Neural, Parallel, and Scientific Computations 20 (2012) 191-214

DYNAMIC HEDGING BY A LARGE PLAYER: FROM THEORY TO PRACTICAL IMPLEMENTATION

AYMERIC KALIFE, XIAOLU TAN, AND LIHANG WANG

AXA, Group Risk Management
aymeric.kalife@axa.com
Ecole Polytechnique
xiaolu.tan@gmail.com
University Paris-Dauphine
lyhangone@hotmail.com

ABSTRACT. Market liquidity risk refers to the degree to which large size transactions can be carried out in a timely fashion with a minimal impact on prices. Emphasized by the G10 report in 1993 and the BIS report in 1997, it is viewed as one factor of destabilization in the financial markets, as illustrated recently by the Asian crisis, the faillure of the hedge fund LTCM during the Russian crisis. So in order to assess welfare implications of portfolio insurance strategies, it would be useful to estimate the dynamic hedging activity in securities markets through a specific parsimonious and realistic model.

In the paper, large traders hold sufficient liquid assets to meet liquidity needs of other traders, and so bear the risk of their imbalanced derivatives portfolio. Their dynamic hedging strategies entail non-linear positive feedback effects, and in turn buying and selling derivatives at prices shifted by an amount that depends on their net holding. And therefore, the replicating equation turns to be a fully nonlinear parabolic PDE, as proposed by Frey [10].

It turns out that such a nonlinear PDE equation may be numerically unstable when using traditional finite-difference methods. Therefore we need some specific adequate numerical implementation in order to solve this equation with significant accuracy and flexibility, while keeping stability. In this respect paper we devised and customized two different numerical methods: one is a refined finite difference method; the other involves the probabilistic scheme proposed by Fahim and al. [9]. In contrast, another method based on Lie algebra and developed by Bordag and al. only provides a generic, albeit analytical, formulation of solutions, and not the specific one consistent with our payoff. Still, that method offers a reference for our proposed methods in terms of numerical accuracy.

Using such a framework, a Large Player is then in a position to take into account those positive feedback effects in dynamic hedging. Lastly, we show how dynamic hedging may directly and endogenously give rise to empirically observed bid-offer spreads.

Received September 15, 2011

1. INTRODUCTION

Due to the finite market elasticity in the presence of large traders, the usual continuous delta-hedging strategy may change the equilibrium price dynamics, as was highlighted in the $G10$ report in 1993 and the B.I.S. report in 1997: "One set of market participants who rely on market liquidity are those firms engaged in dynamic trading strategies, such as dynamic hedging or portfolio insurance. Previous research has highlighted the possibility that such strategies could, at times, have adverse repercussions for market functioning". A very popular dynamic hedging program at that time was portfolio insurance, a dynamic hedging strategy that, as already mentioned, replicates a European put option. These strategies were used to insure security portfolios against a fall in value under a predetermined floor. In theory, they are supposed to protect a portfolio from large depreciations while providing unlimited potential for appreciation. The rapid growth of such programs at that time also has to be seen in the light of portfolio insurance firms aggressively marketing dynamic strategies as a risk management device. These firms specialized in selling dynamic strategies as an alleged substitute for real derivative securities. Lessons learned from the crash teach that synthetic derivatives, in contrast to their real counterparts, may fail to fulfill their promises in practice. One major reason is that typical hedge programs do not incorporate the fact that they themselves, if implemented on a large scale, can cause market conditions they are intended to protect against. This paradox has been widely and increasingly observed since the crach of 1987, and is one factor of destabilization in the financial markets, as illustrated recently by the Asian crisis, the failure of the hedge fund LTCM during the Russian crisis.

It is worth pointing out that the main difference between the stock market crash of 1987 and the crisis in 1998 is the manner in which dynamic hedging became market influencing. In 1987 it was the simultaneous use of portfolio insurance programs by many market participants. In contrast, the near collapse of LTCM in 1998 was primarily caused by the high leverage that LTCM built up. Besides this difference, the absolute amount of money being dynamically hedged is nonetheless comparable. The severity of the LTCM case is illustrated by the fact that in 1998 assets of LTCM alone totalled 125 billion USD compared to 100 billion USD subject to portfolio insurance schemes in 1987 (the share of portfolio insurance induced selling in the futures markets reached peaks of over 27% on October 19 and October 20, in the stock market that proportion had reached 15.6%). It is also noteworthy that LTCM's off-balance volume in derivatives contracts amounted to an incredible 1400 billion USD in notional value.

Market illiquidity is generated by the inability of traders to buy or sell at no cost any quantity according to the definition of Friend and Blume [12]. It refers to the degree at which transaction flows affect asset prices in a market, separately from any change in the economic fundamentals that determine asset values. Thus market liquidity risk is the price risk associated with the execution of large transactions, manifested in a sharp movement of prices against a trader involved in a large purchase or sale of a security. It may be referred to as the degree to which large size transactions can be carried out in a timely fashion with minimal impact on prices.

In financial markets it is common that asset prices are pushed in a direction by comparatively large trades. Liquidity or the lack of it causes a combination of transaction costs and a price slippage impact. Firstly, it has an impact on the transaction price: whereas it may be possible to trade small quantities of an asset at a price which is close to the published mid-price, the larger the trade size, the more levels of market depth will have to be tapped and the further away the average transaction price will deviate from the mid-price. Secondly, liquidity is directly related to the degree of market slippage: large trades of one agent may remove entire price layers and lead market makers to adjust their prices accordingly.

In order to analyze this market liquidity risk, we identify three groups of agents in the market: value investors (or fundamentalists) who hold an asset when they think it is undervalued and short it when it is overvalued; trend followers (or chartists) who hold an asset when the price has been going up and sell it when it has been going down; but in the typical case buy and sell orders do not match. A third category consists of large traders who hold sufficient liquid assets to meet their joint liquidity needs and provide immediacy by holding stocks and options in inventory to cover imbalances in the buy and sell orders.

Consequently large traders, such as financial intermediaries (banks, pension and mutual funds, insurance companies), behave *de facto* as market makers. As observed by Kambhu [15], large dealers on OTC interest rates option market sell 50% more options than they buy. This third group of traders lower the price when buying and raise it when selling: orders are filled at a price that is shifted from the previous price, by an amount that depends on the net order of traders, to avoid mishedging of their portfolios of derivatives. Since markets only have limited liquidity, this will affect the value of the option position through "feedback effects", exaggerate market moves and cause prices to rise above the levels supportable by fundamentals. Large traders may thus move the value of the underlying in an undesired direction because the trade-slippage feeds back into their mark-to-market contract values: buying drives the price up, and selling drives it down.

Therefore we wish to build a model dealing with these specific dynamic hedging problems in a parsimonious and realistic way, giving rise to consistent bid and offer market prices. To account for the potential market impact of large transactions on hedging costs, we would also like to devise specific numerical hedging strategies complying with usual large traders behaviors.

The financial market is here characterized by the interaction of one "large trader" whose action affects prices and many price taking "small traders". The usual fundamental characterization of absence of arbitrage as developed in Delbaen and Schachermayer [7] cannot apply because of the impact on the underlying asset price dynamics. Jarrow [14] finds further restrictive conditions required to exclude "Market Manipulation Strategies", i.e. arbitrage opportunities taking the price impact into account. Jarrow [14] considers a symmetric information structure between players: instead we emphasize here the fact that this equilibrium price only holds with respect to the large trader's situation, given the market impact function and the structure of her option portfolio. Out of this no arbitrage argument, we introduce the positive feedback effects of a continuous dynamic delta hedging strategy by a large trader on the unit option price, giving rise to a nonlinear Black&Scholes PDE.

We describe the financial mechanism behind: whereas among small traders end-user needs in derivatives are roughly balanced across buyers and sellers, large dealers are empirically net writers (sellers) of options, so their exposure to gamma risk is significant, and they have to replicate synthetically the payoff of options by a dynamic hedging self-financing strategy in underlying asset, which bears upper pressures on it, inducing inventory holding costs. And the greater the net demand from the small traders, the higher the market clearing option selling price. The cumulative cost of these "buy high, sell low" hedge adjustments, which equals the value of the option, turns out to be respectively higher than the price-taker B&S price.

Based on the same principle in the context of a positive net options portfolio, we are able to give rise to a lower buying price. As a result we exhibit a parsimonious endogenous bid ask, through an optimal response from the large trader's dynamic hedging to cover her inventory holding costs and risks, which makes the comparison between the large trader and a market maker relevant. The bid-offer spread is an increasing function of two variables: the size of the large trader's net long or short position in options (the inventory), and the underlying asset price volatility depending on the residual net gamma of her portfolio.We insist here on the endogenous way of obtaining the bid offer spread, in contrast to the usual exogeneous transaction costs methods specifications: such costs here stem directly and endogenously from inventory holding costs. To test the accuracy of our approach, we first check that such an endogenous bid ask is consistent with every empirical statistical feature so far observed and we secondly calibrate it to the empirical market bid ask of a specific option.

For the resolution of nonlinear PDEs, we tried several method, including Lie algebra, finite difference numerical method, probabilistic numerical scheme etc. The

Lie algebra method provides closed-formula solutions, however, these solutions may not be compatible with the payoff of our options. By a benchmark from Lie algebra, we showed that the two numerical methods has enough stability and accuracy for such equations. In case of lower dimension, finite difference method is more efficient than the probabilistic scheme.

Thus we have developed an approach generating illiquidity as an endogenous trading cost from the inability of traders to share a common risk perfectly.

The rest of this paper is as follows. In section 2 we introduce the framework of our analysis and the corresponding pricing and hedging of derivatives, taking into account hedging feedback effects. In case of the nonlinear feedback, the pricing equation turns to be nonlinear parabolic PDE. In section 3, we talk about the numerical resolution of nonlinear PDEs. In Section 4 we introduce an endogenous bid-offer spread stemming from those inventory holding costs, consistent with market data and implied volatility smiles. Section 5 contains concluding comments.

2. THE MARKET STRUCTURE SETTING

2.1. **Framework and assumptions required for pricing and hedging in the presence of a large trader.** The financial market is characterized by the interaction of one "large trader" whose action affects prices and many price taking "small traders". The framework of our analysis is based on a continuous-time version of the models proposed by Jarrow [14], further developed by Frey [10], but no previous model involves all the required assumptions to give rise to sensible strategies by the large trader. Moreover we emphasize that our framework holds only with respect to the large trader's information who is the only one to know the number of hedged options, which highlights the information asymmetry between the large trader and the small traders on the market. For simplicity of the argument, we assume the risk-free interest rate is zero.

A1: There are no transaction costs and no short sale restrictions.

A2: Equilibrium price and the reaction function

The observed underlying asset price on the market $\tilde{S}_t$ can be expressed as a function ψ called "reaction function" $\tilde{S}\left(t, S_t, \alpha_t, \gamma_t\right) = \psi\left(t, S_t, \alpha_t, \gamma_t\right)$ where S_t is the underlying price in the absence of the large trader. α is the large trader's position (cadl àg, whose jumps are bounded below, and such that $\alpha_t^+ = \lim_{s \to t} \alpha_s$ is a continuous semimartingale) in the underlying and γ the number of options held by the large trader. ψ is assumed to be smooth enough on $[0, T] \times R^+ \times I_0 \times I_1$ where I_0 and I_1 are open intervals.

A3: The large trader has a procyclical market power: $\frac{\partial}{\partial \alpha}\psi > 0$

A4: The underlying asset price process is independent of the large trader's past holdings. If $\alpha^t = (\alpha_t, \alpha_{t-1}, ..., \alpha_0)$ represents the history vector for α, $\tilde{S}(t, S_t, \alpha^t, \gamma_t) = \psi(t, S_t, \alpha_t, \gamma_t)$

A5: Underlying asset price dynamics $d\tilde{S}_t = \sigma_t \tilde{S}_t dW_t + \rho_t \lambda\left(\tilde{S}_t\right) \tilde{S}_t d\alpha_t$. $\lambda_t(S)$ is a continuous function called 'market liquidity profile', used to retrieve a particular shape of the implied volatility smile. ρ_t represents the intensity of the liquidity impact (a possible choice is the ratio of change in the price of the underlying to notional traded, which is observable given an order book). So $\frac{1}{\rho \lambda_t\left(\tilde{S}_t\right)\tilde{S}_t}$ represents the market depth at time t, the order flow required to move prices by one unit.

In our context, the fundamental characterization of absence of arbitrage as in Delbaen and Schachermayer [7] cannot be applied because of the direct feedback effect in the underlying asset price dynamics. An alternative "no arbitrage condition", called "*No Market Manipulation Strategy*" (MMS) by Jarrow [14] is introduced and based on real wealth (the value as if the holdings were liquidated).

A6: There exists an equivalent martingale measure for the underlying asset in the absence of the large trader

A7: The market operates in the absence of corners, that is the combined effective holding of the underlying and the derivative must not exceed the net supply of the underlying asset.

A8: Synchronous Markets Condition $\psi(t, \alpha_t + \gamma_t \cdot \xi_t, 0) = \psi(t, \alpha_t, \gamma_t)$ where ξ_t is any admissible self-financing replicating strategy ('delta') in underlying assets. This amounts to saying that prices adjust instantaneously across underlying and derivative markets (i.e. price changes in one should be immediately reflected in the other). The large trader may use information mismatches between them to post riskless profits.

Finally we introduce the notion of Market Manipulation Strategies with respect to the information of the large trader (the structure of her portfolio), based on real wealth:

$$
\begin{aligned}
V_0 &= 0 \\
V_T &\geq 0 \ a.s. \\
P\left[V_T > V_0\right] &> 0
\end{aligned}
$$

Proposition 2.1. *Under conditions* A1–A8*, and given the number of options γ_t held by the large trader at time t, and assuming $c \leq \alpha_t + \xi_t \gamma_t$ where $c \in [-\infty, 1]$ there is no MMS in real wealth.*

Proof. The proof is done in the discrete case then extended to the continuous framework through convergence results. We use the self financing property then the definition of real wealth in order to show that the No MMS condition is enforced. More details can be found in Propositions 3.10 and 3.23 of [1]. □

2.2. Pricing and dynamic hedging with respect to the large trader's information.

2.2.1. *The notion of positive feedback effects from the large trader's hedging portfolio strategy on the option price.* In order to maintain an option portfolio's exposure to price risk, the large trader must adjust the hedge position after a price shock to allow for the change in the option's price sensitivity. More precisely, a call option's value increases by an amount smaller than the increase in the value of the underlying asset because there is always some probability that the price of the underlying asset will reverse direction by the time the contract matures, and even fall below the strike price at expiration, rendering the option worthless. But as the underlying asset's price rises further, however, this probability of a worthless outcome becomes smaller, and the option's value becomes more sensitive to changes in the underlying asset's price. To compensate for this increase in the price sensitivity of a call option, a hedge position in the underlying asset must be made larger as well, affecting in return its price process. This mechanism generates the potential for positive feedback in price dynamics because the hedge adjustment is to buy (sell) the underlying asset after its price rises (falls), as the transactions could introduce further upward (downward) pressure on prices after an initial upward (downward) shock to asset prices.

2.2.2. *The underlying asset dynamics in the presence of the large trader.* The large trader's portfolio strategy, denoted ϕ, is assumed to be $C^{1,2}\left([0,T]\times R^{+}\right)$ and $\rho\tilde{S}\phi_S\big(t,\tilde{S}\big)<1\ \forall\big(t,\tilde{S}\big)\in[0,T]\times R^{+}$.

Proposition 2.2. *The portfolio strategy in the underlying assets is denoted*

$$
\begin{aligned}
\alpha_t &= \phi\left(t,\tilde{S}_t\right) \\
d\tilde{S}_t &= v\left(t,\tilde{S}_t\right)\tilde{S}_t dW_t + b\left(t,\tilde{S}_t\right)S_t dt
\end{aligned}
\tag{2.1}
$$

with "adjusted" trend and "feedback" volatility coefficients:

$$
\begin{aligned}
v\left(t,\tilde{S}_t\right) &= \frac{\sigma_t}{1-\rho\lambda\left(\tilde{S}_t\right)\tilde{S}_t\phi_S\left(t,S\right)} \\
b\left(t,\tilde{S}_t\right) &= \frac{\rho\lambda\left(\tilde{S}_t\right)}{1-\lambda\rho\tilde{S}_t\phi_{\tilde{S}}\left(t,\tilde{S}\right)}\times\left(\phi_t\left(t,\tilde{S}\right)+\frac{\sigma_t^2\tilde{S}_t^2\phi_{SS}\left(t,\tilde{S}\right)}{2\left(1-\rho\lambda\left(\tilde{S}_t\right)\tilde{S}_t\phi_S\left(t,\tilde{S}\right)\right)^2}\right)
\end{aligned}
\tag{2.2}
$$

Proof. According to Frey [10], Ito's formula and Assumption A5 in [10] imply that the stockholdings α are a semimartingale. Again by Ito's formula we have

$$
d\alpha_t=\phi_S(t,S_t)dS_t+(\phi_t(s,S_s)+\frac{1}{2}\phi_{SS}(t,S_t)v^2(t,S_t)(S_s)^2)ds \tag{2.3}
$$

Assumption A4 of [10] together with (2.3) now yields the following dynamics for the equilibrium stock price process S

$$dS_t = \sigma S_t dW_t + \rho S_t \phi_S(t, S_t) dS_t + \rho S_t(\phi_t(t, S_t)dt + \frac{1}{2}\phi_{SS}(t, S_t)d\langle S\rangle_t) \tag{2.4}$$

or equivalently

$$(1 - \rho S_t \phi_S(t, S_t))dS_t = \sigma S_t dW_t + \rho S_t(\phi_t(t, S_t)dt + \frac{1}{2}\phi_{SS}(t, S_t)d\langle S\rangle_t) \tag{2.5}$$

Under Assumption A5 in [10] the expression $(1 - \rho S_t \phi_S(t, S_t))$ is strictly positive. Integrating $1/(1 - \rho S_t \phi_S(t, S_t))$ over both sides of (2.5) therefore yields the following explicit form for the equilibrium stock price dynamics

$$dS_t = \frac{\sigma}{1 - \rho S_t \phi_S(t, S_t)} S_t dW_t + \frac{\rho S_t}{1 - \rho S_t \phi_S(t, S_t)}(\phi_t(t, S_t) + \frac{\sigma^2 S_t^2}{(1 - \rho S_t \phi_S(t, S_t))^2})dt \tag{2.6}$$

which proves the Proposition. □

Once again we emphasize that the large trader is the only one to be aware of the specific feedback effect intensity on the option price.

2.2.3. *Perfect replication of a European call option by the large trader.* We consider a usual continuous delta-hedging strategy implemented by the large trader. For simplicity of notation, we assume the risk-free interest rate is zero.

Proposition 2.3. *Under a zero risk-free interest rate (for simplicity of notation), the replicating cost of a γ- option portfolio is the solution of this nonlinear PDE*

$$\begin{cases} u_t(t, S, \gamma) + \frac{1}{2}\frac{1}{(1+\rho\lambda(S) S u_{SS}(t,S,\gamma))^2}\sigma_t^2 S_t^2 u_{SS}(t, S, \gamma) = 0 \\ u(T, S, \gamma) = nh(S) \end{cases} \tag{2.7}$$

where $\alpha\left(t, \tilde{S}, \gamma\right) = \phi\left(t, \tilde{S}, \gamma\right) = -u_S\left(t, \tilde{S}, \gamma\right)$ is the replication of the option portfolio continuous delta hedging strategy and $V_t = u\left(t, \tilde{S}, \gamma\right)$ is the replicating value of a γEuropean call portfolio held by the large trader whose final payoff is $h(S_T)$ at time T per option unit.

Proof. The existence and unicity of the solution of the above nonlinear PDE with its terminal value was proved by Frey [11] through a quasilinear PDE,

$$\begin{aligned} 0 &= \frac{\partial}{\partial t}\phi(t, f) + \frac{1}{2}\eta^2 f^2 (1 + 2\rho\frac{\psi_\alpha}{\psi_f}\frac{\partial\phi}{\partial f})\frac{\partial^2\phi}{\partial f^2} \\ &+ \frac{\eta^2}{\psi_f}\frac{\partial\phi}{\partial f}[f\psi_f - \psi_t + \frac{f^2}{2}\psi_{ff} + \rho\frac{\partial\phi}{\partial f}(f^2\psi_{\alpha f} + f\psi_\alpha) + (\rho\frac{\partial\phi}{\partial f})^2\frac{f^2}{2}\psi_{\alpha\alpha}] \end{aligned} \tag{2.8}$$

with the terminal condition

$$\phi(T, f) = h'(X^\phi(T, f))\forall f > 0 \tag{2.9}$$

According to [11], since the product $\phi(t,f)\frac{\partial X^{\phi}(t,f)}{\partial f}$ converges locally uniformly to $\phi(T,f)\frac{\partial X^{\phi}(T,f)}{\partial f}$ as $t \to T$ we get

$$\phi(T,f)\frac{\partial X^{\phi}(T,f)}{\partial f} = \lim_{t\to T}\phi(t,f)\frac{\partial X^{\phi}(t,f)}{\partial f} = \lim_{t\to T}\frac{\partial H(t,f)}{\partial f} = \frac{\partial H(T,f)}{\partial f} \tag{2.10}$$

On the other hand we have $\frac{\partial}{\partial f}h(X^{\phi}(T,f)) = h'(X^{\phi}(T,f))\frac{\partial}{\partial f}X^{\phi}(T,f)$ and that the terminal condition yields the desired equality of the derivatives. □

Solving such a non linear PDE requires distinguishing long and short positions the sign of the payoff depending on whether the large trader buys or sells options: for a short call we have

$$u\left(T,S,\gamma\right) = -\gamma\left(S-K\right)^{+}$$

whereas for a long call

$$u\left(T,S,\gamma\right) = \gamma\left(S-K\right)^{+}$$

so long and short positions have different values (see later).

Proposition 2.4. *Given the number of options γ_t held by the large trader at time t, and under the assumptions* A1–A9*, we consider an admissible self-financing hedging portfolio strategy $\xi = (\alpha,\beta,\gamma)$ which replicates the derivative, the value of which is denoted by V^{ξ}. Then the MMS condition (with respect to the large trader's information) implies*

$$V_t^{\xi} = c_t \tag{2.11}$$

Proof. We consider a variation in hedging positions and work out the terminal wealth, using the Synchronous Markets Condition. Finally the MMS condition implies the result. □

The feedback effect of the large trader's dynamic hedging portfolio strategy on the unit call option price. Traded options exist only for well established markets, and only for relatively short maturities. For very long dated options, dynamic replication is the only avenue open to traders if they wish to hedge an implicit short put position. Generally speaking, an option can be hedged by taking an offsetting position in the underlying asset, and the required size of this position varies with the price of the underlying asset. This variability of the hedge position results from the varying sensitivity of the option value to the price of the underlying asset as its price changes. Simulations illustrates the fact that when the large trader sells European call option, the European call unit price rises, see Fig 1.

Therefore an apparent paradox arises empirically observed on the markets as regards large traders' transactions: selling a large amount of calls causes the price to rise. In fact when a large amount of options is engaged in such trading strategies,

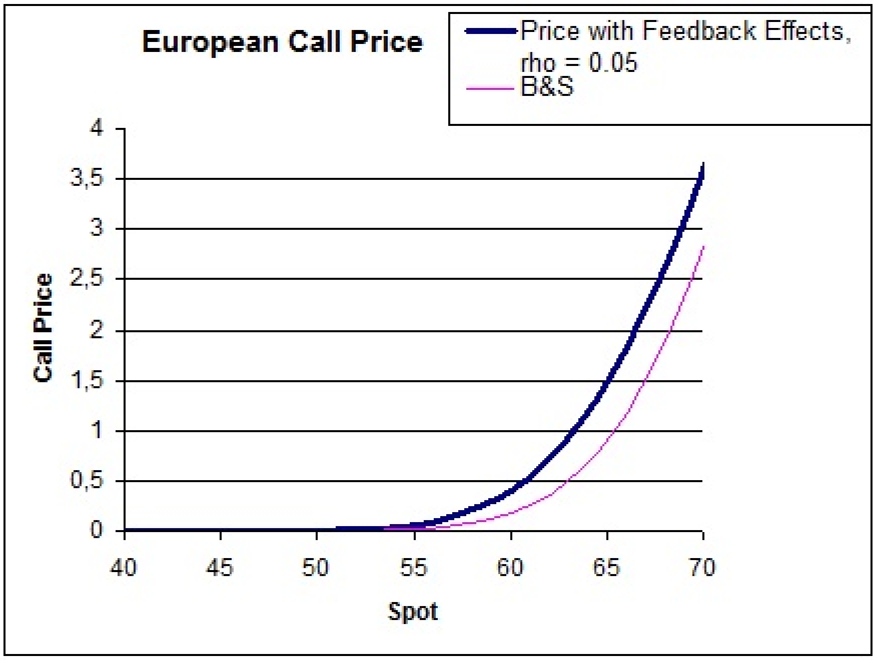

FIGURE 1. $\rho = 0.05$, $\lambda = 1$

the market dynamics may be affected by the trading strategy itself, and hence lead to potentially destabilising price paths.

In order to explain financially such a mechanism, let's analyze the following steps. Actually, as the large trader sells European calls, she has to buy a large amount of the underlying asset in order to hedge synthetically, which makes the underlying asset price rise, thus the short delta decreases, implying a short gamma, so the 'feedback' volatility $\left(\frac{\sigma_t^2}{(1+\rho\lambda(S)Su_{SS}(t,S,\gamma))^2}\right)$ rises. Consequently the option unit price turns out to be higher than the usual price-taker B&S price.

Therefore illiquidity appears as an endogenous trading cost compensating for the sharing of risks measured here by the spot market volatility. Actually, buying with rising prices, the large trader's demand is procyclic. Therefore, the apparent paradox is just a consequence of the positive feedback effect induced by the dynamic hedging of the large trader through its Portfolio Insurance Strategy, designed to protect the capital during a market downturn by replicating option positions. In fact, this *positive feedback effect* stems from the absence of sufficient natural counterparts to meet the demand for puts and calls, where large dealers can meet the demand by selling puts and calls. In doing so, they become short the option; so they can neutralize their net risk exposure by synthetically replicating long option positions, which requires selling as the market falls and buying as it rises, to ensure the hedge position is sufficient to cover the option rising exposure, which introduces transactions large enough to amplify the initial price shock. It generates precisely the kind of vicious feedback loop that destabilizes markets. Best estimates then suggested that around

$100 billion in fundswere following formal portfolio insurance programs, representing around 3% of the pre-cash market value. However, this is almost certainly an underestimate of total selling pressure arising from informal hedging techniques such as stop-loss orders.

Some observers cite the stock market crash of 1987 - which occurred in the absence of any significant change in economic fundamentals - as an example of positive feedback dynamics: the sharp fall was intensified by portfolio insurance strategies that prescribe the sale (purchase) of stocks when prices fall (rise). These mutually reinforcing interactions are characteristic of markets where traders have short decision horizons, or where they operate under external constraints on their decision (due to internally imposed trading limits or under risk management system which circumscribes their actions), which may require positions to be sold for cash when net asset values are low or when a margin call dictates liquidation of trading positions. The Brady Commission's report (1988) attributed the magnitude and the swiftness of the price decline in the 1987 stock market crash to practices such as portfolio insurance and dynamic hedging techniques. The sales dictated by dynamic hedging model amounted to $12 billion, but the actual sales had been around $4 billion. More recently, the dynamic hedging associated with OTC puts has been blamed for several bouts of market instability, notably in 1989, 1991,1997, and 1998. Furthermore, program trading, most commonly used by large traders, currently constitutes about 10 percent of a typical day's volume on the NYSE.

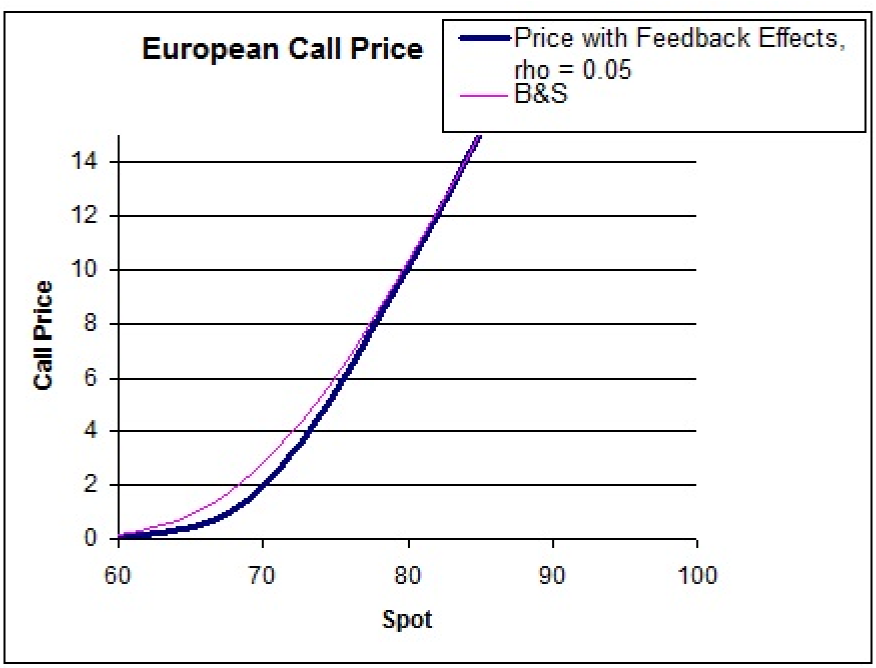

FIGURE 2. $\rho = 0.05$, $\lambda = 1$

Symmetrically, as the large trader buys European calls, she has to sell a large quantity of the underlying asset in order to hedge, which makes the underlying price and its volatility fall, as shown by the PDE (see above), so the option unit price (because the vega of a call is positive) becomes smaller than the usual price-taker B&S price.

The unit option price depends on the total amount of options replicated by the large trader. For the simplicity of simulations, we take $\rho\lambda(S) = \frac{\phi}{S}$where ϕ, assumed to be constant, represents the volume effect on the unit underlying price. We illustrate here the non homogeneity of prices: the larger the absolute payoff, the wider the relative bid-offer spreads. We show through simulations that the feedback effect of the large trader's activity causes the hedge cost to be non-linear in the number of replicated options: the price of the large trader's replicating portfolio, and in turn the equilibrium option price is not proportional to the number of options held by the large trader. In fact, the option unit price increases with the number of options held by the large trader (see an analogous result in Carr, Geman, Madan [4]). More precisely, as illustrated below, the average replicating price is nearly a quasi-linear function of the number of options held, wich implies that the replicating unit price is roughly a quadratic function of the number of options held by the large trader.

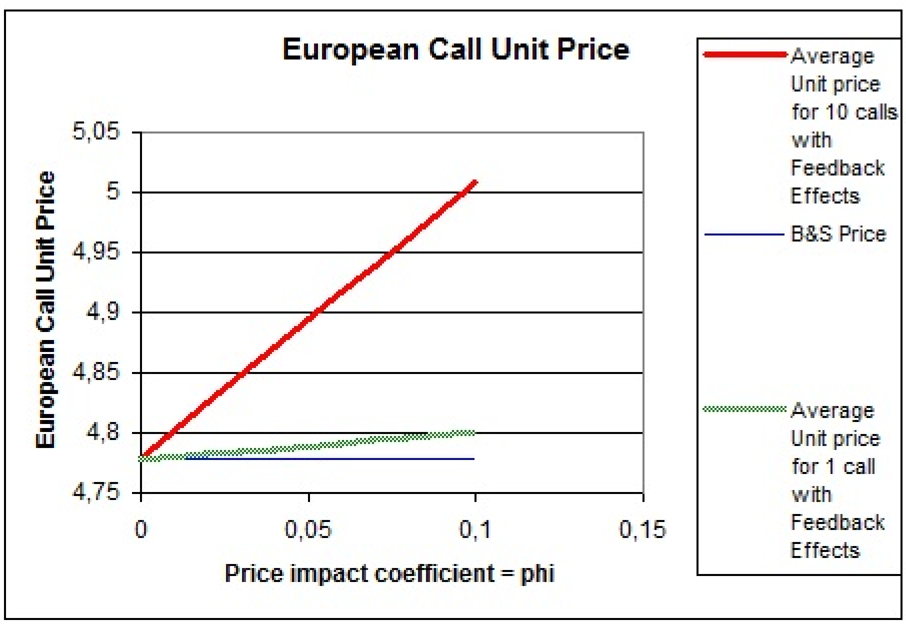

FIGURE 3. Unit 10-call versus 1-call price

As a result, the number of options held by the large trader has an impact on the unit price of the call. But, except for the large trader, no other market participant is aware of this information. Therefore she can use this information asymmetry in order to manipulate prices.

3. RESOLUTION OF NONLINEAR PARABOLIC EQUATION

Frey's model gives a nonlinear parabolic equation (2.7). For the resolution of these nonlinear equations, we tried different methods: Lie algebra, finite difference and the probabilistic scheme of Fahim et al [9].

The method of Lie uses symmetry reduction technique to translate a PDE into some ODEs. However, the solution may not fit our terminal and boundary conditions. This method is not directly useful for our problems. However it gives a reference for other numerical methods.

As for the resolution of linear parabolic equations, the finite difference method discretizes the time and space and approximates the derivatives. Sevcovic [19] proposes an iterative algorithm for the nonlinear equation, which is originally for the evaluation of the American style options in nonlinear model. We use this transformation method to our problems.

We also test the probabilistic scheme proposed in Fahim and al. [9], which has a strong lien with second order BSDE. The methods needs Monte-Carlo simulations to estimate the conditional expectations in scheme.

3.1. **Lie algebra.** The general theory of the Lie algebra's applications to the differential equations can be found in Olver [18]. With this method, Bordag [2] gives invariant solutions to the following Frey equation:

$$\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\sigma^2}{(1-\rho S u_{SS})^2} S^2 \frac{\partial^2 u}{\partial S^2} = 0 \tag{3.1}$$

Theorem 3.1. *The invariant solution of* (2.7) *is given by*

$$\begin{cases} u(t,S) = dS \\ u(t,S) = \frac{3}{\rho} S\left(\log S - \sigma^2 \frac{t}{8}\right) \\ u(t,S) = -\frac{1}{\rho} S\left(\log S - \sigma^2 \frac{t}{8}\right) \end{cases} \tag{3.2}$$

where d is an arbitrary constant.

Proof. This is a direct result from Bordag [2] Theorem 4.3. □

These solutions cannot be used directly to our hedging problems, because each option payoff has a specific terminal condition. However, this result is still useful for us, since it provides the exact solutions which could be benchmark for other numerical methods.

3.2. **Finite difference.** Finite difference method for nonlinear parabolic equation usually uses an explicit approximation for the nonlinear parts, then uses the explicit, implicit scheme or θ-scheme for the linear parts.

Sevcovic [19] proposes a transformation method, which is originally for pricing the American style options in nonlinear model. Here, we implement his method for the European style options.

Given a nonlinear pricing equation

$$\frac{\partial u}{\partial t} + (r-q)S\frac{\partial u}{\partial S} + \frac{1}{2}\sigma^2(t,S,u_S,u_{SS})S^2\frac{\partial^2 u}{\partial S^2} - ru = 0, \tag{3.3}$$

with transformations

$$S = S_0 e^{-x}, \quad \tau = T - t \quad \text{and} \quad \Pi(\tau,x) = u(t,S) - S\frac{\partial u}{\partial S}(t,S),$$

we get

$$\frac{\partial \Pi}{\partial x} = S^2\frac{\partial^2 u}{\partial S^2},$$
$$\frac{\partial^2 \Pi}{\partial x^2} = -2S^2\frac{\partial^2 u}{\partial S^2} - S^3\frac{\partial^3 u}{\partial S^3}$$

and

$$\frac{\partial \Pi}{\partial \tau} = -\frac{\partial u}{\partial t} + S\frac{\partial^2 u}{\partial S \partial t}.$$

Plugging them into (3.3), it gives

$$\begin{aligned}
&\frac{\partial \Pi}{\partial \tau} + (r - q - \frac{\sigma^2}{2})\frac{\partial \Pi}{\partial x} - \frac{1}{2}\frac{\partial}{\partial x}\left(\sigma^2\frac{\partial \Pi}{\partial x}\right) + r\Pi \\
&= \left(S\frac{\partial}{\partial S} - 1\right)\left(\frac{\partial u}{\partial t} + (r-q)S\frac{\partial u}{\partial S} + \frac{1}{2}\sigma^2(t,S,u_S,u_{SS})S^2\frac{\partial^2 u}{\partial S^2} - ru\right) \\
&= 0
\end{aligned}$$

On the other side, given the function Π, there is expression of u in terms of Π. Since

$$\Pi(\tau,x) = u(t,S) - S\frac{\partial u}{\partial S}(t,S)$$

we have

$$\frac{\partial}{\partial S}\left(-\frac{u(t,S)}{S}\right) = \frac{\Pi(\tau,x)}{S^2}$$

then

$$\frac{u(t,S)}{S} - \frac{u(t,SMax)}{SMax} = \int_S^{SMax}\frac{\Pi(\tau,\tilde{x})}{\tilde{S}^2}d\tilde{S} = \int_{XMin}^{x}\frac{\Pi(\tau,\tilde{x})}{S_0}e^{\tilde{x}}d\tilde{x}$$

Finally, we propose a finite difference numerical scheme on Π:

1⟩ For the convective part, we use the exact scheme:

$$\Pi(t_{n+\frac{1}{2}}, x_i) = \Pi\left(t_n, x_i - (r-q)\Delta t\right)$$

2⟩ For the diffusion part, we use an implicit scheme:

$$\begin{aligned}
\frac{\Pi_i^{n+1} - \Pi_i^{n+\frac{1}{2}}}{\Delta t} \quad &- \quad \frac{\tilde{\sigma}_i^2}{2}\frac{\Pi_{i+1}^{n+1} - \Pi_{i-1}^{n+1}}{2\Delta x} + r\Pi_i^{n+1} \\
&- \quad \frac{1}{2}\left(\tilde{\sigma}_i^2\frac{\Pi_{i+1}^{n+1} - \Pi_i^{n+1}}{\Delta x^2} - \tilde{\sigma}_{i-1}^2\frac{\Pi_i^{n+1} - \Pi_{i-1}^{n+1}}{\Delta x^2}\right) = 0
\end{aligned}$$

Finally, to test the stability and convergence of the numerical scheme, we compare it to the exact solution from (3.2). As shown in Figure 4, it converges very well to theoretical solutions.

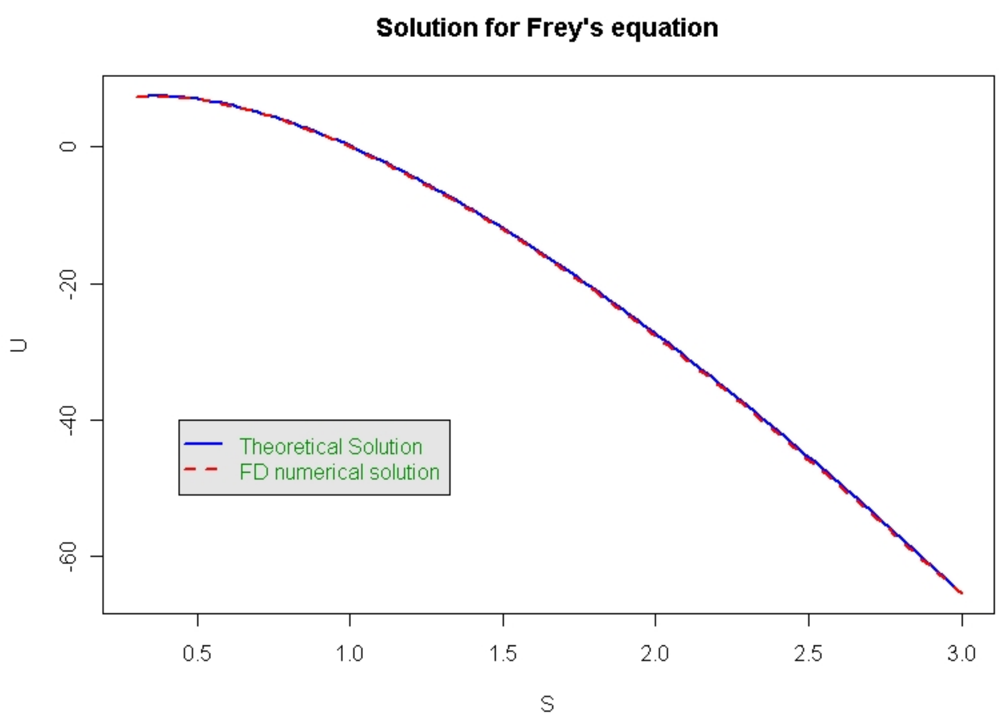

FIGURE 4. Comparaison of the theoretical solution and numerical solution for Frey's equation.

3.3. **BSDE and non-linear parabolic equations.** With its relation to nonlinear parabolic PDEs, the BSDE (Forward Backward Stochastic Differential Equation) can be viewed as an extension of Feynman-Kac formula. As a result, we can resolve a nonlinear parabolic equation by the resolution of BSDE.

Let $(\Omega, \mathcal{F}, (\mathcal{F})_{0\leq t\leq T}, \mathbb{P})$ be a probability space generated by a d-dimensional standard Brownian motion W. A FBSDE (Forward-Backward Stochastic Differential Equation) is the equations:

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t \qquad \text{with } X_0 = x \tag{3.4}$$

$$-dY_t = f(t, X_t, Y_t, Z_t)dt - Z_t dW_t \quad \text{with } \ Y_T = \Phi(X_T) \tag{3.5}$$

Given Lipschitz conditions on the coefficients, we have the existence and uniqueness of the solution (X,Y,Z) in a Hilbert space $\mathbb{H}^2_T(\mathbb{R}^d) \times \mathbb{H}^2_T(\mathbb{R}^d) \times \mathbb{H}^2_T(\mathbb{R}^{n\times d})$. (For details, see [8] or [17] for example.)

Suppose that the semilinear parabolic differential equation

$$\partial_t u(t,s) + b(t,x)D_x u + \frac{1}{2}\sigma\sigma^T \cdot D^2_{xx}u(t,s) + f(t,s,u,\sigma(t,s) * \partial_s u) = 0 \tag{3.6}$$

with terminal condition $u(T,s) = \Phi(s)$ have a smooth solution in $\mathcal{C}^{1,2}([0,T]\times\mathbb{R})$, and

$$|u(t,s)| + |\sigma(t,s) * \partial_s u(t,s)| \leq C(1+|s|),$$

then $(Y_s, Z_s) = (u(s,X_s), \partial_x u(s,X_s)\sigma(s,X_s))$ solves the FBSDE (3.4, 3.5).

For fully nonlinear parabolic equations, Cheridito et al. [6] and later Soner et al. [20] developed a second-order BSDE to establish a relation with it.

Analytical solutions of nonlinear PDEs or BSDEs are not usually available, we need numerical resolutions. With discretization on time inteval and the approximations of derivatives by conditional expectations, we get a numerical scheme for BSDE. Concretely, we discretize firstly the interval $[0,T]$ into N parts, then with terminal condition $Y^N_{t_N} = \Phi(X^N_{t_N})$, we solve the dynamic programming equation:

$$Y^N_{t_k} = E_{t_k}(Y^N_{t_{k+1}}) + \Delta t \cdot E_{t_k} f(t_k, X^N_{t_k}, Y^N_{t_{k+1}}, Z^N_{t_k})$$

$$\Delta t \cdot Z^N_{t_k} = E_{t_k}(Y^N_{t_{k+1}}\Delta W_k)$$

where $E_{t_k}(\cdot) = E(\cdot|\mathcal{F}_{t_k})$ denotes the conditional expectation.

Fahim et al. [9] proposed a similar numerical scheme for fully nonlinear parabolic equation. Consider the nonlinear equation:

$$-\mathcal{L}v - F(\cdot, v, Dv, D^2v) = 0, \tag{3.7}$$

where

$$\mathcal{L}v := \partial_t v + \mu \cdot Dv + \frac{1}{2}\sigma\sigma^T \cdot D^2 v \quad \text{is the linear part,}$$

and F is the nonlinear part. For $\hat{X}^{t,x}_h = x + \mu(t,x)\Delta t + \sigma(t,x)\Delta W$, they propose scheme

$$v^h(t,x) = E(v^h(t+h, \hat{X}^{t,x}_h)) + F(t,x,\mathcal{D}_h v),$$

where

$$\mathcal{D}_h v = E[v(\hat{X}^{t,x}_h)H^h_i],\ i = 0,1,2$$

with

$$H_0 = 1,\ H_1 = (\sigma^T)^{-1}\frac{W_h}{h},\ H_2 = (\sigma^T)^{-1}\frac{W_h W_h^T - h1_d}{h^2}\sigma^{-1}.$$

For the calculus of this conditional expectation, Bouchard and Touzi [3] proposed a Malliavin calculus techniques, while Gobet et al. [13] proposed a regression technique. The idea of regression technique is to mimic the formula

$$E(G|\mathcal{F}_t) = \arg\min_{k\in\mathcal{F}_t} E\left((G-k)^2\right)$$

with simulations.

4. DYNAMIC HEDGING STRATEGIES BY A LARGE PLAYER

4.1. The inventory holding cost component of the option bid-offer spread. Because the large player will buy low and sell high through the feedback effect from her dynamic hedging strategy, a bid-offer spread for derivatives endogenously naturally arises, serving as a revenue source as well as a risk insurance buffer. Furthermore large imbalances through higher hedging ratios lead to price movements and to a higher volatility, which increases the ask-price and decreases the bid-price, so widen the bid-offer spread.

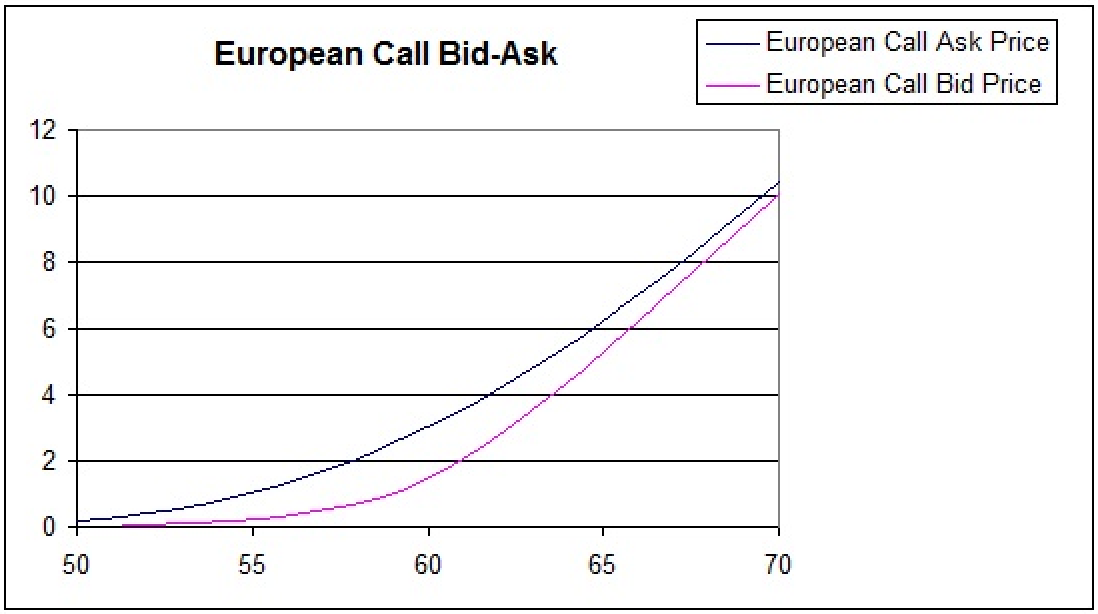

Figure 5. Unit 10-call versus 1-call price

Therefore the large player accepts orders to buy and sell an option at the bid and ask prices which are adjusted to cover her average costs. A number of theoretical models specify the cost component of the large player's bid/ask spread. Stoll [21] and [22] posit that large player's costs fall into three categories: order-processing costs, inventory-holding costs, and adverse selection costs. The first category costs are largely fixed, at least in the short run, and their contribution decreases with the trading volume, so we will neglect them in our setting. The second category, inventory holding costs, are the costs that a large player incurs while carrying positions acquired in supplying investors with immediacy of exchange (i.e. liquidity). If the net position in derivatives is non zero, market makers can hedge the value of their options inventory using underlying assets. In fact, in the absence of sufficient natural counterparts to meet the demand for puts and calls, large dealers meet the demand by selling puts and calls, which increases the size of their net inventory. They can neutralize their risk exposure by synthetically replicating option positions. These costs of hedging the inventory need to be covered by the revenues from market making, i.e. by the bid-offer spread, and they will depend both on the size of the inventory and on the price-change in the derivative (the higher the price-change risk, the higher the bid-offer spread). Limiting inventory cost is all the more a key issue as the large trader is short gamma, characterizing a procycling behavior (buy high, sell low), especially when the underlying asset price evolves inside a corridor.

Furthermore, as shown above, this replicating cost grows more than linearly with respect to the number of replicated options, the necessary compensation of bearing price risk hence under an increasing constraint on the large player as her inventory is less and less balanced. This explains the widening bid-offer spread as the number of replicated options increases. This is consistent with Vijh [23] who reports that large option trades cause a widening of the bid-offer spread in the option market.

In summary the bid-offer spread is an increasing function of two variables: the size of the large trader's net long or short position in options, and market price volatility. Kim, Ko and Noh [16] show that the holding of undesired risky positions account for a very large proportion of the bid-offer spread, so that transaction costs arise from the trader's inability to share risk under no constraints with the rest of the market.

The other essential component of the bid-offer spread is linked to the third category of large player costs, the adverse selection costs, which are presented later.

4.2. **A parsimonious endogenous bid-offer spread consistent with market data.** As observed before, the hedging strategy of a large trader naturally induces the empirically observed spread between bid and offer prices. As a result, we account for bid-offer spreads endogenously, stemming directly from the limited market depth and the hedging behavior of large players. In order to test the accuracy of our approach, we look at several key issues such as pricing accuracy, volatility smile and bid-offer spread statistical features.

First we show that such a model improves pricing. We use IBM call options traded on the CBOE during 1995, assuming that all option writers use delta hedging, to estimate the parameter $(\lambda\rho)$ of the large trader, whose stock holding is the aggregate hedging position of all option writers, i.e. the number of written options.. The risk free rate is assumed to be 3%, approximating the rate on the month treasury bill during 1995.

$$\mathbb{E}\left[\lambda\rho\right] = 0.30$$

Second, such a complete market model is likely to give rise to a volatility smile due to the random "feedback" volatility. Financially speaking such an anomaly has often been linked to the net buying pressure of calls and puts used for hedging purpose, in agreement with liquidity features: in a downward trending market liquidity is usually with far more large moves.Thus the large player's hedging activity may play quite a significant role. Choosing $\rho\lambda(S) = \frac{\phi}{S}$ gives rise to a skew

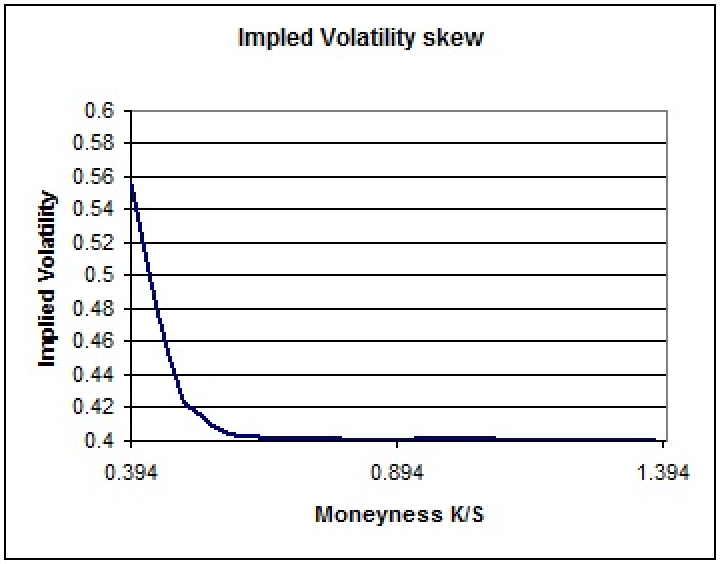

FIGURE 6. ss

In order to get various and flexible "smirk" profiles we specify the function $\lambda(S)$ by taking a decreasing function, with a minimum around the forward price S_0

$$\lambda(S) = 1 + (S - S_0)^2 \left(a_1 1_{\{S \le S_0\}} + a_2 1_{\{S \ge S_0\}}\right) \tag{4.1}$$

Based on S&P 500 index call options from July to December 1990, we estimate ρ, a_1, a_2 through calibration on options prices, and obtain

$$\begin{aligned} \rho &= 0.017 \\ a_1 &= 0.236 \\ a_2 &= 0.007 \end{aligned} \tag{4.2}$$

From this we get the implied volatility smile through the Raphson-New on algorithm

Whereas most models obtain the smile by exogenously altering the volatility structure of the underlying price process, such as stochastic volatility models, here no assumptions are needed on the underlying price volatility; the smile pattern rather occurs endogenously as a consequence of the market structure.

Third, our model is consistent with empirically observed statistical features of bid-offer spreads (see for instance [5]):

- the bid price is strictly higher than the ask price, and the spread between B&S and bid prices is higher than the spread between Black & Scholes and ask prices:

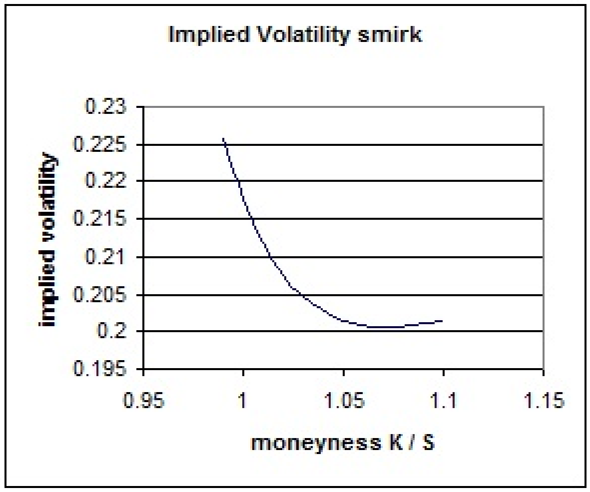

FIGURE 7. ss

the option value is closer to the bid than to the ask quote. This illustrates the fact that the market impacts of buying and selling are different.

- the higher the underlying asset price, the greater the investment in inventory, the carrying cost, and in turn the spread. Moreover the larger the liquidity parameter ρ, the lower the market liquidity of the underlying, and the wider the bid-offer spreads on the derivative.
- The option value is closer to the bid than ask quote and the degree of the asymmetry increases as the moneyness of the option decreases or increases:

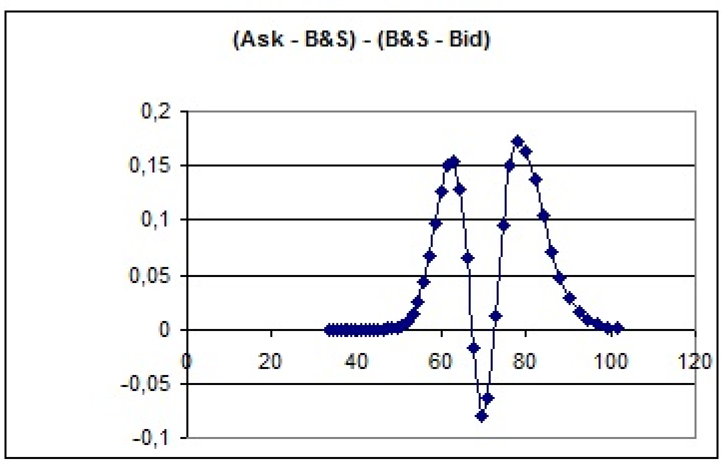

FIGURE 8. ss

- The ask quote changes more than the bid quote, the same for their standard deviations and the standard deviations of their changes
- The ratios of these two quotes and of their changes are also in accordance with empirical facts: the slope coefficients decline as the moneyness (S/K for calls) of the options declines

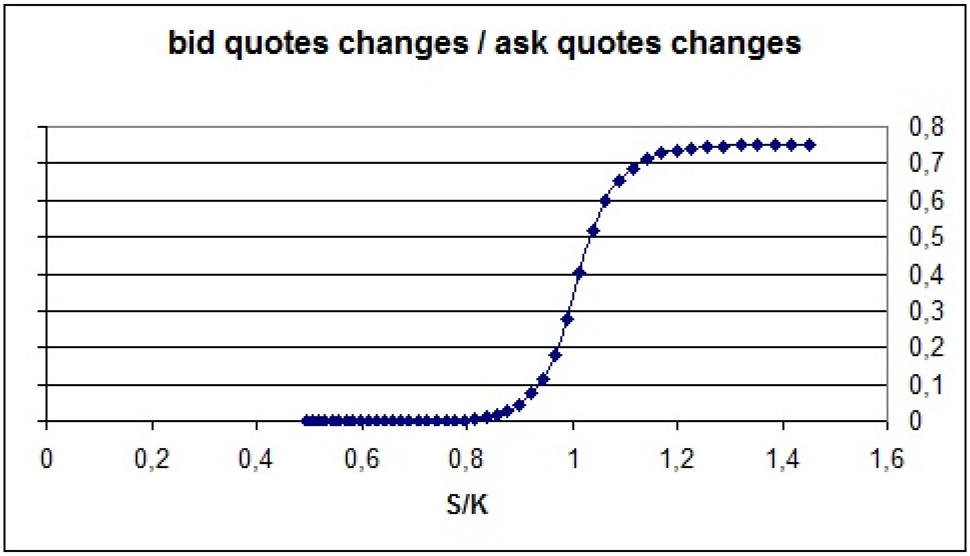

FIGURE 9. ss

- Both quotes standard deviations decrease with respect to the quantity. K/S in the case of a call whereas it increases for puts. So the more out-the-money an option, the less variable its value, thus implying a volatility smile.

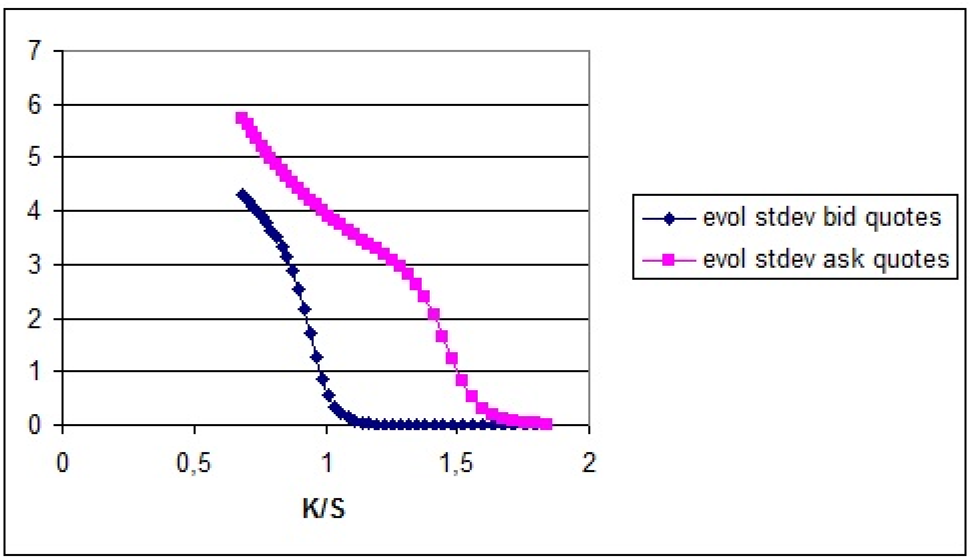

FIGURE 10. ss

5. CONCLUSION

In this paper illiquidity has been introduced in the optimization of the large trader as an inability to trade and share risk without changing the market price.

Actually transaction costs and market slippage aspects of market liquidity risks have been isolated through a parsimonious complete market model involving feedback hedging effects. A nonlinear feedback leads to a nonlinear pricing equation. We tested a finite difference scheme as well as a probabilistic scheme for the numerical resolution of such nonlinear equation.

Thus we integrated this feedback effect in dynamic hedging into a market framework and extended it to give rise to an endogenously and empirically consistent bid-offer spread. Specifically, this model accounts for essential features of illiquidity: 1) Illiquidity trading costs are directly associated with the volatility of the spot price through the feedback effects, something that had to be assumed previously. 2) Illiquidity is also associated with the net volume generated by hedging pressures, a result common to the microstructure literature.

REFERENCES

[1] J. Bierbaum. Uber die rckwirkung von han- delsstrategienausgewhlter investoren auf wertpapierpreisprozesse. *PhD Dissertation*, 1997.

[2] Bordag and R. Frey. Nonlinear option pricing models for illiquid markets: scaling properties and explicit solutions. *ArXiv e-prints*, August 2007.

[3] Bruno Bouchard and Nizar Touzi. Discrete-time approximation and Monte-Carlo simulation of backward stochastic differential equations. *Stochastic Process. Appl.*, 111(2):175–206, 2004.

[4] P. Carr, H. Geman, and D. Madan. Pricing and hedging in incomplete markets. *Journal of financial economics*, 2001.

[5] Chung P. Johnson H. Chan, K. The intreday behavior of bid-ask spreads for nyse stocks and cboe options. *Journal of Financial and Quantitative Analysis*, 30:329–346, 1995.

[6] Patrick Cheridito, H. Mete Soner, Nizar Touzi, and Nicolas Victoir. Second-order backward stochastic differential equations and fully nonlinear parabolic PDEs. *Comm. Pure Appl. Math.*, 60(7):1081–1110, 2007.

[7] F. Delbaen and W. Schachermayer. A general version of the fundamental theorem of asset pricing. *Mathematische Annalen*, 300:463–520, 1994.

[8] N. El Karoui, S. Peng, and M. C. Quenez. Backward stochastic differential equations in finance. *Math. Finance*, 7(1):1–71, 1997.

[9] N. Touzi. Fahim A and X. Warin. A probabilistic numerical scheme for fully nonlinear pdes. *Annals of Applied Probability.*

[10] R. Frey. The pricing and hedging of options in finetely elastic markets. *Working Paper, University of Bonn*, 1996.

[11] R. Frey. Perfect option hedging for a large trader. *Finance and Stochastics*, 2:115–141, 1998.

[12] I. Friend and M.E. Blume. The demand for risky assets. *American Economic Review*, 65, 1975.

[13] Emmanuel Gobet, Jean-Philippe Lemor, and Xavier Warin. A regression-based Monte Carlo method to solve backward stochastic differential equations. *Ann. Appl. Probab.*, 15(3):2172–2202, 2005.

[14] R. Jarrow. Derivative securities markets, market manipulation and option pricing theory. *Journal of financial and quantitative analysis*, 29:241–261, 1994.

[15] J. Kambhu. Dealers' hedging of interest rate options in the us dollar fixed income market. *Federal Reserve Bank of New York Economic Policy Review*, 4:35–57, 1998.

[16] J. Kim, K. Ko, and K. Noh. Time-varying bid-ask components of nikkei 225 index futures on simex. *Working Paper, Korean Security Research Institute*, 2000.

[17] J. Ma and J.Yong. *Forward-Backward Stochastic Differential Equations and their aplications*. Springer.

[18] Peter J. Olver. *Applications of Lie groups to differential equations*, volume 107 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1993.

[19] Daniel Sevcovic. An iterative algorithm for evaluating approximations to the optimal exercise boundary for a nonlinear black-scholes equation, 2007.

[20] H. Mete Soner, Nizar Touzi, and Jianfeng Zhang. Wellposedness of second order backward sdes,. *Probability Theory and Related Fields,*.

[21] H. Stoll. The pricing of security dealer services: An empirical study of nasdaq stocks. *Journal of Finance*, pages 1153–1172, 1978.

[22] H. Stoll. Inferring the components of the bid-ask spread: Theory and empirical tests. *Journal of Finance*, 44:116–134, 1992.

[23] A. Vijh. Liquidity of the cbor options. *Journal of Finance*, 45:1157–1179, 1990.