt{\mu^2 + 4\varepsilon}}{2\varepsilon}.$$

The numerical rates of convergence are computed using the standard formula

$$r_N = \log_2(E^N_{\epsilon,\mu} / E^{2N}_{\epsilon,\mu}),$$

Table 1. Numerical maximum errors ($E^N_{\epsilon,\mu}$) and rate of convergence (r_N), when applied to examples for various values of ε and μ, without using artificial viscosity.

ε, μ	$N = 32$	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$
Example 1						
$10^{-4}, 10^{-1}$	7.158E-001	6.622E-001	5.219E-001	3.068E-001	1.176E-001	2.984E-002
	0.1123	0.3435	0.7665	1.3834	1.9786	
$10^{-6}, 10^{-1}$	8.106E-001	8.523E-001	8.725E-001	8.794E-001	8.761E-001	8.611E-001
	-0.0724	-0.0338	-0.0114	0.0054	0.0249	
$10^{-8}, 10^{-1}$	8.116E-001	8.544E-001	8.769E-001	8.883E-001	8.940E-001	8.968E-001
	-0.0741	-0.0375	-0.0186	-0.0092	-0.0045	
$10^{-10}, 10^{-1}$	8.116E-001	8.545E-001	8.769E-001	8.884E-001	8.942E-001	8.971E-001
	-0.0743	-0.0373	-0.0188	-0.0094	-0.0047	
Example 2						
$2^{-8}, 2^{-3}$	3.737E-002	1.090E-002	2.421E-003	5.985E-004	1.490E-004	3.725E-005
	1.7776	2.1707	2.0162	2.0060	2.0000	
$2^{-12}, 2^{-3}$	1.883E-001	1.798E-001	1.449E-001	8.708E-002	3.408E-002	8.790E-003
	0.0666	0.3113	0.7346	1.3534	1.9550	
$2^{-16}, 2^{-3}$	2.107E-001	2.266E-001	2.325E-001	2.300E-001	2.183E-001	1.935E-001
	-0.1050	-0.0371	0.0156	0.0753	0.1740	
$2^{-20}, 2^{-3}$	2.121E-001	2.299E-001	2.393E-001	2.439E-001	2.455E-001	2.448E-001
	-0.1163	-0.0578	-0.0275	-0.0094	0.0041	
Example 3						
$2^{-10}, 2^{-4}$	8.092E-001	6.836E-001	4.879E-001	2.486E-001	6.451E-002	1.422E-004
	0.2433	0.4866	0.9728	1.9462	8.8255	
$2^{-12}, 2^{-4}$	8.092E-001	6.836E-001	4.879E-001	2.486E-001	6.451E-002	1.966E-003
	0.2433	0.4866	0.9728	1.9462	5.0362	
$2^{-20}, 2^{-4}$	8.159E-001	8.828E-001	9.164E-001	9.284E-001	9.236E-001	8.998E-001
	-0.1137	-0.0539	-0.0188	0.0075	0.0377	
$2^{-25}, 2^{-4}$	8.174E-001	8.861E-001	9.234E-001	9.426E-001	9.520E-001	9.560E-001
	-0.1164	-0.0595	-0.0297	-0.0143	-0.0060	

where $E^N_{\epsilon,\mu}$ is the maximum absolute error at mesh points for given ϵ, μ and N.

The numerical results are displayed in Tables 1-2, which clearly indicates that the proposed method based on a B-spline collocation using fitting viscosity parameter is more accurate. For Examples 2 and 3, we plot the graphs (taking $N = 128$) using uniform mesh with/without fitting factors. It can be seen, that there is an oscillation near left boundary layer using uniform mesh with no fitted factor, however, one can see that how the proposed method tackles this problem very well using fitting factor.

From Figures 2a)-2b) of Example 1, it is observed that, when the ratio (ϵ/μ) increases the width of the left boundary layer increases and in the case of right

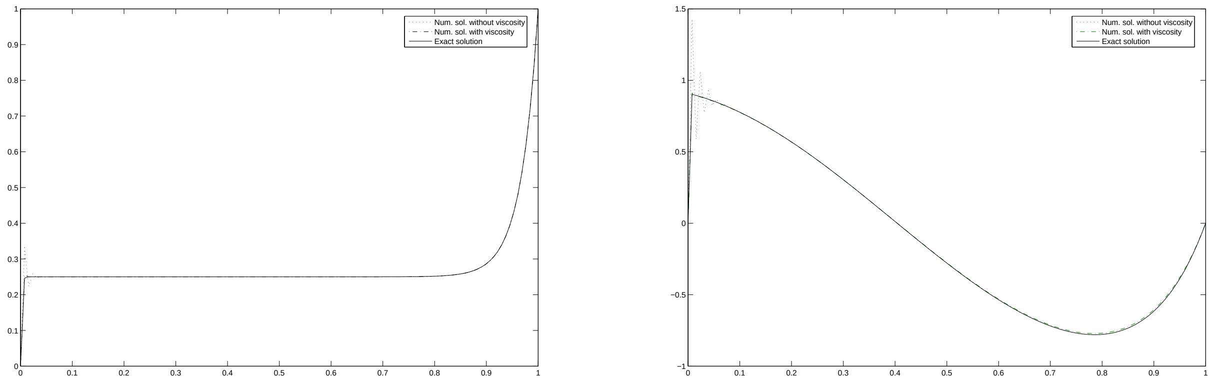

FIGURE 1. Solution profiles with/without artificial viscosity of a) Example 2 for $\varepsilon = 2^{-12}, \mu = 2^{-4}$ and $N = 128$ and b) Example 3 for $\varepsilon = 10^{-4}$, $\mu = 10^{-1}$ and $N = 128$.

boundary layer, the width increases as the μ increases, i.e., they are of width order $O(\epsilon/\mu)$ and $O(\mu)$ respectively. This clearly shows the effect of ϵ, μ on the boundary layers.

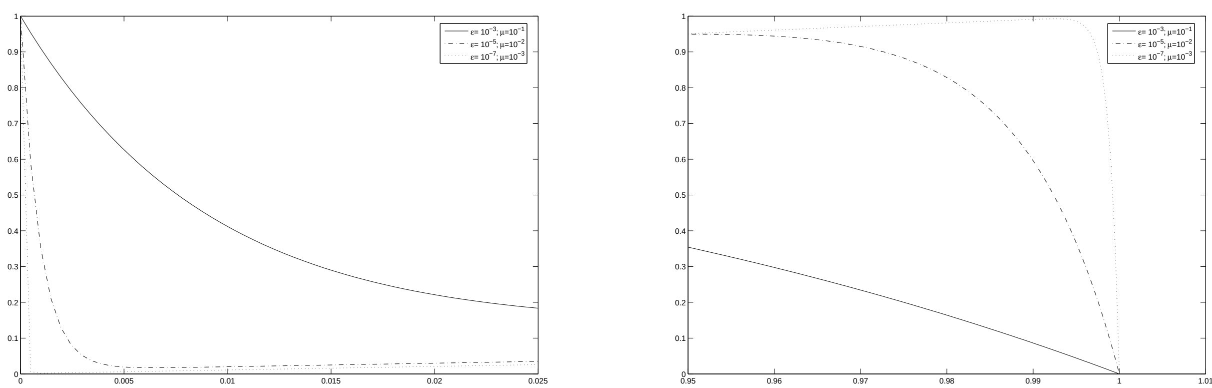

FIGURE 2. a) Left hand and b) Right hand solution profiles of Example 1 for various values of ε and μ.

8. **CONCLUSION**

In this paper, we presented an approximate method based on collocation for solving two-parameter singularly perturbed two point boundary value problems. Collocation methods are investigated because of their simplicity and inherent efficiency for applications to linear boundary value problems. These methods can be closely related to Galerkin methods, hence to finite-element methods, as they are much easier and more efficient for computing. It is relatively simple to collocate the solution at the mesh points. Results obtained using artificial viscosity algorithm are more accurate

Table 2. Numerical maximum errors ($E^N_{\epsilon,\mu}$) and rate of convergence (r_N), when applied to examples for various values of ε and μ, using artificial viscosity.

ε, μ	$N = 32$	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$
Example 1						
$10^{-4}, 10^{-1}$	4.740E-002	2.438E-002	1.102E-002	5.449E-003	2.117E-003	5.622E-004
	0.9592	1.1456	1.0161	1.3640	1.9129	
$10^{-6}, 10^{-1}$	5.134E-002	2.831E-002	1.491E-002	7.640E-003	3.854E-003	1.921E-003
	0.8588	0.9250	0.9646	0.9872	1.0045	
$10^{-8}, 10^{-1}$	5.138E-002	2.835E-002	1.495E-002	7.680E-003	3.894E-003	1.961E-003
	0.8579	0.9232	0.9610	0.9799	0.9897	
$10^{-10}, 10^{-1}$	5.138E-002	2.835E-002	1.495E-002	7.680E-003	3.895E-003	1.962E-003
	0.8579	0.9232	0.9610	0.9795	0.9893	
Example 2						
$2^{-8}, 2^{-3}$	1.111E-002	3.311E-003	7.982E-004	2.010E-004	5.033E-005	1.259E-005
	1.7465	2.0524	1.9896	1.9977	1.9991	
$2^{-12}, 2^{-3}$	4.689E-002	2.501E-002	1.159E-002	4.321E-003	1.280E-003	3.370E-004
	0.9068	1.1096	1.4234	1.7552	1.9253	
$2^{-16}, 2^{-3}$	5.084E-002	2.896E-002	1.553E-002	7.971E-003	3.944E-003	1.862E-003
	0.8119	0.8990	0.9622	1.0151	1.0828	
$2^{-20}, 2^{-3}$	5.109E-002	2.921E-002	1.578E-002	8.223E-003	4.196E-003	2.114E-003
	0.8066	0.8884	0.9404	0.9707	0.9890	
Example 3						
$2^{-10}, 2^{-4}$	3.930E-002	1.216E-002	2.764E-003	6.899E-004	1.715E-004	4.293E-005
	1.6924	2.1373	2.0023	2.0082	1.9982	
$2^{-12}, 2^{-4}$	5.121E-002	2.970E-002	1.276E-002	3.597E-003	8.370E-004	2.055E-004
	0.7860	1.2188	1.8268	2.1035	2.0261	
$2^{-20}, 2^{-4}$	7.084E-002	4.022E-002	2.165E-002	1.123E-002	5.691E-003	2.831E-003
	0.8167	0.8935	0.9470	0.9806	1.0074	
$2^{-25}, 2^{-4}$	7.093E-002	4.031E-002	2.174E-002	1.132E-002	5.780E-003	2.920E-003
	0.8153	0.8908	0.9415	0.9697	0.9851	

then the stated existing method with same numbers of nodal points. Also, collocation with B-splines leads to banded matrices as opposed to full matrices using polynomials, trigonometric functions and other well-known nonpiecewise approximates. We discussed the solution behavior for different values of ϵ and μ. Three numerical examples have been furnished to demonstrate the applicability of the method.

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