

THE BIOLOGICAL COMPETITION PROBLEM

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ABSTRACT. This paper addresses the biological competition stability problem among organisms of the same or different species associated with the need for a common resource that occurs in a limited supply relative to demand by considering it as a discrete event dynamical system. Timed Petri nets are a graphical and mathematical modeling tool applicable to discrete event dynamical systems in order to represent its states evolution. Lyapunov stability theory provides the required tools needed to aboard the stability problem for discrete event dynamical systems modeled with timed Petri nets. By proving boundedness one confirms a dominant oscillating behavior of both organisms dynamics performance. However, the oscillating frequency results to be unknown. This inconvenience is overcome by considering a specific recurrence equation, in the max-plus algebra.

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1. INTRODUCTION

Consider the biological competition stability problem among organisms of the same or different species associated with the need for a common resource that occurs in a limited supply relative to demand. In other words, competition better defined as interaction occurs when the capability of the environment to supply resources is smaller than the potential biological requirement so that organisms interfere with each other. More concisely we can say that competition is the interaction between individuals of the same or different species leading to an increase of the fitness of one at the expense of the other. Plants, for example, often compete for access to a limited supply of nutrients, water, sunlight, and space. Therefore, two species cannot indefinitely coexist if they are limited by the same predator. If two competitors try to occupy the same realized niche, one species will eliminate the other [1]. In the study of this type of problems Lotka-Volterra models as well as evolutionary game theory concepts have been used [2, 3]. This paper proposes a new modeling and analysis methodology which consists in considering the competition as a discrete event dynamical system. Timed Petri nets are a graphical and mathematical modeling tool applicable

to discrete event systems in order to represent its states evolution where the timing at which the state changes is taken into consideration. Lyapunov stability theory provides the required tools needed to aboard the stability problem for discrete event system modeled with timed Petri nets whose mathematical model is given in terms of difference equations [6]. Employing Lyapunov methods, a sufficient condition for the stabilization problem is also obtained. It is shown that it is possible to restrict the discrete event systems state space in such a way that boundedness is guaranteed. By proving boundedness one confirms a dominant oscillating behavior of both organisms dynamics performance. However, the oscillating frequency results to be unknown. This inconvenience is overcome by considering a specific recurrence equation, in the max-plus algebra, which is assigned to the the timed Petri net graphical model. The main contribution of the paper consists in combining Lyapunov theory with max-plus algebra to study the biological interaction stability problem among organisms treated as discrete event dynamical systems modeled with timed Petri nets. The paper is organized as follows. In section 2, Lyapunov theory for discrete event modeled with Petri nets is addressed. Section 3, presents Max-Plus algebra. In section 4, the stability for discrete event dynamical systems modeled with timed Petri nets is given. Section 5, discusses the biological competition stability problem. Finally, the paper ends with some conclusions and with the trees lyrics [8].

2. LYAPUNOV STABILITY AND STABILIZATION OF DISCRETE EVENT SYSTEMS MODELED WITH PETRI NETS [6]

NOTATION: $N = \{0, 1, 2, \dots\}$, $R_+ = [0, \infty)$, $N_{n_0}^+ = \{n_0, n_0 + 1, \dots, n_0 + k, \dots\}$, $n_0 \geq 0$. Given $x, y \in R^n$, $x \leq y$ is equivalent to $x_i \leq y_i, \forall i$. A function $f(n, x)$, $f : N_{n_0}^+ \times R^n \rightarrow R^n$ is called nondecreasing in x if given $x, y \in R^n$ such that $x \geq y$ and $n \in N_{n_0}^+$ then, $f(n, x) \geq f(n, y)$. Consider systems of first ordinary difference equations given by

$$(2.1) \quad x(n+1) = f[n, x(n)], x(n_0) = x_0, n \in N_{n_0}^+$$

where $n \in N_{n_0}^+$, $x(n) \in R^n$ and $f : N_{n_0}^+ \times R^n \rightarrow R^n$ is continuous in $x(n)$.

Definition 1. The n vector valued function $\Phi(n, n_0, x_0)$ is said to be a solution of (2.1) if $\Phi(n_0, n_0, x_0) = x_0$ and $\Phi(n+1, n_0, x_0) = f(n, \Phi(n, n_0, x_0))$ for all $n \in N_{n_0}^+$.

Definition 2. The system (2.1) is said to be i). Practically stable, if given (λ, A) with $0 < \lambda < A$, then

$$|x_0| < \lambda \Rightarrow |x(n, n_0, x_0)| < A, \forall n \in N_{n_0}^+, n_0 \geq 0;$$

ii). Uniformly practically stable, if it is practically stable for every $n_0 \geq 0$.

Definition 3. A continuous function $\alpha : [0, \infty) \rightarrow [0, \infty)$ is said to belong to class \mathcal{K} if $\alpha(0) = 0$ and it is strictly increasing.

Consider a vector Lyapunov function $v(n, x(n))$, $v : N_{n_0}^+ \times R^n \rightarrow R_+^p$ and define the variation of v relative to (2.1) by

$$(2.2) \quad \Delta v = v(n+1, x(n+1)) - v(n, x(n))$$

Then, the following result concerns the practical stability of (1).

Theorem 4. Let $v : N_{n_0}^+ \times R^n \rightarrow R_+^p$ be a continuous function in x , define the function $v_0(n, x(n)) = \sum_{i=1}^p v_i(n, x(n))$ such that satisfies the estimates

$$(2.3) \quad b(|x|) \leq v_0(n, x(n)) \leq a(|x|); a, b \in \mathcal{K}, \Delta v(n, x(n)) \leq w(n, v(n, x(n)))$$

for $n \in N_{n_0}^+$, $x(n) \in R^n$, where $w : N_{n_0}^+ \times R_+^p \rightarrow R^p$ is a continuous function in the second argument. Assume that $g(n, e) \triangleq e + w(n, e)$ is nondecreasing in e , $0 < \lambda < A$ are given and finally that $a(\lambda) < b(A)$ is satisfied. Then, the practical stability properties of

$$(2.4) \quad e(n+1) = g(n, e(n)), e(n_0) = e_0 \geq 0.$$

imply the practical stability properties of system (2.1).

Corollary 5. In Theorem 4: *i).* If $w(n, e) \equiv 0$ we get uniform practical stability of (1) which implies structural stability. *ii).* If $w(n, e) = -c(e)$, for $c \in \mathcal{K}$, we get uniform practical asymptotic stability of (2.1).

Definition 6. A Petri net is a 5-tuple, $PN = \{P, T, F, W, M_0\}$ where: $P = \{p_1, p_2, \dots, p_m\}$ is a finite set of places, $T = \{t_1, t_2, \dots, t_n\}$ is a finite set of transitions, $F \subset (P \times T) \cup (T \times P)$ is a set of arcs, $W : F \rightarrow N_1^+$ is a weight function, $M_0 : P \rightarrow N$ is the initial marking, $P \cap T = \emptyset$ and $P \cup T \neq \emptyset$.

Definition 7. The clock structure associated with a place $p_i \in P$ is a set $\mathbf{V} = \{V_i : p_i \in P\}$ of clock sequences $V_i = \{v_{i,1}, v_{i,2}, \dots\}$, $v_{i,k} \in R^+$, $k = 1, 2, \dots$

The positive number $v_{i,k}$, associated to $p_i \in P$, called holding time, represents the time that a token must spend in this place until its outputs enabled transitions $t_{i,1}, t_{i,2}, \dots$, fire. We partition P into subsets P_0 and P_h , where P_0 is the set of places with zero holding time, and P_h is the set of places that have some holding time.

Definition 8. A timed Petri net is a 6-tuple $TPN = \{P, T, F, W, M_0, \mathbf{V}\}$ where $\{P, T, F, W, M_0\}$ are as before, and $\mathbf{V} = \{V_i : p_i \in P\}$ is a clock structure. A timed Petri net is a timed event petri net when every $p_i \in P$ has one input and one output transition, in which case the associated clock structure set of a place $p_i \in P$ reduces to one element $V_i = \{v_i\}$

Notice that if $W(p, t) = \alpha$ (or $W(t, p) = \beta$) then, this is often represented graphically by α , (β) arcs from p to t (to p) each with no numeric label.

Let $M_k(p_i)$ denote the marking (i.e., the number of tokens) at place $p_i \in P$ at time k and let $M_k = [M_k(p_1), \dots, M_k(p_m)]^T$ denote the marking (state) of PN at time k . A transition $t_j \in T$ is said to be enabled at time k if $M_k(p_i) \geq W(p_i, t_j)$ for all $p_i \in P$ such that $(p_i, t_j) \in F$. It is assumed that at each time k there exists at least one transition to fire. If a transition is enabled then, it can fire. If an enabled transition $t_j \in T$ fires at time k then, the next marking for $p_i \in P$ is given by

$$(2.5) \quad M_{k+1}(p_i) = M_k(p_i) + W(t_j, p_i) - W(p_i, t_j).$$

Let $A = [a_{ij}]$ denote an $n \times m$ matrix of integers (the incidence matrix) where $a_{ij} = a_{ij}^+ - a_{ij}^-$ with $a_{ij}^+ = W(t_i, p_j)$ and $a_{ij}^- = W(p_j, t_i)$. Let $u_k \in \{0, 1\}^n$ denote a firing vector where if $t_j \in T$ is fired then, its corresponding firing vector is $u_k = [0, \dots, 0, 1, 0, \dots, 0]^T$ with the one in the j^{th} position in the vector and zeros everywhere else. The nonlinear difference matrix equation describing the dynamical behavior represented by a PN is:

$$(2.6) \quad M_{k+1} = M_k + A^T u_k$$

where if at step k , $a_{ij}^- < M_k(p_j)$ for all $p_i \in P$ then, $t_i \in T$ is enabled and if this $t_i \in T$ fires then, its corresponding firing vector u_k is utilized in the difference equation to generate the next step. Notice that if M' can be reached from some other marking M and, if we fire some sequence of d transitions with corresponding firing vectors u_0, u_1, \dots, u_{d-1} we obtain that

$$(2.7) \quad M' = M + A^T u, u = \sum_{k=0}^{d-1} u_k.$$

Let $(N_{n_0}^m, d)$ be a metric space where $d : N_{n_0}^m \times N_{n_0}^m \rightarrow R_+$ is defined by

$$d(M_1, M_2) = \sum_{i=1}^m \zeta_i |M_1(p_i) - M_2(p_i)|; \zeta_i > 0$$

and consider the matrix difference equation which describes the dynamical behavior of the discrete event system modeled by a PN , see (2.7).

Proposition 9. *Let PN be a Petri net. PN is uniform practical stable if there exists a Φ strictly positive m vector such that*

$$(2.8) \quad \Delta v = u^T A \Phi \leq 0$$

Moreover, PN is uniform practical asymptotic stable if the following equation holds

$$(2.9) \quad \Delta v = u^T A \Phi \leq -c(e), c \in \mathcal{K}$$

Lemma 10. *Let suppose that Proposition 9 holds then,*

$$(2.10) \quad \Delta v = u^T A \Phi \leq 0 \Leftrightarrow A \Phi \leq 0$$

Remark 11. Notice that since the state space of a TPN is contained in the state space of the same now not timed PN, stability of PN implies stability of the TPN.

2.1. Lyapunov Stabilization.

Definition 12. Let PN be a Petri net. PN is said to be stabilizable if there exists a firing transition sequence with transition count vector u such that system (2.7) remains bounded.

Proposition 13. *Let PN be a Petri net. PN is stabilizable if there exists a firing transition sequence with transition count vector u such that the following equation holds*

$$(2.11) \quad \Delta v = A^T u \leq 0$$

Remark 14. By fixing a particular u , which satisfies (2.11), the state space is restricted to those markings that are finite.

3. MAX-PLUS ALGEBRA [4, 5, 7]

3.1. Basic Definitions. NOTATION: $\epsilon = -\infty$, $e = 0$, $\mathbb{R}_{\max} = \mathbb{R} \cup \{\epsilon\}$, $\underline{n} = 1, 2, \dots, n$. Let $a, b \in \mathbb{R}_{\max}$ and define the operations \oplus and \otimes by: $a \oplus b = \max(a, b)$ and $a \otimes b = a + b$.

Definition 15. The set \mathbb{R}_{\max} with the two operations \oplus and \otimes is called a max-plus algebra and is denoted by $\mathfrak{R}_{\max} = (\mathbb{R}_{\max}, \oplus, \otimes, \epsilon, e)$.

Definition 16. A semiring is a nonempty set R endowed with two operations \oplus_R , \otimes_R , and two elements ϵ_R and e_R such that: \oplus_R is associative and commutative with zero element ϵ_R , \otimes_R is associative, distributes over \oplus_R , and has unit element e_R , ϵ_R is absorbing for \otimes_R i.e., $a \otimes_R \epsilon = \epsilon_R \otimes a = a$, $\forall a \in R$.

In addition if \otimes_R is commutative then R is called a commutative semiring, and if \oplus_R is such that $a \oplus_R a = a$, $\forall a \in R$ then it is called idempotent.

Theorem 17. *The max-plus algebra $\mathfrak{R}_{\max} = (\mathbb{R}_{\max}, \oplus, \otimes, \epsilon, e)$ has the algebraic structure of a commutative and idempotent semiring.*

3.2. Matrices and Graphs. Let $\mathbb{R}_{\max}^{n \times n}$ be the set of $n \times n$ matrices with coefficients in \mathbb{R}_{\max} with the following operations: The sum of matrices $A, B \in \mathbb{R}_{\max}^{n \times n}$, denoted $A \oplus B$ is defined by: $(A \oplus B)_{ij} = a_{ij} \oplus b_{ij} = \max(a_{ij}, b_{ij})$ for i and $j \in \underline{n}$. The product of matrices $A \in \mathbb{R}_{\max}^{n \times l}$, $B \in \mathbb{R}_{\max}^{l \times n}$, denoted $A \otimes B$ is defined by: $(A \otimes B)_{ik} = \bigotimes_{j=1}^l (a_{ij} \otimes b_{jk})$ for i and $k \in \underline{n}$. Let $\mathcal{E} \in \mathbb{R}_{\max}^{n \times n}$ denote the matrix with all its elements equal to ϵ and denote by $E \in \mathbb{R}_{\max}^{n \times n}$ the matrix which has its diagonal elements equal to e and all the other elements equal to ϵ . Then, the following result can be stated.

Theorem 18. *The 5-tuple $\mathfrak{R}_{\max}^{n \times n} = (\mathbb{R}_{\max}^{n \times n}, \oplus, \otimes, \mathcal{E}, E)$ has the algebraic structure of a noncommutative idempotent semiring.*

Definition 19. Let $A \in \mathbb{R}_{\max}^{n \times n}$ and $k \in \mathbb{N}$ then the k -th power of A denoted by $A^{\otimes k}$ is defined by: $A^{\otimes k} = A \otimes A \otimes \cdots \otimes A$, (k times), where $A^{\otimes 0}$ is set equal to E .

Definition 20. A matrix $A \in \mathbb{R}_{\max}^{n \times n}$ is said to be regular if A contains at least one element distinct from ϵ in each row.

Definition 21. Let \mathcal{N} be a finite and non-empty set and consider $\mathcal{D} \subseteq \mathcal{N} \times \mathcal{N}$. The pair $G = (\mathcal{N}, \mathcal{D})$ is called a directed graph, where \mathcal{N} is the set of elements called nodes and \mathcal{D} is the set of ordered pairs of nodes called arcs. A directed graph $G = (\mathcal{N}, \mathcal{D})$ is called a weighted graph if a weight $w(i, j) \in \mathbb{R}$ is associated with any arc $(i, j) \in \mathcal{D}$.

Let $A \in \mathbb{R}_{\max}^{n \times n}$ be any matrix, a graph $\mathcal{G}(A)$, called the communication graph of A , can be associated as follows. Define $\mathcal{N}(A) = \underline{n}$ and a pair $(i, j) \in \underline{n} \times \underline{n}$ will be a member of $\mathcal{D}(A) \Leftrightarrow a_{ji} \neq \epsilon$, where $\mathcal{D}(A)$ denotes the set of arcs of $\mathcal{G}(A)$.

Definition 22. A path from node i to node j is a sequence of arcs $p = \{(i_k, j_k) \in \mathcal{D}(A)\}_{k \in \underline{m}}$ such that $i = i_1, j_k = i_{k+1}$, for $k < m$ and $j_m = j$. The path p consists of the nodes $i = i_1, i_2, \dots, i_m, j_m = j$ with length m denoted by $|p|_1 = m$. In the case when $i = j$ the path is said to be a circuit. A circuit is said to be elementary if nodes i_k and i_l are different for $k \neq l$. A circuit consisting of one arc is called a self-loop.

Let us denote by $P(i, j; m)$ the set of all paths from node i to node j of length $m \geq 1$ and for any arc $(i, j) \in \mathcal{D}(A)$ let its weight be given by a_{ij} then the weight of a path $p \in P(i, j; m)$ denoted by $|p|_w$ is defined to be the sum of the weights of all the arcs that belong to the path. The average weight of a path p is given by $|p|_w / |p|_1$. Given two paths, as for example, $p = ((i_1, i_2), (i_2, i_3))$ and $q = ((i_3, i_4), (i_4, i_5))$ in $\mathcal{G}(A)$ the concatenation of paths $\circ : \mathcal{G}(A) \times \mathcal{G}(A) \rightarrow \mathcal{G}(A)$ is defined as $p \circ q = ((i_1, i_2), (i_2, i_3), (i_3, i_4), (i_4, i_5))$. The communication graph $\mathcal{G}(A)$ and powers of matrix A are closely related as it is shown in the next theorem.

Theorem 23. *Let $A \in \mathbb{R}_{\max}^{n \times n}$, then $\forall k \geq 1: [A^{\otimes k}]_{ji} = \max\{|p|_w : p \in P(i, j; k)\}$, where $[A^{\otimes k}]_{ji} = \epsilon$ in the case when $P(i, j; k)$ is empty i.e., no path of length k from node i to node j exists in $\mathcal{G}(A)$.*

Definition 24. Let $A \in \mathbb{R}_{\max}^{n \times n}$ then define the matrix $A^+ \in \mathbb{R}_{\max}^{n \times n}$ as: $A^+ = \bigoplus_{k=1}^{\infty} A^{\otimes k}$. Where the element $[A^+]_{ji}$ gives the maximal weight of any path from j to i . If in addition one wants to add the possibility of staying at a node then one must include matrix E in the definition of matrix A^+ giving rise to its Kleene star representation defined by: $A^* = \bigoplus_{k=0}^{\infty} A^{\otimes k}$.

Lemma 25. Let $A \in \mathbb{R}_{\max}^{n \times n}$ be such that any circuit in $\mathcal{G}(A)$ has average circuit weight less than or equal to ϵ . Then it holds that: $A^* = \bigoplus_{k=0}^{n-1} A^{\otimes k}$.

Definition 26. Let $G = (\mathcal{N}, \mathcal{D})$ be a graph and $i, j \in \mathcal{N}$, node j is reachable from node i , denoted as $i\mathcal{R}j$, if there exists a path from i to j . A graph G is said to be strongly connected if $\forall i, j \in \mathcal{N}, j\mathcal{R}i$. A matrix $A \in \mathbb{R}_{\max}^{n \times n}$ is called irreducible if its communication graph is strongly connected, when this is not the case matrix A is called reducible.

Definition 27. Let $G = (\mathcal{N}, \mathcal{D})$ be a not strongly connected graph and $i, j \in \mathcal{N}$, node j communicates with node i , denoted as $i\mathcal{C}j$, if either $i = j$ or $i\mathcal{R}j$ and $j\mathcal{R}i$.

The relation $i\mathcal{C}j$ defines an equivalence relation in the set of nodes, and therefore a partition of \mathcal{N} into a disjoint union of subsets, the equivalence classes, $\mathcal{N}_1, \mathcal{N}_2, \dots, \mathcal{N}_q$ such that $\mathcal{N} = \mathcal{N}_1 \cup \mathcal{N}_2 \cup \dots \cup \mathcal{N}_q$ or $\mathcal{N} = \bigcup_{i \in \mathcal{N}} [i]$; $[i] = \{j \in \mathcal{N} : i\mathcal{C}j\}$.

Given the above partition, it is possible to focus on subgraphs of G denoted by $G_r = (\mathcal{N}_r, \mathcal{D}_r)$; $r \in \underline{q}$ where \mathcal{D}_r denotes the subset of arcs, which belong to \mathcal{D} , that have both the begin node and end node in \mathcal{N}_r . If $\mathcal{D}_r \neq \emptyset$, the subgraph $G_r = (\mathcal{N}_r, \mathcal{D}_r)$ is known as a maximal strongly connected subgraph of G .

Definition 28. The reduced graph $\tilde{G} = (\tilde{\mathcal{N}}, \tilde{\mathcal{D}})$ of G is defined by setting $\tilde{\mathcal{N}} = \{[i_1], [i_2], \dots, [i_q]\}$ and $([i_r], [i_s]) \in \tilde{\mathcal{D}}$ if $r \neq s$ and there exists an arc $(k, l) \in \mathcal{D}$ for some $k \in [i_r]$ and $l \in [i_s]$.

Let A_{rr} denote the matrix by restricting A to the nodes in $[i_r]$ $\forall r \in \underline{q}$ i.e., $[A_{rr}]_{kl} = a_{kl} \forall k, l \in [i_r]$. Then $\forall r \in \underline{q}$ either A_{rr} is irreducible or is equal to ϵ . Therefore since by construction the reduced graph does not contain any circuits, the original reducible matrix A after a possible relabeling of the nodes in $G(A)$, can be written as:

$$(3.1) \quad A = \begin{pmatrix} A_{11} & A_{12} & \cdots & \cdots & A_{1q} \\ \mathcal{E} & A_{22} & \cdots & \cdots & A_{2q} \\ \mathcal{E} & \mathcal{E} & A_{33} & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \mathcal{E} & \mathcal{E} & \cdots & \mathcal{E} & A_{qq} \end{pmatrix}$$

with matrices A_{sr} $1 \leq s < r \leq q$, where each finite entry in A_{sr} corresponds to an arc from a node in $[i_r]$ to a node in $[i_s]$.

Definition 29. Let $A \in \mathbb{R}_{\max}^{n \times n}$ be a reducible matrix then, the block upper triangular given by (3.1) is said to be a normal form of matrix A .

3.2.1. *Spectral Theory and Linear Equations.*

Definition 30. Let $A \in \mathbb{R}_{\max}^{n \times n}$ be a matrix. If $\mu \in R_{\max}$ is a scalar and $v \in R_{\max}^n$ is a vector that contains at least one finite element such that: $A \otimes v = \mu \otimes v$ then, μ is called an eigenvalue and v an eigenvector.

Let $\mathcal{C}(A)$ denote the set of all elementary circuits in $\mathcal{G}(A)$ and write: $\lambda = \max_{p \in \mathcal{C}(A)} \frac{|p|_w}{|p|_1}$ for the maximal average circuit weight. Notice that since $\mathcal{C}(A)$ is a finite set, the maximum is attained (which is always the case when matrix A is irreducible). In case $\mathcal{C}(A) = \emptyset$ define $\lambda = \epsilon$.

Definition 31. A circuit $p \in G(A)$ is said to be critical if its average weight is maximal. The critical graph of A , denoted by $G^c(A) = (\mathcal{N}^c(A), \mathcal{D}^c(A))$, is the graph consisting of those nodes and arcs that belong to critical circuits in $G(A)$.

Theorem 32. *If $A \in \mathbb{R}_{\max}^{n \times n}$ is irreducible, then there exists one and only one finite eigenvalue (with possible several eigenvectors). This eigenvalue is equal to the maximal average weight of circuits in $G(A)$ $\lambda(A) = \max_{p \in \mathcal{C}(A)} \frac{|p|_w}{|p|_1}$*

Theorem 33. *Let $A \in \mathbb{R}_{\max}^{n \times n}$ and $b \in \mathbb{R}_{\max}^n$. If the communication graph $G(A)$ has maximal average circuit weight less than or equal to ϵ , then $x = A^* \otimes b$ solves the equation $x = (A \otimes x) \oplus b$. Moreover, if the circuit weights in $G(a)$ are negative then, the solution is unique.*

3.3. **Max-Plus recurrence equations for timed event Petri Nets.**

Definition 34. Let $A_m \in \mathbb{R}_{\max}^{n \times n}$ for $0 \leq m \leq M$ and $x(m) \in \mathbb{R}_{\max}^n$ for $-M \leq m \leq -1$; $M \geq 0$. Then, the recurrence equation: $x(k) = \bigoplus_{m=0}^M A_m \otimes x(k - m)$; $k \geq 0$ is called an M th order recurrence equation.

Theorem 35. *The M th order recurrence equation, given by equation $x(k) = \bigoplus_{m=0}^M A_m \otimes x(k - m)$; $k \geq 0$, can be transformed into a first order recurrence equation $x(k + 1) = A \otimes x(k)$; $k \geq 0$ provided that A_0 has circuit weights less than or equal to zero.*

With any timed event Petri net, matrices $A_0, A_1, \dots, A_M \in \mathbb{N}^n \times \mathbb{N}^n$ can be defined by setting $[A_m]_{jl} = a_{jl}$, where a_{jl} is the largest of the holding times with respect to all places between transitions t_l and t_j with m tokens, for $m = 0, 1, \dots, M$, with M

equal to the maximum number of tokens with respect to all places. Let $x_i(k)$ denote the k th time that transition t_i fires, then the vector $x(k) = (x_1(k), x_2(k), \dots, x_m(k))^T$, called the state of the system, satisfies the M th order recurrence equation: $x(k) = \bigoplus_{m=0}^M A_m \otimes x(k-m)$; $k \geq 0$. Now, assuming that all the hypothesis of theorem (35) are satisfied, and setting $\hat{x}(k) = (x^T(k), x^T(k-1), \dots, x^T(k-M+1))^T$, equation $x(k) = \bigoplus_{m=0}^M A_m \otimes x(k-m)$; $k \geq 0$ can be expressed as: $\hat{x}(k+1) = \hat{A} \otimes \hat{x}(k)$; $k \geq 0$, which is known as the standard autonomous equation.

4. THE SOLUTION TO THE STABILITY PROBLEM FOR DISCRETE EVENT DYNAMICAL SYSTEMS MODELED WITH TIMED PETRI NETS

Definition 36. A TPN is said to be stable if all the transitions fire with the same proportion i.e., if there exists $q \in \mathbb{N}$ such that

$$(4.1) \quad \lim_{k \rightarrow \infty} \frac{x_i(k)}{k} = q, \forall i = 1, \dots, n$$

This means that in order to obtain a stable *TPN* all the transitions have to be fired q times. It will be desirable to be more precise and know exactly how many times. The answer to this question is given next.

Lemma 37. Consider the recurrence relation $x(k+1) = A \otimes x(k)$, $k \geq 0$, $x(0) = x_0 \in \mathbb{R}^n$ arbitrary. A an irreducible matrix and $\lambda \in \mathbb{R}$ its eigenvalue then,

$$(4.2) \quad \lim_{k \rightarrow \infty} \frac{x_i(k)}{k} = \lambda, \forall i = 1, \dots, n$$

Proof. Let v be an eigenvector of A such that $x_0 = v$ then,

$$x(k) = \lambda^{\otimes k} \otimes v \Rightarrow x(k) = k\lambda + v \Rightarrow \frac{x(k)}{k} = \lambda + \frac{v}{k} \Rightarrow \lim_{k \rightarrow \infty} \frac{x_i(k)}{k} = \lambda$$

□

Now starting with an unstable *TPN*, collecting the results given by: proposition (13), what has just been discussed about recurrence equations for *TPN* at the end of subsection (3.3) and the previous lemma (37) plus theorem (32), the solution to the problem is obtained.

5. THE BIOLOGICAL COMPETITION STABILITY PROBLEM

Consider a biological interaction system whose *TPN* model is depicted in Fig 1.

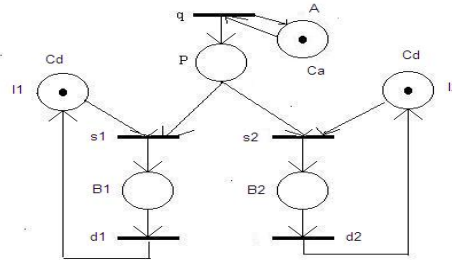


Figure 1. Timed Petri net model

Where the events (transitions) that drive the system are: q: resource to be consumed, s1, s2: consuming starts, d1,d2: the resource has been consumed. The places that represent the states are: A: the resource is active, P: the resource is available to be consumed, B1, B2: the resource is being consumed, I1, I2: the organisms are idle. The holding times associated to the places A and I1, I2 are Ca and Cd respectively, (with $Ca > Cd$) i.e., limited resource supply. The incidence matrix that represents the PN model is

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

Therefore since there does not exist a Φ strictly positive m vector such that $A\Phi \leq 0$ the sufficient condition for stability is not satisfied, (moreover, the PN (TPN) is unbounded since by the repeated firing of q, the marking in P grows indefinitely). However, by taking $u = [k, k/2, k/2, k/2, k/2]; k > 0$ (but unknown) we get that $A^T u \leq 0$. Therefore, the PN is stabilizable which implies that the TPN is stable. Now, let us proceed to determine the exact value of k . From the TPN model we obtain that:

$$A_0 = \begin{pmatrix} \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon \end{pmatrix} \text{ and } A_1 = \begin{pmatrix} Ca & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & Cd & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & Cd \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \end{pmatrix} \text{ which implies}$$

$$A_0^* = \begin{pmatrix} 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ 0 & 0 & \varepsilon & \varepsilon & \varepsilon \\ 0 & \varepsilon & 0 & \varepsilon & \varepsilon \\ 0 & 0 & \varepsilon & 0 & \varepsilon \\ 0 & \varepsilon & 0 & \varepsilon & 0 \end{pmatrix}, \text{ leading to:}$$

$$\hat{A} = A_0^* \otimes A_1 = \begin{pmatrix} Ca & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ Ca & \varepsilon & \varepsilon & Cd & \varepsilon \\ Ca & \varepsilon & \varepsilon & \varepsilon & Cd \\ Ca & 0 & \varepsilon & Cd & \varepsilon \\ Ca & \varepsilon & \varepsilon & \varepsilon & Cd \end{pmatrix}$$

Therefore, $\lambda(A) = \max_{p \in \mathcal{C}(A)} \frac{|p|_w}{|p|_1} = \max\{Ca, Cd\} = Ca$. This means that in order for the *TPN* to be stable and work properly the speed at which the two organisms consume has to be equal to Ca which is attained by taking $k = Ca$, i.e., the resource has to be equally shared between the two organisms.

6. CONCLUSIONS

The main contribution of the paper consists in combining Lyapunov theory with max-plus algebra to study the biological interaction stability problem treated as a discrete event dynamical system modeled with timed Petri nets.

The Trees Lyrics

There is unrest in the forest, There is trouble with the trees, For the maples want more sunlight And the oaks ignore their pleas.

The trouble with the maples, (And they're quite convinced they're right) They say the oaks are just too lofty And they grab up all the light. But the oaks can't help their feelings If they like the way they're made. And they wonder why the maples Can't be happy in their shade.

There is trouble in the forest, And the creatures all have fled, As the maples scream "Oppression!" And the oaks just shake their heads.

So the maples formed a union And demanded equal rights. "The oaks are just too greedy; We will make them give us light." Now there's no more oak oppression, For they passed a noble law, And the trees are all kept equal By hatchet, axe, and saw.

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