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AN EFFICIENT NUMERICAL METHOD FOR A SYSTEM OF SINGULARLY PERTURBED REACTION-DIFFUSION EQUATIONS

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ABSTRACT. Often, when singularly perturbed problems are solved on non-uniform meshes, solutions are decomposed into their regular and singular components. The analysis of the underlying method is conducted accordingly. In this paper, we consider a system of two coupled singularly perturbed reaction-diffusion equations. This type of systems are encountered in various fields of applied science such as chemical kinetics and predator-prey population dynamics. The presence of a small positive parameter multiplying the highest derivative in each equation leads to two overlapping and interacting boundary layers. We propose a fitted operator finite difference method to solve such systems. We show that the fitted operator satisfies a maximum principle. A rigorous error analysis (without decomposition of the solution) shows that the method is second order uniformly convergent. The theoretical results are confirmed through computational investigations.

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1. INTRODUCTION

The abundance of research on singularly perturbed problems (SPPs) over the last few decades is motivated by the singular aspect of their solutions: they vary sharply in small layers. This behavior is due to the presence of a small parameter multiplying the highest derivative in the differential equation of the underlying problem. Standard numerical methods have failed to resolve optimally these problems. As an alternative, fitted numerical methods have been extensively used. More information about numerical methods for SPPs can be found in [4, 15, 16, 18, 19, 22] and the references therein.

In this paper, a system of two coupled singularly perturbed reaction-diffusion equations is considered. More precisely, we consider the problem of finding $\vec{u} \in C^2(\bar{\Omega})$, such that for all $x \in \Omega = (0,1)$,

$$\mathbf{L}_{\varepsilon_1,\varepsilon_2}\vec{u} \equiv \begin{pmatrix} -\varepsilon_1 \frac{d^2}{dx^2} & 0 \\ 0 & -\varepsilon_2 \frac{d^2}{dx^2} \end{pmatrix} \vec{u}(x) + \mathbf{A}(x)\vec{u}(x) = \vec{f}(x), \tag{1.1}$$

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$$\vec{u}(0) = (u_1(0), u_2(0))^T, \quad \vec{u}(1) = (u_1(1), u_2(1))^T \tag{1.2}$$

where

$$\vec{u}(x) = \begin{pmatrix} u_1(x) \\ u_2(x) \end{pmatrix}, \quad \mathbf{A}(x) = \begin{pmatrix} a_{11}(x) & a_{12}(x) \\ a_{21}(x) & a_{22}(x) \end{pmatrix}, \quad \vec{f}(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix}.$$

It is assumed that the functions $a_{ij}(x), f_i(x),\ i, j = 1, 2$ are sufficiently smooth and that $0 < \varepsilon_1 \le \varepsilon_2 \le 1$. Moreover, we assume that

$$\text{(A1)} \qquad a_{11}(x) > |a_{12}(x)|, \quad a_{22}(x) > |a_{21}(x)|, \quad x \in \bar{\Omega} \tag{1.3}$$

$$\text{(A2)} \qquad a_{12}(x) \le 0, \quad a_{21}(x) \le 0, \quad x \in \bar{\Omega}. \tag{1.4}$$

While assumption (A1) is made to assure the diagonal dominance, assumption (A2) guarantees that $\mathbf{L}_{\varepsilon_1,\varepsilon_2}$ satisfies the standard maximum principle.

Such problems are encountered in many real life situations, for instance in modeling of diffusion processes complicated by chemical reactions (for example in the context of the Brusselator model) [1, 13] or in predator-prey population dynamics [5].

Over many years, several methods have been developed to solve SPPs. However, limited efforts have been devoted to singularly perturbed systems of equations. Below, we mention few examples. In [23], Shishkin considered singularly perturbed boundary-value problems for systems of elliptic and parabolic equations on an infinite strip. Some theoretical bounds on the solution and its derivatives were established. A classical finite difference method on Shishkin mesh was developed by Matthews et al. [12] to solve systems of the type (1.1)–(1.2). Madden and Stynes [10] constructed a piecewise-uniform mesh that is invariant of the usual Shishkin mesh. They developed a first order accurate central finite difference scheme. A numerical method on a Shishkin mesh was also studied by Linß and Madden [6]. Xenophontos and Oberbroeckling [25] investigated finite element methods while Clavero *et al.* [2] examined a hybrid finite difference scheme of HODIE type.

The works cited above involved systems of two equations. Recently, systems of an arbitrary number of equations were investigated by Linß and Madden [7] and Rao and Kumar [21]. In the former paper, discrete Green's functions were utilized to establish the properties of a central difference scheme on certain layer-adapted meshes while in the later, a high order Schwarz domain decomposition method was developed.

More works on systems of coupled singularly perturbed equations can be found in [9, 11, 24].

All the works mentioned above treat systems of singularly perturbed equations on non-uniform meshes. For the case of meshes of Shishkin type, the idea is to divide the domain of the problem into a number of subintervals, on each of which the mesh is constrained to be uniform. For a single reaction-diffusion two-point boundary value problem, three subintervals are required because two boundary layers are expected. In the case of a system such as (1.1), the domain will be divided into five subintervals due to the presence of two overlapping and interacting layers near both ends of the intervals. The solution is decomposed into its smooth and singular components. This situation leads to a lengthy and sometimes complicated analysis.

Recently, fitted operator finite difference methods (FOFDMs) have attracted a lot of attention for SPPs because of their simplicity: Their analysis is simple due to the use of uniform meshes. As examples, Lubuma and Patidar [8] and Patidar [20] successfully used these methods for scalar ODEs. Munyakazi and Patidar [18, 19] later extended these methods to elliptic PDEs in two dimensions. However, up to the best of our knowledge, no attempt has been made so far to use this type of methods to solve systems of coupled singularly perturbed equations. To fill this gap, we develop a FOFDM to solve the problem (1.1)–(1.2).

The FOFDM will automatically resolve the layers without having to decompose the solution. As a consequence, the analysis will be simpler. Another advantage is that, this type of methods seems to be more accurate (e.g., see comparison of results in [18, 19]).

Very often, knowledge of the properties of the analytical solution of a problem is necessary in the analysis of appropriate numerical methods. In Section 2, some qualitative issues pertaining to the problem are provided. Bounds on the solution and its derivatives, adapted from [2], are also presented.

Section 3 is concerned with the construction and analysis of the numerical method. In this section, the properties of the discrete operator are presented. In particular, we prove that this operator satisfies a discrete maximum principle as well as a stability result.

The error analysis of the proposed FOFDM is dealt with in Section 4. We prove that the method is second order uniformly convergent. Our theoretical findings are confirmed through computational investigations in Section 5. Finally, Section 6 is devoted to a brief discussion of the results of this paper.

2. **THEORETICAL ESTIMATES**

In this section, we present results which are necessary for the existence, uniqueness and well-posedness of the solution of the continuous problem (1.1)–(1.2).

Before we proceed, we adopt the following notation:

$$||\xi|| = \max_{\bar{\Omega}} |\xi(x)|, \quad ||\vec{\xi}|| = \max_{i}\{||\xi_i||\}, \quad \vec{\xi} = (\xi_1, \xi_2)^T.$$

Lemma 2.1 (Maximum Principle)**.** *Assume* (A1) *and* (A2). *Let* $\vec{\Psi} = (\Psi_1, \Psi_2)$ *be any sufficiently smooth function such that* $\vec{\Psi}(0) \geq \vec{0}$, $\vec{\Psi}(1) \geq \vec{0}$ *and* $\mathbf{L}_{\varepsilon_1,\varepsilon_2}\vec{\Psi} \geq \vec{0}, \forall x \in \Omega$. *Then,* $\vec{\Psi}(x) \geq \vec{0}, \forall x \in \bar{\Omega}$.

Proof. See [12]. □

We show below, using this principle, that the solution of the problem (1.1)–(1.2) is bounded.

Lemma 2.2. *Let* $\vec{u}(x)$ *be the solution of* $\mathbf{L}_{\varepsilon_1,\varepsilon_2}\vec{u}(x) = \vec{f}(x)$. *Then* $\vec{u}(x)$ *satisfies*

$$||\vec{u}|| \leq \frac{1}{\alpha}||\vec{f}|| + ||\vec{u}(0)|| + ||\vec{u}(1)||,$$

where $\alpha = \min_\Omega\{a_{11}(x) + a_{12}(x), a_{21}(x) + a_{22}(x)\}$.

Proof. We construct two vector-valued barrier functions $\vec{\Pi}^{\pm}$ defined by

$$\vec{\Pi}^{\pm}(x) = \frac{||\vec{f}(x)||}{\alpha} + ||\vec{u}(0)|| + ||\vec{u}(1)|| \pm \vec{u}(x). \tag{2.1}$$

We have

$$\vec{\Pi}^{\pm}(0) \geq \vec{0} \text{ and } \vec{\Pi}^{\pm}(1) \geq \vec{0}.$$

Furhermore,

$$\mathbf{L}_{\varepsilon_1,\varepsilon_2}\vec{\Pi}^{\pm}(x) = \begin{pmatrix} \pm f_1(x) + \frac{a_{11}(x)+a_{12}(x)}{\alpha}||\vec{f}(x)|| + (a_{11}(x)+a_{12}(x))(||\vec{u}(0)||+||\vec{u}(0)||) \\ \pm f_2(x) + \frac{a_{21}(x)+a_{22}(x)}{\alpha}||\vec{f}(x)|| + (a_{21}(x)+a_{22}(x))(||\vec{u}(0)||+||\vec{u}(0)||) \end{pmatrix}$$

If follows that $\mathbf{L}_{\varepsilon_1,\varepsilon_2}\vec{\Pi}^{\pm}(x) \geq \vec{(0)}, \forall x \in \Omega$. Thus, by the maximum principle,

$$\vec{\Pi}^{\pm}(x) \geq \vec{0}, \quad \forall x \in \bar{\Omega}.$$ □

For the purpose of the error analysis of the numerical method which we propose in next section, we present bounds on the solution and its derivatives in the lemma below. These bounds are adapted from [2].

In what follows,

$$\mathcal{B}_{\varepsilon_1}(x) = \exp\left(-x\sqrt{\alpha/\varepsilon_1}\right) + \exp\left(-(1-x)\sqrt{\alpha/\varepsilon_1}\right),$$

$$\mathcal{B}_{\varepsilon_2}(x) = \exp\left(-x\sqrt{\alpha/\varepsilon_2}\right) + \exp\left(-(1-x)\sqrt{\alpha/\varepsilon_2}\right).$$

Lemma 2.3. *Let $\vec{u}(x) = (u_1, u_2)^T$ be the solution of the problem* (1.1)–(1.2). *There exists a constant C, independent of ε_1 and ε_2, such that*

$$\begin{aligned}
|u_1(x)| &\leq C\left[1+\mathcal{B}_{\varepsilon_2}(x)\right],\\
|u_2(x)| &\leq C\left[1+\mathcal{B}_{\varepsilon_2}(x)\right],\\
|u_1^{(k)}(x)| &\leq C\left[1+{\varepsilon_1}^{-k/2}\mathcal{B}_{\varepsilon_1}(x)+{\varepsilon_2}^{-k/2}\mathcal{B}_{\varepsilon_2}(x)\right], \ \ \textit{for} \ \ k=1,2,3,4.\\
|u_1^{(k)}(x)| &\leq C\left[1+{\varepsilon_1}^{(4-k)/2}\varepsilon_2+{\varepsilon_1}^{-k/2}\mathcal{B}_{\varepsilon_1}(x)+{\varepsilon_2}^{-k/2}\mathcal{B}_{\varepsilon_2}(x)\right], \ \ \textit{for} \ \ k=5,6.\\
|u_2^{(k)}(x)| &\leq C\left[1+\varepsilon_2^{-k/2}\mathcal{B}_{\varepsilon_2}(x)\right], \ \ k=1,2,\\
|u_2^{(k)}(x)| &\leq C\left[1+{\varepsilon_2}^{-1}\left({\varepsilon_1}^{(2-k)/2}\mathcal{B}_{\varepsilon_1}(x)+{\varepsilon_2}^{(2-k)/2}\mathcal{B}_{\varepsilon_2}(x)\right)\right], \ \ \textit{for} \ \ k=3,4,5,6.
\end{aligned}$$

3. THE DISCRETE PROBLEM

In this section, we propose a numerical method for the system of BVPs (1.1)–(1.2). Let n be a positive integer. Consider the following partition of the interval $[0,1]$:

$$x_0 = 0, \ \ x_j = x_0 + jh, \ \ j = 1(1)n, \ \ h = x_j - x_{j-1}, \ \ x_n = 1.$$

We denote this mesh by Σ_n. In the rest of this paper, we use the notation $v_j = v(x_j)$ for any function $v(x)$ and denote the approximation of the solution $\vec{u}(x) = (u_1(x), u_2(x))^T$ by the unknown $\vec{U} = (U_1, U_2)^T$.

Using the theory of difference equations [14], we construct the following scheme on the mesh Σ_n:

$$-\varepsilon_1 \frac{U_{1j-1} - 2U_{1j} + U_{1j+1}}{\varphi_j^2} + a_{11,j}U_{1j} = f_{1,j} - a_{12,j}U_{2j}, \ \ j = 1(1)n-1, \tag{3.1}$$

$$-\varepsilon_2 \frac{U_{2j-1} - 2U_{2j} + U_{2j+1}}{\psi_j^2} + a_{22,j}U_{2j} = f_{2,j} - a_{21,j}U_{1j}, \ \ j = 1(1)n-1, \tag{3.2}$$

with

$$U_{10} = u_1(0), \ U_{1n} = u_1(1), \tag{3.3}$$

and

$$U_{20} = u_2(0), \ U_{2n} = u_2(1). \tag{3.4}$$

The denominator functions are given by

$$\varphi_j \equiv [\varphi_j(h, \varepsilon_1)] := \frac{2}{(\rho_j)} \sinh\left(\frac{\rho_j h}{2}\right) \tag{3.5}$$

and

$$\psi_j \equiv [\psi_j(h, \varepsilon_2)] := \frac{2}{(\lambda_j)} \sinh\left(\frac{\lambda_j h}{2}\right), \tag{3.6}$$

with

$$\rho_j = \sqrt{a_{11,j}/\varepsilon_1} \text{ and } \lambda_j = \sqrt{a_{22,j}/\varepsilon_2}. \tag{3.7}$$

It is easy to see that $\varphi_j(h, \varepsilon_1) = h + \mathcal{O}\left(\frac{h^3}{\varepsilon_1}\right)$, and $\psi_j(h, \varepsilon_2) = h + \mathcal{O}\left(\frac{h^3}{\varepsilon_2}\right)$.

Thus the discrete formulation of our problem reads: Find $\vec{U}_j = (U_{1j}, U_{2j})^T$ for $j = 1, 2, \ldots, n$ such that

$$\mathbf{L}^h_{\varepsilon_1,\varepsilon_2}\vec{U}_j \equiv \begin{pmatrix} -\varepsilon_1\tilde{\delta} & 0 \\ 0 & -\varepsilon_2\bar{\delta} \end{pmatrix}\vec{U}_j + \mathbf{A}(x_j)\vec{U}_j = \vec{f}_j, \tag{3.8}$$

with the boundary conditions

$$\vec{U}_1 = \vec{u}(0), \quad \vec{U}_n = \vec{u}(1), \tag{3.9}$$

where $\tilde{\delta}$ and $\bar{\delta}$ are the discrete operators approximating the second derivatives in equations (3.1) and (3.2).

Equations (3.1) and (3.3) lead to the system of linear equations

$$AU_1 = F_1 \tag{3.10}$$

where A is the tridiagonal matrix and F_1 the column-vector defined by

$$\begin{aligned}
A_{jj} &= \frac{2\varepsilon_1}{\varphi_j^2} + a_{11,j}, \quad j = 1(1)n-1 \\
A_{j,j+1} &= -\frac{\varepsilon_1}{\varphi_j^2}, \quad j = 1(1)n-2 \\
A_{j,j-1} &= -\frac{\varepsilon_1}{\varphi_j^2}, \quad j = 2(1)n-1 \\
F_{1,1} &= f_{1,1} - a_{12,1}U_{21} + \frac{\varepsilon_1}{\varphi_1^2}U_{10}; \\
F_{1,j} &= f_{1,j} - a_{12,j}U_{2j}, \quad j = 2(j)n-2, \\
F_{1,n-1} &= f_{1,n-1} - a_{12,n-1}U_{2n-1} + \frac{\varepsilon_1}{\varphi_{n-1}^2}U_{1n}.
\end{aligned}$$

Similarly, equations (3.2) and (3.4) lead to the system of linear equations

$$BU_2 = F_2 \tag{3.11}$$

where B is the tridiagonal matrix and F_2 the column-vector defined by

$$\begin{aligned}
B_{jj} &= \frac{2\varepsilon_2}{\psi_j^2} + a_{22,j}, \quad j = 1(1)n-1 \\
B_{j,j+1} &= -\frac{\varepsilon_2}{\psi_j^2}, \quad j = 1(1)n-2 \\
B_{j,j-1} &= -\frac{\varepsilon_2}{\psi_j^2}, \quad j = 2(1)n-1 \\
F_{2,1} &= f_{2,1} - a_{21,1}U_{11} + \frac{\varepsilon_2}{\psi_1^2}U_{20}, \\
F_{2,j} &= f_{2,j} - a_{21,j}U_{1j}, \quad j = 2(j)n-2, \\
F_{2,n-1} &= f_{2,n-1} - a_{21,n-1}U_{1n-1} + \frac{\varepsilon_2}{\psi_{n-1}^2}U_{2n}.
\end{aligned}$$

In the rest of the paper, C denotes various positive constants independent of the parameters ε_1 and ε_2 and of the mesh spacing h and may take different values in different equations and inequalities. Also, the first and second components of $\mathbf{L}^h_{\varepsilon_1,\varepsilon_2}\vec{\xi}_j$ are respectively denoted by $L^h_{\varepsilon_1,\varepsilon_2}\xi_{1j}$ and $L^h_{\varepsilon_1,\varepsilon_2}\xi_{2j}$.

The discrete operator $\mathbf{L}^h_{\varepsilon_1,\varepsilon_2}$ satisfies the following lemmas:

Lemma 3.1 (Discrete maximum principle)**.** *Let* $\{\vec{\phi}_j\} = \{(\phi_{1j}, \phi_{2j})\}$ *be any mesh function satisfying* $\vec{\phi}_0 \geq 0$, $\vec{\phi}_n \geq 0$, *and* $\mathbf{L}^h_{\varepsilon_1,\varepsilon_2}\vec{\phi}_j \geq 0$, $j = 1(1)n-1$. *Then* $\vec{\phi}_j \geq 0$, $\forall j = 0(1)n$.

Proof. Let k and l be indices such that $\phi_{1k} = \min_j \phi_{1j}$, $\phi_{2l} = \min_j \phi_{2j}$, $\forall 0 \leq j \leq n$. Assume that $\phi_{1k} < 0$. Furthermore assume, without loss of generality, that $\phi_{1k} < \phi_{2l}$. It is easily seen that $k \neq 0$ and $k \neq n$. Also, it is clear that $\phi_{1k+1} - \phi_{1k} \geq 0$ and $\phi_{1k} - \phi_{1k-1} \leq 0$. In this way, the first component of $\mathbf{L}^h_{\varepsilon_1,\varepsilon_2}\vec{\phi}_k$ reads

$$-\varepsilon_1 \frac{\phi_{1k-1} - 2\phi_{1k} + \phi_{1k+1}}{\varphi_k^2} + a_{11}(x_k)\phi_{1k} + a_{12}(x_k)\phi_{2k}.$$

We rewrite this expression as follows

$$-\frac{\varepsilon_1}{\varphi_k^2}\left[(\phi_{1k-1} - \phi_{1k}) + (\phi_{1k+1} - \phi_{1k})\right] + [a_{11}(x_k) + a_{12}(x_k)]\phi_{1k} + [\phi_{2k} - \phi_{1k}]a_{12}(x_k) < 0.$$

This contradiction ends the proof. □

Lemma 3.2. If $\vec{Z}_j = (Z_{1j}, Z_{2j})^T$ is any mesh function such that $\vec{Z}_0 = \vec{0}$ and $\vec{Z}_n = \vec{0}$, then

$$|\vec{Z}_i| \leq \frac{1}{\alpha} \max_{1 \leq j \leq n-1} |\mathbf{L}^h_{\varepsilon_1,\varepsilon_2}\vec{Z}_j|, \quad 0 \leq i \leq n.$$

Proof. Let

$$N = \frac{1}{\alpha} \max_{1 \leq j \leq n-1} |\mathbf{L}_{\varepsilon_1,\varepsilon_2}\vec{Z}_j|$$

and $(\vec{\Psi}^{\pm})_j$ be the mesh functions defined by

$$({\Psi_k}^{\pm})_j = N \pm Z_{kj}, \quad k = 1, 2.$$

It is clear that $(\vec{\Psi}^{\pm})_0 \geq \vec{0}$ and $(\vec{\Psi}^{\pm})_n \geq \vec{0}$. The first component of $\mathbf{L}^h_{\varepsilon_1,\varepsilon_2}(\vec{\Psi}^{\pm})_j$ is

$$\begin{aligned} L^h_{\varepsilon_1,\varepsilon_2}\Psi_{1j}^{\pm} &= -\varepsilon_1 \frac{(N \pm Z_{1j-1}) - 2(N \pm Z_{1j}) + (N \pm Z_{1j+1})}{\varphi_j^2} \\ &\quad + a_{11}(x_j)(N \pm Z_{1j}) + a_{12}(N \pm Z_{2j}) \\ &= \frac{a_{11}(x_j) + a_{12}(x_j)}{\alpha} \max_{1\leq j\leq n-1} |\mathbf{L}^h_{\varepsilon_1,\varepsilon_2}\vec{Z}_j| \pm L^h_{\varepsilon_1,\varepsilon_2} Z_{1j} \end{aligned}$$

Since $0 < \alpha \leq a_{11}(x_j) + a_{12}(x_j)$, we have $L^h_{\varepsilon_1,\varepsilon_2}\Psi_{1j}^{\pm} \geq 0$. Similarly, $L^h_{\varepsilon_1,\varepsilon_2}\Psi_{2j}^{\pm} \geq 0$. By virtue of the discrete maximum principle (Lemma 3.1) we have $(\vec{\Psi}^{\pm})_j \geq 0$ for $0 \leq j \leq n$ and this ends the proof. □

Lemma 3.3. *For a fixed mesh and for integers k, we have*

$$\lim_{\varepsilon\to 0} \max_{1\leq j\leq n-1} \frac{\exp(-Cx_j/\sqrt{\varepsilon})}{\varepsilon^{k/2}} = 0 \quad \textit{and}$$

$$\lim_{\varepsilon\to 0} \max_{1\leq j\leq n-1} \frac{\exp(-C(1-x_j)/\sqrt{\varepsilon})}{\varepsilon^{k/2}} = 0.$$

Proof. See [17]. □

We are now ready to analyze the error of the proposed method.

4. ERROR ANALYSIS

The local truncation error of the scheme for the first component of the solution is

$$L^h_{\varepsilon_1,\varepsilon_2}(u_{1j} - U_{1j}) = -\varepsilon_1 u''_{1j} + \varepsilon_1 \frac{u_{1j-1} - 2u_{1j} + u_{1j+1}}{\varphi_j^2} \tag{4.1}$$

Using Taylor expansions of u_{1j-1}, u_{1j+1} and φ_j, we obtain

$$\begin{aligned} &L^h_{\varepsilon_1,\varepsilon_2}(u_{1j} - U_{1j}) \\ &\quad = -\varepsilon_1 \left[u''_{1j} - \left(\frac{1}{h^2} - \frac{\rho_j^2}{12} + \frac{\rho_j^4 h^2}{240} \right) \left(h^2 u''_{1j} + \frac{h^4}{24} \left({u_1}^{(iv)}(\xi_1) + {u_1}^{(iv)}(\xi_2) \right) \right) \right], \end{aligned}$$

where $\xi_1 \in (x_{j-1}, x_j)$ and $\xi_2 \in (x_j, x_{j+1})$. Making use of the first equation of (3.7), rearranging and applying the triangular inequality, we obtain:

$$\begin{aligned} \left| L^h_{\varepsilon_1,\varepsilon_2}(u_{1j} - U_{1j}) \right| &\leq \frac{\varepsilon_1 h^2}{12} \left| {u_1}^{(iv)}(\xi_j) \right| + \frac{a_{11,j} h^2}{12} \left| u''_{1j} \right| + \frac{a_{11,j} h^4}{144} \left| {u_1}^{(iv)}(\xi_j) \right| \\ &\quad + \frac{a_{11,j}^2 h^4}{240} \left| \frac{u''_{1j}}{\varepsilon_1} \right| + \frac{a_{11,j}^2 h^6}{2880} \left| \frac{{u_1}^{(iv)}(\xi_j)}{\varepsilon_1} \right| \end{aligned} \tag{4.2}$$

where $\xi_j \in (x_{j-1}, x_{j+1})$. The use of lemmas (2.3) and (3.3) shows immediately that $|u_1{}''_j| \le C$ and $|u_1{}^{(iv)}(\xi_j)| \le C$. Also, we note that

$$\left|\frac{u_1{}''_j}{\varepsilon_1}\right| \le C\left[\frac{\varepsilon_1 + \exp\left(-x_j\sqrt{\frac{\alpha}{\varepsilon_1}}\right) + \exp\left(-(1-x_j)\sqrt{\frac{\alpha}{\varepsilon_1}}\right)}{\varepsilon_1^2} + \frac{\exp\left(-x_j\sqrt{\frac{\alpha}{\varepsilon_2}}\right) + \exp\left(-(1-x_j)\sqrt{\frac{\alpha}{\varepsilon_2}}\right)}{\varepsilon_1\varepsilon_2}\right].$$

But, for all $s \in (0, x_j)$, we have

$$\exp(-x_j\sqrt{\alpha/\varepsilon_1}) < \exp(-s\sqrt{\alpha/\varepsilon_1}).$$

When ε_1 approaches zero, $\exp(-x_j\sqrt{\alpha/\varepsilon_1})$ will tend to zeros faster than $\exp(-s\sqrt{\alpha/\varepsilon_1})$, thus widening the gap between these two quantities. It follows that

$$\varepsilon_1 + \exp(-x_j\sqrt{\alpha/\varepsilon_1}) < \exp(-s\sqrt{\alpha/\varepsilon_1}).$$

In this way, we have $\left|u_1{}''_j/\varepsilon_1\right| \le C$. In a similar way, we can prove that $|u_1{}^{(iv)}(\xi_j)/\varepsilon_1| \le C$.

Inequality (4.2) then leads to

$$|L^h_{\varepsilon_1,\varepsilon_2}(u_{1j} - U_{1j})| \le Ch^2. \tag{4.3}$$

The local truncation error for the second component of the solution is bounded as follows.

$$\begin{aligned} \left|L^h_{\varepsilon_1,\varepsilon_2}(u_{2j} - U_{2j})\right| \;\le\; & \frac{\varepsilon_2 h^2}{12}\left|u_2{}^{(iv)}(\zeta_j)\right| + \frac{a_{22,j}h^2}{12}\left|u_2{}''_j\right| + \frac{a_{22,j}h^4}{144}\left|u_2{}^{(iv)}(\zeta_j)\right| \\ & + \frac{a_{22,j}^2 h^4}{240}\left|\frac{u_2{}''_j}{\varepsilon_2}\right| + \frac{a_{22,j}^2 h^6}{2880}\left|\frac{u_2{}^{(iv)}(\zeta_j)}{\varepsilon_2}\right|, \end{aligned} \tag{4.4}$$

where $\zeta_j \in (x_{j-1}, x_{j+1})$. Note that we have made use of the second equation of (3.7) in the above inequality. Following similar arguments as the ones used for inequality (4.2), we obtain

$$|L^h_{\varepsilon_1,\varepsilon_2}(u_{2j} - U_{2j})| \le Ch^2. \tag{4.5}$$

From (4.3) and (4.5), we see that

$$|\mathbf{L}^h_{\varepsilon_1,\varepsilon_2}(\vec{u} - \vec{U})_j| \le Ch^2. \tag{4.6}$$

Finally, by virtue of Lemma 3.2, we establish our main result which we summarize in the theorem below.

Theorem 4.1. *Let $\vec{u}$ be the analytical solution of problem* (1.1)–(1.2) *and $\vec{U}$ be the approximate solution obtained via the scheme* (3.1)–(3.4). *Then $\vec{U}$ converges uniformly to $\vec{u}$ and the convergence is quadratic. In other words there exists a constant*

C independent of ε_1, ε_2 *and* h *such that*

$$\sup_{0<\varepsilon_1,\varepsilon_2\leq 1} \max_{0\leq j\leq n} |(\vec{u}-\vec{U})_j| \leq Ch^2. \tag{4.7}$$

5. NUMERICAL RESULTS

In our numerical experiments, we will use the following iterative process

$$\mathbf{T_1}\vec{Z}^{(k)} = \vec{f}_1 - \mathbf{D_1}\vec{W}^{(k)}, \tag{5.1}$$

$$\mathbf{T_2}\vec{W}^{(k+1)} = \vec{f}_2 - \mathbf{D_2}\vec{Z}^{(k)}, \tag{5.2}$$

where $\mathbf{T_1}$ and $\mathbf{T_2}$ are the tridiagonal matrices A and B in equations (3.10) and (3.11), respectively and the diagonal matrices $\mathbf{D_1}$ and $\mathbf{D_2}$ are such that:

$$(\mathbf{D_1})_{j,j} = a_{12}(x_j), \quad (\mathbf{D_2})_{j,j} = a_{21}(x_j).$$

The Lemma below, the proof of which follows similar steps to the proof of Lemma 8 in [12], guarantees the convergence of the above iterative scheme.

Lemma 5.1. *Assume that* (A1) *holds. Then the iterative scheme* (5.1)–(5.2) *converges to the solution of the matrix system* (3.8)–(3.9).

In order to illustrate our theoretical findings summarized in Theorem 4.1, we consider the following numerical problem.

$$-\varepsilon_1 u_1''(x) + 2(x+1)^2 u_1(x) - (1+x^3)u_2(x) = 2\exp(x),$$
$$-\varepsilon_2 u_2''(x) - 2\cos\left(\frac{\pi x}{4}\right) u_1(x) + 2.2\exp(1-x)u_2(x) = 10x+1,$$

with the boundary conditions $u_1(0) = u_1(1) = u_2(0) = u_2(1) = 0$.

This test example was considered in [12] for $0 < \varepsilon_1 < \varepsilon_2 = 1$ and in [10] for $0 < \varepsilon_1 \leq \varepsilon_2 \leq 1$.

Due to the fact that the exact solution is not available, the maximum errors at all mesh points are calculated using the formulas

$$E{1}^n_{\varepsilon_1,\varepsilon_2} := \max_{0\leq j\leq n} |U_1{}^n_j - U_1{}^{2n}_{2j}|$$

and

$$E{2}^n_{\varepsilon_1,\varepsilon_2} := \max_{0\leq j\leq n} |U_2{}^n_j - U_2{}^{2n}_{2j}|$$

for the first and second components of the solution, respectively and where $\vec{U}^{2n}$ is the approximate solution obtained via the scheme (3.1)–(3.4) on the mesh Σ_{2n}.

The numerical rates of convergence are computed using the formula [3]:

$$r_k \equiv r^k_{\varepsilon_1,\varepsilon_2} := \log_2(E^{n_k}_{\varepsilon_1,\varepsilon_2}/E^{2n_k}_{\varepsilon_1,\varepsilon_2}), \quad k = 1, 2, \ldots$$

In the computations, $\vec{W}$ was initialized to $(0.1, 0.1, \ldots, 0.1)^T$ and the stopping criteria were set to be

$$||\vec{W}^{(k+1)} - \vec{W}^{(k)}|| < 10^{-15} \text{ and } ||\vec{Z}^{(k+1)} - \vec{Z}^{(k)}|| < 10^{-15}.$$

6. DISCUSSION OF RESULTS AND FUTURE PLANS

This paper dealt with the construction and analysis of a fitted operator finite difference method (FOFDM) for systems of singularly perturbed reaction-diffusion equations.

We proved that the FOFDM satisfies a discrete maximum principle. A stability result on this discrete operator was established as a consequence of the maximum principle. We showed that the numerical method is second order accurate uniformly convergent. This result is confirmed through our computational investigations. In our theoretical analysis, we assumed that $0 < \varepsilon_1 \leq \varepsilon_2 \leq 1$. However, in our numerics, both scenarios $\varepsilon_1 < \varepsilon_2$ and $\varepsilon_1 > \varepsilon_2$ were considered.

Numerical results for the case $\varepsilon_1 < \varepsilon_2 = 1$, are presented in the tables 1–4. The first two tables show the computed maximum errors in the approximation of the first component u_1 and the second component u_2 of the solution, respectively. The corresponding rates of convergence are given in tables 3 and 4. Results for other cases are also tabulated following the same pattern: Tables 5–8 for $\varepsilon_1 = 1 > \varepsilon_2$, tables 9–12 for the case $\varepsilon_1 < \varepsilon_2 = 2^{-4}$, tables 13–16 for the case $\varepsilon_1 < \varepsilon_2 = 2^{-4}$, and tables 17–20 for the case $\varepsilon_1 = 2^{-4} > \varepsilon_2$.

A comparison of results presented in tables 1 and 2 to the results in [12] and in [10] shows the superiority of our proposed method. Furthermore, while [2] gives an almost third order method, it turns out that our method performs better in that one needs only 512 subintervals to get a maximum error of magnitude 10^{-6} (see Table 10) whereas 4096 subintervals must be used to expect similar degree of accuracy in [2] (see Table 5 therein).

Currently, we are investigating the possibility of extending the proposed approach to solve system of time-dependent parabolic partial differential equations.

TABLE 1. Maximum errors ($\varepsilon_1 < \varepsilon_2 = 1$) for u_1.

ε_1	n=8	n=16	n=32	n=64	n=128	n=256	n=512	n=1024
2^{-4}	3.98E-03	1.03E-03	2.65E-04	6.63E-05	1.66E-05	4.15E-06	1.04E-06	2.59E-07
2^{-8}	4.38E-03	3.45E-03	9.80E-04	2.66E-04	6.70E-05	1.68E-05	4.20E-06	1.05E-06
2^{-28}	8.67E-05	2.22E-05	5.64E-06	1.41E-06	3.53E-07	8.82E-08	2.21E-08	5.51E-09
2^{-32}	8.67E-05	2.22E-05	5.64E-06	1.41E-06	3.53E-07	8.82E-08	2.21E-08	5.51E-09
2^{-34}	8.67E-05	2.22E-05	5.64E-06	1.41E-06	3.53E-07	8.82E-08	2.21E-08	5.51E-09
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2^{-60}	8.67E-05	2.22E-05	5.64E-06	1.41E-06	3.53E-07	8.82E-08	2.21E-08	5.51E-09

TABLE 2. Maximum errors ($\varepsilon_1 < \varepsilon_2 = 1$) for u_2.

ε_1	n=8	n=16	n=32	n=64	n=128	n=256	n=512	n=1024
2^{-4}	1.23E-03	3.18E-04	8.03E-05	2.01E-05	5.03E-06	1.26E-06	3.15E-07	7.86E-08
2^{-8}	7.76E-04	3.17E-04	8.97E-05	2.32E-05	5.84E-06	1.46E-06	3.66E-07	9.15E-08
2^{-28}	3.36E-04	8.38E-05	2.12E-05	5.29E-06	1.32E-06	3.31E-07	8.27E-08	2.07E-08
2^{-32}	3.36E-04	8.38E-05	2.12E-05	5.29E-06	1.32E-06	3.31E-07	8.27E-08	2.07E-08
2^{-34}	3.36E-04	8.38E-05	2.12E-05	5.29E-06	1.32E-06	3.31E-07	8.27E-08	2.07E-08
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2^{-60}	3.36E-04	8.38E-05	2.12E-05	5.29E-06	1.32E-06	3.31E-07	8.27E-08	2.07E-08

TABLE 3. Rates of convergence r_k ($\varepsilon_1 < \varepsilon_2 = 1$) for u_1, $n_k = 8 \times 2^{k-1}, k = 1(1)7$.

ε_1	r_1	r_2	r_3	r_4	r_5	r_6	r_7
2^{-4}	1.944	1.966	1.997	1.999	2.000	2.000	2.000
2^{-8}	3.475	1.814	1.882	1.989	1.996	1.999	2.000
2^{-28}	1.967	1.975	2.000	2.000	2.000	2.000	2.001
2^{-32}	1.967	1.975	2.000	2.000	2.000	2.000	2.000
2^{-34}	1.967	1.975	2.000	2.000	2.000	2.000	2.000
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2^{-60}	1.967	1.975	2.000	2.000	2.000	2.000	2.000

TABLE 4. Rates of convergence r_k ($\varepsilon_1 < \varepsilon_2 = 1$) for u_2, $n_k = 8 \times 2^{k-1}, k = 1(1)7$.

ε_1	r_1	r_2	r_3	r_4	r_5	r_6	r_7
2^{-4}	1.946	1.986	1.997	1.999	2.000	2.000	2.000
2^{-8}	1.290	1.823	1.953	1.988	1.997	1.999	2.000
2^{-28}	2.006	1.983	2.000	2.000	2.000	2.000	2.000
2^{-32}	2.006	1.983	2.000	2.000	2.000	2.000	2.000
2^{-34}	2.006	1.983	2.000	2.000	2.000	2.000	2.000
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2^{-60}	2.006	1.983	2.000	2.000	2.000	2.000	2.000

Table 5. Maximum errors ($\varepsilon_1 = 1 > \varepsilon_2$) for u_1.

ε_2	n=8	n=16	n=32	n=64	n=128	n=256	n=512	n=1024
2^{-6}	3.17E-03	9.58E-04	2.51E-04	6.35E-05	1.59E-05	3.98E-06	9.96E-07	2.49E-07
2^{-10}	1.79E-03	8.91E-04	3.68E-04	1.06E-04	2.73E-05	6.88E-06	1.72E-06	4.31E-07
2^{-26}	2.46E-03	6.28E-04	1.58E-04	3.95E-05	9.86E-06	2.47E-06	6.17E-07	1.51E-07
2^{-28}	2.46E-03	6.28E-04	1.58E-04	3.95E-05	9.86E-06	2.47E-06	6.17E-07	1.51E-07
2^{-30}	2.46E-03	6.28E-04	1.58E-04	3.95E-05	9.86E-06	2.47E-06	6.17E-07	1.54E-07
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
2^{-60}	2.46E-03	6.28E-04	1.58E-04	3.95E-05	9.86E-06	2.47E-06	6.17E-07	1.54E-07

Table 6. Maximum errors ($\varepsilon_1 = 1 > \varepsilon_2$) for u_2.

ε_2	n=8	n=16	n=32	n=64	n=128	n=256	n=512	n=1024
2^{-4}	7.34E-03	1.88E-03	4.74E-04	1.19E-04	2.98E-05	7.46E-06	1.87E-06	4.66E-07
2^{-6}	1.38E-02	4.42E-03	1.14E-03	2.94E-04	7.36E-05	1.84E-05	4.60E-06	1.15E-06
2^{-26}	1.45E-03	3.65E-04	9.16E-05	2.29E-05	5.73E-06	1.43E-06	3.58E-07	8.93E-08
2^{-28}	1.45E-03	3.65E-04	9.16E-05	2.29E-05	5.73E-06	1.43E-06	3.58E-07	8.96E-08
2^{-30}	1.45E-03	3.65E-04	9.16E-05	2.29E-05	5.73E-06	1.43E-06	3.58E-07	8.96E-08
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
2^{-60}	1.45E-03	3.65E-04	9.16E-05	2.29E-05	5.73E-06	1.43E-06	3.58E-07	8.96E-08

Table 7. Rates of convergence r_k ($\varepsilon_1 = 1 > \varepsilon_2$) for u_1, $n_k = 8 \times 2^{k-1}, k = 1(1)7$.

ε_2	r_1	r_2	r_3	r_4	r_5	r_6	r_7
2^{-6}	1.725	1.934	1.984	1.995	1.999	2.000	2.000
2^{-10}	1.006	1.274	1.804	1.952	1.988	1.997	1.999
2^{-26}	1.969	1.994	1.998	2.000	2.000	2.000	2.029
2^{-28}	1.969	1.994	1.998	2.000	2.000	2.000	2.000
2^{-30}	1.969	1.994	1.998	2.000	2.000	2.000	2.000
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
2^{-60}	1.969	1.994	1.998	2.000	2.000	2.000	2.000

Table 8. Rates of convergence r_k ($\varepsilon_1 = 1 > \varepsilon_2$) for u_2, $n_k = 8 \times 2^{k-1}, k = 1(1)7$.

ε_2	r_1	r_2	r_3	r_4	r_5	r_6	r_7
2^{-4}	1.962	1.991	1.991	1.999	1.999	2.000	2.000
2^{-6}	1.639	1.950	1.961	1.997	1.999	2.000	2.000
2^{-26}	1.985	1.996	1.998	2.000	2.000	2.000	2.005
2^{-28}	1.985	1.996	1.998	2.000	2.000	2.000	2.000
2^{-30}	1.985	1.996	1.998	2.000	2.000	2.000	2.000
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
2^{-60}	1.985	1.996	1.998	2.000	2.000	2.000	2.000

TABLE 9. Maximum errors ($\varepsilon_1 < \varepsilon_2 = 2^{-4}$) for u_1.

ε_1	n=8	n=16	n=32	n=64	n=128	n=256	n=512	n=1024
2^{-6}	1.55E-02	5.07E-03	1.41E-03	3.60E-04	9.04E-05	2.27E-05	5.66E-06	1.42E-06
2^{-14}	2.79E-03	7.21E-04	5.38E-04	9.34E-04	5.10E-04	1.41E-04	3.73E-05	9.39E-06
2^{-22}	2.79E-03	7.20E-04	1.82E-04	4.55E-05	1.14E-05	2.85E-06	1.20E-05	3.80E-05
2^{-26}	2.79E-03	7.20E-04	1.82E-04	4.55E-05	1.14E-05	2.85E-06	7.11E-07	1.78E-07
2^{-30}	2.79E-03	7.20E-04	1.82E-04	4.55E-05	1.14E-05	2.85E-06	7.11E-07	1.78E-07
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2^{-60}	2.79E-03	7.20E-04	1.82E-04	4.55E-05	1.14E-05	2.85E-06	7.11E-07	1.78E-07

TABLE 10. Maximum errors ($\varepsilon_1 < \varepsilon_2 = 2^{-4}$) for u_2.

ε_1	n=8	n=16	n=32	n=64	n=128	n=256	n=512	n=1024
2^{-6}	1.84E-02	5.23E-03	1.36E-03	3.43E-04	8.59E-05	2.15E-05	5.37E-06	1.34E-06
2^{-14}	1.17E-02	3.03E-03	7.79E-04	2.50E-04	8.94E-05	2.46E-05	6.31E-06	1.59E-06
2^{-22}	1.17E-02	3.03E-03	7.64E-04	1.92E-04	4.80E-05	1.20E-05	3.05E-06	1.18E-06
2^{-26}	1.17E-02	3.03E-03	7.64E-04	1.92E-04	4.80E-05	1.20E-05	3.00E-06	7.50E-07
2^{-30}	1.17E-02	3.03E-03	7.64E-04	1.92E-04	4.80E-05	1.20E-05	3.00E-06	7.50E-07
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2^{-60}	1.17E-02	3.03E-03	7.64E-04	1.92E-04	4.80E-05	1.20E-05	3.00E-06	7.50E-07

TABLE 11. Rates of convergence r_k ($\varepsilon_1 < \varepsilon_2 = 2^{-4}$) for u_1, $n_k = 8 \times 2^{k-1}, k = 1(1)7$.

ε_1	r_1	r_2	r_3	r_4	r_5	r_6	r_7
2^{-6}	1.61	1.84	1.97	1.99	2.00	2.00	2.00
2^{-14}	1.95	0.42	-0.79	0.87	1.85	1.92	1.99
2^{-26}	1.95	1.99	2.00	2.00	2.00	2.00	2.00
2^{-30}	1.95	1.99	2.00	2.00	2.00	2.00	2.00
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2^{-60}	1.95	1.99	2.00	2.00	2.00	2.00	2.00

TABLE 12. Rates of convergence r_k ($\varepsilon_1 < \varepsilon_2 = 2^{-4}$). for u_2, $n_k = 8 \times 2^{k-1}, k = 1(1)7$

ε_1	r_1	r_2	r_3	r_4	r_5	r_6	r_7
2^{-6}	1.81	1.95	1.99	2.00	2.00	2.00	2.00
2^{-14}	1.95	1.96	1.64	1.48	1.86	1.96	1.99
2^{-26}	1.95	1.99	2.00	2.00	2.00	2.00	2.00
2^{-30}	1.95	1.99	2.00	2.00	2.00	2.00	2.00
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2^{-60}	1.95	1.99	2.00	2.00	2.00	2.00	2.00

TABLE 13. Maximum errors ($\varepsilon_1 < \varepsilon_2 = 2^{-12}$) for u_1.

ε_1	n=32	n=64	n=128	n=256	n=512	n=1024
2^{-18}	6.38E-03	5.62E-03	3.60E-03	3.46E-03	1.61E-03	5.59E-04
2^{-26}	6.38E-03	5.62E-03	1.65E-03	4.55E-04	1.15E-04	2.88E-05
2^{-28}	6.38E-03	5.62E-03	1.65E-03	4.55E-04	1.15E-04	2.88E-05
2^{-30}	6.38E-03	5.62E-03	1.65E-03	4.55E-04	1.15E-04	2.88E-05
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2^{-60}	6.38E-03	5.62E-03	1.65E-03	4.55E-04	1.15E-04	2.88E-05

TABLE 14. Maximum errors ($\varepsilon_1 =< \varepsilon_2 = 2^{-12}$) for u_2.

ε_1	n=32	n=64	n=128	n=256	n=512	n=1024
2^{-18}	2.59E-02	2.27E-02	6.80E-03	3.12E-03	1.07E-03	2.99E-04
2^{-26}	2.59E-02	2.27E-02	6.64E-03	1.83E-03	4.62E-04	1.16E-04
2^{-28}	2.59E-02	2.27E-02	6.64E-03	1.83E-03	4.62E-04	1.16E-04
2^{-30}	2.59E-02	2.27E-02	6.64E-03	1.83E-03	4.62E-04	1.16E-04
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2^{-60}	2.59E-02	2.27E-02	6.64E-03	1.83E-03	4.62E-04	1.16E-04

TABLE 15. Rates of convergence r_k ($\varepsilon_1 < \varepsilon_2 = 2^{-12}$) for u_1, $n_k = 32 \times 2^{k-1}, k = 1(1)6$.

ε_1	r_1	r_2	r_3	r_4	r_5	r_6
2^{-18}	0.18	0.65	0.06	1.11	1.52	1.86
2^{-26}	0.18	1.77	1.86	1.98	2.00	-0.21
2^{-28}	0.18	1.77	1.86	1.98	2.00	2.00
2^{-30}	0.18	1.77	1.86	1.98	2.00	2.00
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2^{-60}	0.18	1.77	1.86	1.98	2.00	2.00

TABLE 16. Rates of convergence r_k ($\varepsilon_1 < \varepsilon_2 = 2^{-12}$) for u_2, $n_k = 32 \times 2^{k-1}, k = 1(1)6$.

ε_1	r_1	r_2	r_3	r_4	r_5	r_6
2^{-18}	0.19	1.74	1.12	1.55	1.84	1.91
2^{-26}	0.20	1.77	1.86	1.98	2.00	1.97
2^{-28}	0.20	1.77	1.86	1.98	2.00	2.00
2^{-30}	0.20	1.77	1.86	1.98	2.00	2.00
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2^{-60}	0.20	1.77	1.86	1.98	2.00	2.00

TABLE 17. Maximum errors ($\varepsilon_1 = 2^{-4} > \varepsilon_2$) for u_1.

ε_2	n=8	n=16	n=32	n=64	n=128	n=256	n=512	n=1024
2^{-6}	2.41E-02	7.46E-03	2.05E-03	5.18E-04	1.30E-04	3.26E-05	8.14E-06	2.04E-06
2^{-16}	6.47E-03	1.71E-03	4.12E-04	3.67E-04	3.22E-04	1.18E-04	3.28E-05	8.43E-06
2^{-26}	6.47E-03	1.71E-03	4.29E-04	1.07E-04	2.68E-05	6.70E-06	1.67E-06	4.02E-07
2^{-28}	6.47E-03	1.71E-03	4.29E-04	1.07E-04	2.68E-05	6.70E-06	1.67E-06	4.19E-07
2^{-30}	6.47E-03	1.71E-03	4.29E-04	1.07E-04	2.68E-05	6.70E-06	1.67E-06	4.19E-07
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
2^{-60}	6.47E-03	1.71E-03	4.29E-04	1.07E-04	2.68E-05	6.70E-06	1.67E-06	4.19E-07

TABLE 18. Maximum errors ($\varepsilon_1 = 2^{-4} > \varepsilon_2$) for u_2.

ε_2	n=8	n=16	n=32	n=64	n=128	n=256	n=512	n=1024
2^{-6}	3.81E-02	1.07E-02	2.89E-03	7.27E-04	1.82E-04	4.56E-05	1.14E-05	2.85E-06
2^{-8}	3.65E-02	2.17E-02	5.97E-03	1.53E-03	3.84E-04	9.64E-05	2.41E-05	6.03E-06
2^{-26}	4.01E-03	1.04E-03	2.60E-04	6.51E-05	1.63E-05	4.07E-06	1.02E-06	2.40E-07
2^{-30}	4.01E-03	1.04E-03	2.60E-04	6.51E-05	1.63E-05	4.07E-06	1.02E-06	2.54E-07
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
2^{-60}	4.01E-03	1.04E-03	2.60E-04	6.51E-05	1.63E-05	4.07E-06	1.02E-06	2.54E-07

TABLE 19. Rates of convergence r_k ($\varepsilon_1 = 2^{-4} > \varepsilon_2$) for u_1, $n_k = 8 \times 2^{k-1}, k = 1(1)7$.

ε_2	r_1	r_2	r_3	r_4	r_5	r_6	r_7
2^{-6}	1.69	1.87	1.98	1.99	2.00	2.00	2.00
2^{-16}	1.92	2.06	0.17	0.19	1.45E	1.85	1.96
2^{-26}	1.92	2.00	2.00	2.00	2.00	2.00	2.06
2^{-28}	1.92	2.00	2.00	2.00	2.00	2.00	2.00
2^{-30}	1.92	2.00	2.00	2.00	2.00	2.00	2.00
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
2^{-60}	1.92	2.00	2.00	2.00	2.00	2.00	2.00

TABLE 20. Rates of convergence r_k ($\varepsilon_1 = 2^{-4} > \varepsilon_2$) for u_2, $n_k = 8 \times 2^{k-1}, k = 1(1)7$.

ε_2	r_1	r_2	r_3	r_4	r_5	r_6	r_7
2^{-6}	1.83	1.89	1.99	2.00	2.00	2.00	2.00
2^{-8}	0.75	1.86	1.97	1.99	2.00	2.00	2.00
2^{-26}	1.95	2.00	2.00	2.00	2.00	2.00	2.08
2^{-30}	1.95	2.00	2.00	2.00	2.00	2.00	2.00
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
2^{-60}	1.95	2.00	2.00	2.00	2.00	2.00	2.00

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