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FLUCTUATIONS OF A NONMONOTONE MARKED POINT PROCESS UNDER A TIME CONSTRAINT

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ABSTRACT. The present article investigates a trivariate delayed recurrent process, which models the behavior of a nonmonotone financial instrument observed at random times. We are able to find explicitly the distributions of the highest value of the instrument prior to its first drop and the time when it takes place prior to the expiration of an option on which this stock is written. We also provide numerical illustrations.

Key Words: random walk analysis, stock market, fluctuation theory, marked point process, first passage time, stochastic finance, American option

AMS Subject Classification: 82B41, 60G51, 60G55, 60G57, 91A10, 91A05, 91A60, 60K05

1. Introduction

In this article we continue to study nonmonotone stochastic processes observed upon random times started in Dshalalow and Robinson [5]. Most relevant real-world situations that inspired us to write this article stem from financial markets. They include the time of the first drop or increase and the exit time from a particular set, to name a few. Crossing times turn out to be of a lesser problem for some processes if they are observed continuously. A bigger challenge is posed with observations being restricted to some particular (mostly random) times.

Another situation worth mentioning can take place when analyzing random polynomials. Much of work on random polynomials is concentrated around finding real roots and some on level crossings [7]. The first passage time of crossing a predefined deterministic or random threshold by a random polynomial (at "real time" or upon selected random times) would agree with our present research offering useful alternatives to applications of our methods.

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In [5] we found the probability distribution function (PDF) of the highest value of a financial instrument prior to its first decrease and the time when this takes place. In various applications, such as option trading, of further interest is to know the distribution of this moment and the value of a stock prior to some reference time, in particular, the expiration date of an option on which this stock has been written.

Consequently, adhering yet another active element (namely, the observation process with respect to the expiration time) is the focus of our investigation. The corresponding probabilistic information we obtain will be of interest to stock market analysts and option traders.

Our chief goal is to offer an explicit formula for the PDF's of the exit time and the stock value for a special case and thereby demonstrate analytical tractability of an underlying general result from [3].

We use game-theoretical principles [4, 6] (see also related papers [11, 12]) and fluctuations [2, 9, 10] to obtain the explicit results, as well as operational calculus throughout.

2. **Preliminaries**

All processes below will be considered on a filtered probability space $(\Omega, \mathcal{F}(\Omega), (\mathcal{F}_t; t \geq 0), P)$.

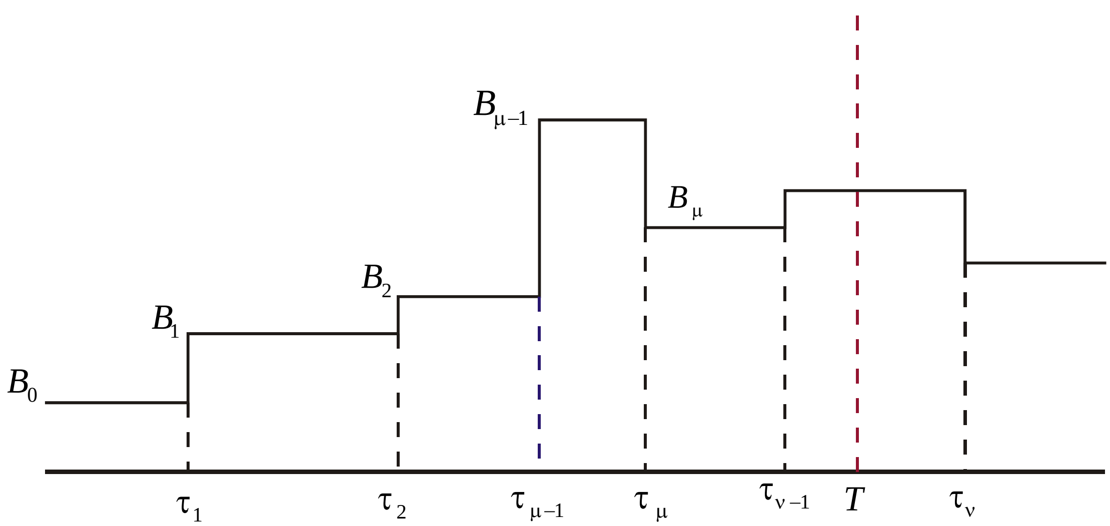

Figure 1

Consider a stock observed at random moments of time $\tau_0, \tau_1, \tau_2 \ldots$ with respective values $B_0, B_1, B_2, \ldots$ as shown in Figure 1 above. The stock varies continuously, but the observation process can make it arbitrarily crude (or fine) and as the result, it becomes a piece-wise linear jump process.

We formalize it as the marked point process

$$(\mathcal{B},\mathcal{T}) = \sum_{k\geq 0} Y_k \varepsilon_{\tau_k} \tag{2.1}$$

(Y_k are associated marks or increments $B_\mu - B_{\mu-1}, B_0 = Y_0$ and ε_a is a point mass) and we assume that $(\mathcal{B},\mathcal{T})$ is a delayed marked renewal process with the position dependent marking. We are concerned with an American option written on this stock, which has an expiration date, say a fixed positive real number T. Suppose we are interested in probability that the stock drops for the first time while increasing prior to the expiration date T, as it can be an optimal time to exercise the option.

Let us introduce the following reference indices:

$$\mu = \min\{m \geq 1 : Y_m < 0\}, \tag{2.2}$$

where $Y_0 > 0$ a.s.

$$\nu = \min\{n \geq 1 : \tau_n = \Delta_0 + \ldots + \Delta_n \geq T\} \tag{2.3}$$

Furthermore, we introduce the following auxiliary process

$$A_m := X_0 + \ldots + X_m \tag{2.4}$$

with

$$X_k = \begin{cases} 0, & Y_k \geq 0 \\ 1, & Y_k < 0 \end{cases} \tag{2.5}$$

and adhere it to $(\mathcal{B},\mathcal{T})$ upon $\mathcal{T}$ to form the bivariate marked point process with position dependent marking:

$$(\mathcal{A},\mathcal{B},\mathcal{T}) = \sum_{k\geq 0} (X_k,\, Y_k)\varepsilon_{\tau_k} \tag{2.6}$$

adapted to the filtration $(\mathcal{F}_t)$.

The process $(\mathcal{A},\mathcal{B},\mathcal{T})$ should normally evolve until one of the two events take place:

(i) the stock drops for the first time, i.e. $Y_\mu < 0$
(ii) τ_ν crosses T upon the first observation epoch.

Thus, the part of $(\mathcal{A},\mathcal{B},\mathcal{T})$ we are interested in is restricted to

$$(\mathcal{A},\mathcal{B},\mathcal{T})^* = \sum_{k\geq 0}^{\min\{\mu,\nu\}} (X_k, Y_k)\varepsilon_{\tau_k} \tag{2.7}$$

with the history prior to $\mathcal{F}_{\min\{\tau_\mu,\tau_\nu\}}$.

Suppose the following associated joint transforms

$$\gamma(u,v,\theta) := E[u^X e^{vY-\theta\Delta}], |u| \le 1, \quad \mathrm{Re}(\theta) \ge 0 \tag{2.8}$$

$$\gamma_0(u,v,\theta) := E[u^{X_0} e^{vY_0-\theta\Delta_0}], |u| \le 1, \quad \mathrm{Re}(\theta) \ge 0 \tag{2.9}$$

are known or can be derived. Here

$$X_i \in [X], \quad Y_i \in [Y], \quad \Delta_i \in [\Delta],$$

with $[\cdot]$ is an associated equivalence class of r.v.'s. With (2.4)–(2.5), equation (2.2) can be rewritten as

$$\mu = \min\{m \ge 0 : A_m = X_0 + \cdots + X_m \ge 1\}. \tag{2.10}$$

The problem we set, in light of (2.2)–(2.7), can be modeled by the following functional

$$\Phi := \Phi(z,u,w,v,\vartheta,\theta) = E\left[z^{A_{\mu-1}} u^{A_\mu} e^{wB_{\mu-1}} e^{vB_\mu} e^{-\vartheta\tau_{\mu-1}} e^{-\theta\tau_\mu} \mathbb{1}_{\{\mu<\nu\}}\right], \tag{2.11}$$

where $\mathbb{1}_A$ is the indicator function of set A.

Included in the functional are the following key exit r.v.'s:

B_μ - the observed stock value when it drops for the first time
$B_{\mu-1}$ - the highest stock value prior to its first drop
τ_μ - the observed first drop time
$\tau_{\mu-1}$ - the observed time of the highest stock value

Obviously, $\mathbb{1}_{\{\mu<\nu\}}$ is stochastically equivalent to $\mathbb{1}_{\{\tau_\mu<\tau_\nu\}}$ and thus it will serve as a substitute. This event $\{\tau_\mu < \tau_\nu\}$ means that the stock drops before the moment that exceeds the expiration date for the first time. Thus the functional Φ is related to the σ-algebra $\mathcal{F}_{\min\{\tau_\mu,\tau_\nu\}}$.

We will investigate the marginal functional

$$\phi(w,\vartheta) := \Phi(1,1,w,1,\vartheta,0) = E\left[e^{wB_{\mu-1}} e^{-\vartheta\tau_{\mu-1}} \mathbb{1}_{\{\mu<\nu\}}\right] \tag{2.12}$$

which gives the joint transform of the highest stock price before the first drop and prior to the expiration date at T of an option on which the stock is written.

Let us introduce the operator

$$\mathcal{D}_x^k \varphi(x,y) = \begin{cases} \lim_{x\to 0} \frac{1}{k!} \frac{\partial^k}{\partial x^k} \left[\frac{1}{1-x}\varphi(x,y)\right], & k \ge 0 \\ 0, & k < 0 \end{cases} \tag{2.13}$$

to be applied to a function $\varphi(x,y)$ analytic at zero in x.

We will also consider the Laplace-Carson transform

$$\mathcal{LC}_p(\cdot)(s) := s\int_{p=0}^{\infty} e^{-sp}(\cdot)dp, \quad \mathrm{Re}(s) > 0, \tag{2.14}$$

whose inverse obviously is

$$\mathcal{LC}_s^{-1}(\bullet)(p) = \mathfrak{L}^{-1}(\bullet\frac{1}{s}), \tag{2.15}$$

where $\mathfrak{L}^{-1}$ denotes the inverse of the Laplace transform.

The following main result is due to Dshalalow [3].

Theorem 2.1. *Let* $(\mathcal{A}, \mathcal{B}, \mathcal{T})$ *be a bivariate delayed marked renewal process with position dependent marking defined in* (2.1)–(2.10). *Let*

$$\Phi = E\big[z^{A_{\mu-1}} u^{A_\mu} e^{wB_{\mu-1}} e^{vB_\mu} e^{-\vartheta\tau_{\mu-1}} e^{-\theta\tau_\mu} \mathbb{1}_{\{\mu<\nu\}}\big].$$

Then,

$$\begin{aligned}\Phi = \mathcal{D}_x^{N-1}\mathcal{L}C_q^{-1}\Big\{&\gamma_0(u,v,\theta+q) - \gamma_0(ux,v,\theta+q)\\ &+ \frac{\gamma_0(zux, w+v, \vartheta+\theta+q)}{1-\gamma(zux, w+v, \vartheta+\theta+q)}\left(\gamma(u,v,\theta+q) - \gamma(ux,v,\theta+q)\right)\Big\}(T)\end{aligned} \tag{2.16}$$

where $\gamma(u, v, \theta)$ *and* $\gamma_0(u, v, \theta)$ *are defined in* (2.8) *and* (2.9).

As we mentioned it earlier, we let $N = 1$ which means the process will be observed prior to its first drop. Furthermore, as we are concerned with the marginal functional specified in (2.12), we set $z = u = 1$, $v = \theta = 0$. We also let the time observation and the initial stock value being constants, namely

$$\tau_0 = 0 \text{ and } Y_0 \geq 0 \text{ a.s.} \tag{2.17}$$

Since $N = 1$ and $\mathcal{D}_x^0\varphi(x, y) = \varphi(x, y)$, the $\mathcal{D}$ operator is eliminated and we obtain

(2.18)

$$\phi(w,\vartheta) := E\big[e^{wB_{\mu-1}} e^{-\vartheta\tau_{\mu-1}} \mathbb{1}_{\{\mu<\nu\}}\big]$$

(2.19)

$$= \mathcal{L}C_q^{-1}\Big\{\gamma_0(1,0,q) - \gamma_0(0,0,q) + \frac{\gamma_0(0,w,\vartheta+q)}{1-\gamma(0,w,\vartheta+q)}\big[\gamma(1,0,q) - \gamma(0,0,q)\big]\Big\}(T)$$

Under assumptions (2.17) we have

$$\gamma_0(z, v, \theta) = E\big[u^{X_0} e^{-vY_0} e^{-\theta\tau_0}\big] = e^{-vY_0} \tag{2.20}$$

$$\gamma_0(0,0,q) = E\big[\mathbb{1}_{Y_0 \geq 0\}}\big] = 1. \tag{2.21}$$

(2.18) requires the knowledge of

$$\gamma(0,w,\vartheta+q) = E\big[\mathbb{1}_{\{Y_1\geq 0\}} e^{-wY_1} e^{-(\vartheta+q)\Delta_1}\big]. \tag{2.22}$$

If Δj is exponentially distributed with parameter δ and Y_i has the Laplace distribution with the pdf

$$f_Y(x) = \frac{\lambda}{2}e^{-\lambda|x|}, x \in \mathbb{R}, \tag{2.23}$$

then we have

$$E\left[\mathbb{1}_{Y_1\geq 0\}}e^{-wY_1}e^{-(\vartheta+q)\Delta_1}\right]=\frac{\lambda}{2(\lambda+w)}\frac{\delta}{\delta+\vartheta+q}. \tag{2.24}$$

Furthermore,

$$\gamma(1,0,q)=E\left[\mathbb{1}^{X_0}e^{-0Y_0}e^{-q\Delta_1}\right]=\frac{\delta}{\delta+q} \tag{2.25}$$

$$\gamma(0,0,q)=E\left[\mathbb{1}_{\{Y_1\geq 0\}}\right]\frac{\delta}{\delta+q}=\frac{1}{2}\frac{\delta}{\delta+q}. \tag{2.26}$$

After substituting (2.20)–(2.26) in (2.18), $\phi(w,\vartheta)$ turns to

$$\phi(w,\vartheta)=\mathcal{L}C_q^{-1}\left\{\frac{1}{1-\frac{\lambda}{2(\lambda+w)}\frac{\delta}{\delta+\vartheta+q}}\left[\frac{\delta}{\delta+q}-\frac{1}{2}\frac{\delta}{\delta+q}\right]\right\}e^{-wY_0}(T) \tag{2.27}$$

With

$$\Lambda=\frac{\lambda}{2(\lambda+w)}, \tag{2.28}$$

$\phi(w,\vartheta)$ reads

$$\begin{aligned}\phi(w,\vartheta)&=e^{-wY_0}\mathcal{L}C_q^{-1}\left\{\frac{1}{1-\Lambda\frac{\delta}{\delta+\vartheta+q}}\left[\frac{\delta}{\delta+q}-\frac{1}{2}\frac{\delta}{\delta+q}\right]\right\}(T)\\&=\frac{\delta}{2}e^{-wY_0}\mathcal{L}C_q^{-1}\left\{\frac{1}{\delta+q}+\frac{\Lambda}{(\delta+q)[\delta(1-\Lambda)+\vartheta+q]}\right\}(T).\end{aligned}$$

Dividing $\phi(w,\vartheta)$ by q (which leads us to the inverse Laplace transform $\mathcal{L}_q^{-1}$), and then using the partial fraction decomposition yields

$$\begin{aligned}\frac{1}{q}\phi(w,\vartheta)&=\frac{\delta}{2}e^{-wY_0}\mathcal{L}_q^{-1}\left\{\frac{1}{q\delta}-\frac{1}{\delta(\delta+q)}+\frac{\Lambda}{q[\delta(1-\Lambda)+\vartheta]}-\frac{\Lambda}{(\delta+q)\,(\vartheta-\delta\Lambda)}\right.\\&\qquad\left.+\frac{\delta\Lambda}{[\delta(\Lambda-1)-\vartheta][\delta\Lambda-\vartheta][\delta(1-\Lambda)+\vartheta+q]}\right\}(T)\\&=\frac{\delta}{2}e^{-wY_0}\left\{\frac{1}{\delta}+\frac{\Lambda}{[\delta(1-\Lambda)+\vartheta]}-\frac{1}{\delta}e^{-\delta T}-\frac{\Lambda}{(\vartheta-\delta\Lambda)}e^{-\delta T}\right.\\&\qquad\left.+\frac{\delta\Lambda}{[\delta(\Lambda-1)-\vartheta][\delta\Lambda-\vartheta]}e^{-[\delta(1-\Lambda)+\vartheta]T}\right\}\end{aligned}$$

after restoring the original notation introduced in (2.28)

$$\begin{aligned}&=\frac{\delta}{2}e^{-wY_0}\left\{\frac{1}{\delta}+\frac{\frac{\lambda}{2(\lambda+w)}}{\delta\left(1-\frac{\lambda}{2(\lambda+w)}\right)+\vartheta}-\frac{1}{\delta}e^{-\delta T}-\frac{\frac{\lambda}{2(\lambda+w)}}{\vartheta-\frac{\lambda\delta}{2(\lambda+w)}}e^{-\delta T}\right.\\&\qquad\left.+\frac{\delta\frac{\lambda}{2(\lambda+w)}}{\left[\delta\left(\frac{\lambda}{2(\lambda+w)}-1\right)-\vartheta\right]\left[\delta\frac{\lambda}{2(\lambda+w)}-\vartheta\right]}e^{-\left[\delta\left(1-\frac{\lambda}{2(\lambda+w)}\right)+\vartheta\right]T}\right\}.\end{aligned} \tag{2.29}$$

3. The Marginal PDF of $B_{\mu-1}$

To find the marginal Laplace-Stieltjes transform of $B_{\mu-1}$ we let $\vartheta = 0$ in (2.29) followed by the Laplace inverse transform:

$$\begin{aligned} E[e^{wB_{\mu-1}}\mathbb{1}_{\mu<\nu}] &= \phi(w,0) \\ &= \frac{\delta}{2}e^{-wY_0}\Bigg\{\frac{1}{\delta} + \frac{\frac{\lambda}{2(\lambda+w)}}{\delta\left(1-\frac{\lambda}{2(\lambda+w)}\right)} - \frac{1}{\delta}e^{-\delta T} + \frac{\frac{\lambda}{2(\lambda+w)}}{\frac{\lambda\delta}{2(\lambda+w)}}e^{-\delta T} \\ &\quad + \frac{\delta\frac{\lambda}{2(\lambda+w)}}{\left[\delta\left(\frac{\lambda}{2(\lambda+w)}-1\right)\right]\left[\delta\frac{\lambda}{2(\lambda+w)}\right]}e^{-\left[\delta\left(1-\frac{\lambda}{2(\lambda+w)}\right)\right]T}\Bigg\} \\ &= \frac{1}{2}e^{-wY_0} + \frac{\lambda}{4(\frac{\lambda}{2}+w)}e^{-wY_0} - \left[1-\frac{w}{(\lambda+2w)}\right]e^{-wY_0}e^{-\left[\delta\left(1-\frac{\lambda}{2(\lambda+w)}\right)\right]T} \end{aligned} \tag{3.1}$$

(3.1) would yield the probability density function of $B_{\mu-1}$ jointly with the event $\{\mu<\nu\}$. To get the PDF (probability distribution function) we first divide (3.1) by w:

$$\begin{aligned} \frac{1}{w}E[e^{wB_{\mu-1}}\mathbb{1}_{\mu<\nu}] &= \frac{1}{2w}e^{-wY_0} + \frac{\lambda}{4w(\frac{\lambda}{2}+w)}e^{-wY_0} \\ &\quad - \left[\frac{1}{w} - \frac{1}{(\lambda+2w)}\right]e^{-wY_0}e^{-\left[\delta\left(1-\frac{\lambda}{2(\lambda+w)}\right)\right]T} \end{aligned} \tag{3.2}$$

and then use the partial fractions decomposition to have

$$\begin{aligned} \frac{1}{w}E[e^{wB_{\mu-1}}\mathbb{1}_{\mu<\nu}] &= \frac{1}{2w}e^{-wY_0} + \left[\frac{1}{2w} - \frac{1}{2(\frac{\lambda}{2}+w)}\right]e^{-wY_0} \\ &\quad - \left[\frac{1}{w} - \frac{1}{2(\frac{\lambda}{2}+w)}\right]e^{-wY_0}e^{-\left[\delta\left(1-\frac{\lambda}{2(\lambda+w)}\right)\right]T}. \end{aligned} \tag{3.3}$$

After some algebra, (3.3) reduces to

$$\begin{aligned} \frac{1}{w}E[e^{wB_{\mu-1}}\mathbb{1}_{\mu<\nu}] &= \frac{1}{w}e^{-wY_0} - \frac{1}{2(\frac{\lambda}{2}+w)}e^{-wY_0} - e^{-\delta T}\frac{1}{w}e^{-wY_0}e^{\frac{\frac{\lambda\delta}{2}T}{\lambda+w}} \\ &\quad + \frac{1}{2}e^{-\delta T}\frac{1}{\frac{\lambda}{2}+w}e^{-wY_0}e^{\frac{\frac{\lambda\delta}{2}T}{\lambda+w}}. \end{aligned} \tag{3.4}$$

To continue with the Laplace inverse of (3.4) we need the following

Lemma 3.1. *For three real numbers: $a>0$, b, c, it holds true that*

$$\mathcal{L}_y^{-1}\left\{\frac{exp\left(\frac{a}{y+b}\right)}{y+c}\right\}(q) = e^{-bq}I_0\left(2\sqrt{aq}\right) + (b-c)e^{-cq}\int_{u=0}^{q} e^{-(b-c)u}I_0\left(2\sqrt{au}\right)du, \tag{3.5}$$

where $I_0(x)$ is the modified Bessel function of order zero.

Proof. It is readily seen that

$$\frac{1}{y+c}=\frac{y+b}{(y+c)\,(y+b)}=\frac{y+c+b-c}{(y+c)\,(y+b)}=\frac{b-c}{(y+c)\,(y+b)}+\frac{1}{y+b} \tag{3.6}$$

Thus

$$\mathcal{L}_y^{-1}\Big\{\frac{\exp\left(\frac{a}{y+b}\right)}{y+c}\Big\}(q)=\mathcal{L}_y^{-1}\Big\{\frac{\exp\left(\frac{a}{y+b}\right)}{y+b}\Big\}+(b-c)\,\mathcal{L}_y^{-1}\Big\{\frac{\exp\left(\frac{a}{y+b}\right)}{y+b}\frac{1}{y+c}\Big\}. \tag{3.7}$$

In the first term due to Bateman and Erdélyi [1],

$$\mathcal{L}_y^{-1}\Big\{\frac{\exp\left(\frac{a}{y+b}\right)}{y+b}\Big\}=e^{-bq}I_0\left(2\sqrt{aq}\right). \tag{3.8}$$

In the second term, the expression $\frac{\exp\left(\frac{a}{y+b}\right)}{y+b}\frac{1}{y+c}$ can be regarded as a product of two Laplace transforms: $\frac{\exp\left(\frac{a}{y+b}\right)}{y+b}$ and $\frac{1}{y+c}$. Consequently, the inverse of $\frac{\exp\left(\frac{a}{y+b}\right)}{y+b}\frac{1}{y+c}$ will be the convolution of their respective inverses and thus it yields:

$$\begin{aligned}(b-c)\,\mathcal{L}_y^{-1}\Big\{\frac{\exp\left(\frac{a}{y+b}\right)}{y+b}\frac{1}{y+c}\Big\}&=(b-c)\,e^{-cq}\int_{u=0}^{q}e^{c(q-u)}e^{-bu}I_0\left(2\sqrt{au}\right)du\\&=(b-c)e^{-cq}\int_{u=0}^{q}e^{-(b-c)u}I_0\left(2\sqrt{au}\right)du.\end{aligned} \tag{3.9}$$

So, we are done with the proof of the lemma. □

Applying the Laplace inverse transform to $\frac{1}{w}E[e^{wB_{\mu-1}}\mathbb{1}_{\{\mu<\nu\}}]$ and using Lemma 1 we have

$$\begin{aligned}F_{B_{\mu-1}}(y)&:=P\{B_{\mu-1}\le y,\tau_\mu<\tau_\nu\}=\mathcal{L}_w^{-1}\Big\{\frac{1}{w}E\big[e^{wB_{\mu-1}}\mathbb{1}_{\{\mu<\nu\}}\big]\Big\}(y)\\&=\mathbb{1}_{(Y_0,\infty)}(y)\Big\{1-\frac{1}{2}e^{-\frac{\lambda}{2}(y-Y_0)}-e^{-\delta T}e^{-\lambda(y-Y_0)}I_0\Big(2\sqrt{\frac{\lambda\delta T}{2}\,(y-Y_0)}\Big)\\&\quad-\lambda e^{-\delta T}\int_{t=0}^{y-Y_0}e^{-\lambda t}I_0\Big(2\sqrt{\frac{\lambda\delta T}{2}t}\Big)dt\\&\quad+\frac{1}{2}e^{-\delta T}e^{-\lambda(y-Y_0)}I_0\Big(2\sqrt{\frac{\lambda\delta T}{2}\,(y-Y_0)}\Big)\\&\quad+\frac{\lambda}{4}e^{-\delta T}e^{-\frac{\lambda}{2}(y-Y_0)}\int_{t=0}^{y-Y_0}e^{-\frac{\lambda t}{2}}I_0\Big(2\sqrt{\frac{\lambda\delta T}{2}t}\Big)dt\Big\}.\end{aligned} \tag{3.10}$$

After simplifications we have

$$\begin{aligned} F_{B_{\mu-1}}(y) &= \mathcal{L}_w^{-1}\Big\{\frac{1}{w}E\big[e^{wB_{\mu-1}}\mathbb{1}_{\mu<\nu}\big]\Big\}(y) \\ &= \mathbb{1}_{(Y_0,\infty)}(y)\Big\{1-\frac{1}{2}e^{-\frac{\lambda}{2}(y-Y_0)}-\frac{1}{2}e^{-\delta T}e^{-\lambda(y-Y_0)}I_0\Big(2\sqrt{\frac{\lambda\delta T}{2}(y-Y_0)}\Big) \\ &\quad -\lambda e^{-\delta T}\int_{t=0}^{y-Y_0}e^{-\lambda t}I_0\Big(2\sqrt{\frac{\lambda\delta T}{2}t}\Big)dt \\ &\quad +\frac{\lambda}{4}e^{-\delta T}e^{-\frac{\lambda}{2}(y-Y_0)}\int_{t=0}^{y-Y_0}e^{-\frac{\lambda t}{2}}I_0\Big(2\sqrt{\frac{\lambda\delta T}{2}t}\Big)dt\Big\}. \end{aligned} \tag{3.11}$$

Notice here that

$$F_{B_{\mu-1}}(Y_0)=1-\frac{1}{2}-\frac{1}{2}e^{-\delta T}-0+0=\frac{1}{2}-\frac{1}{2}e^{-\delta T}>0.$$

The following assertion is a useful validation of (3.11).

Proposition 3.2. *$F_{B_{\mu-1}}(y)$ converges to $1-e^{-\frac{\delta}{2}T}$ as $y\to\infty$.*

Proof. It can be readily seen that as y approaches ∞, the second term in (3.11), $\frac{1}{2}e^{-\frac{\lambda}{2}(y-Y_0)}$ runs to 0. Next, $e^{-By}I_0\big(2\sqrt{Ay}\big)>0$ for all $y>0$ and $A>0, B>0$. Due to Dshalalow and Robinson [5, Lemma 2, formula (3.2))],

$$\int_{y=0}^{\infty}e^{-By}I_0\big(2\sqrt{Ay}\big)dy=\frac{1}{B}e^{\frac{A}{B}}. \tag{3.12}$$

Because the above integral converges we can conclude that the third term of (3.11),

$$\frac{1}{2}e^{-\delta T}e^{-\lambda(y-Y_0)}I_0\Big(2\sqrt{\frac{\lambda\delta T}{2}(y-Y_0)}\Big),$$

approaches 0 as y approaches ∞. The fourth term, however, approaches $e^{-\frac{\delta}{2}T}$, because due to (3.12),

$$\int_{t=0}^{\infty}e^{-\lambda t}I_0\Big(2\sqrt{\frac{\lambda\delta T}{2}t}\Big)dt=\frac{1}{\lambda}e^{\frac{\delta T}{2}}$$

and

$$\frac{1}{\lambda}e^{\frac{\delta T}{2}}\lambda e^{-\delta T}=e^{-\frac{\delta}{2}T}.$$

The fifth term of (3.11),

$$\frac{\lambda}{4}e^{-\delta T}e^{-\frac{\lambda}{2}(y-Y_0)}\int_{t=0}^{y}e^{-\frac{\lambda t}{2}}I_0\Big(2\sqrt{\frac{\lambda\delta T}{2}t}\Big)dt,$$

approaches 0, as $y\to\infty$, since $e^{-\frac{\lambda}{2}(y-Y_0)}\to 0$, and

$$\int_{t=0}^{\infty}e^{-\frac{\lambda t}{2}}I_0\Big(2\sqrt{\frac{\lambda\delta T}{2}t}\Big)dt=\frac{2}{\lambda}e^{\delta T}$$

due to (3.12). So we are left with $1-e^{-\frac{\delta}{2}T}$. □

In summary,

Theorem 3.3. *Under the condition of* (2.1)–(2.23)*, the marginal PDF for* $B_{\mu-1}$ *of the highest value* $B_{\mu-1}$ *of* $\mathcal{B}$ *prior to the first drop before time* T *satisfies formula* (3.11)*. In addition,* $F_{B_{\mu-1}}(y)$ *converges to* $1-e^{-\frac{\delta}{2}T}$ *as* $y\to\infty$*.*

4. The Marginal PDF of $\tau_{\mu-1}$

To find the marginal PDF of $\tau_{\mu-1}$ we let $w=0$ in (2.29) and then obtain the Laplace inverse transform:

$$\begin{aligned} E[e^{-\vartheta\tau_{\mu-1}}\mathbb{1}_{\mu<\nu}] = \phi(0,\vartheta) &= \frac{\delta}{2}\Big\{\frac{1}{\delta}+\frac{\frac{1}{2}}{[\delta(1-\frac{1}{2})+\vartheta]}-\frac{1}{\delta}e^{-\delta T}-\frac{\frac{1}{2}}{(\vartheta-\frac{\delta}{2})}e^{-\delta T} \\ &\quad + \frac{\delta\frac{1}{2}}{[\delta(\frac{1}{2}-1)-\vartheta][\delta\frac{1}{2}-\vartheta]}e^{-[\delta\left(1-\frac{1}{2}\right)+\vartheta]T}\Big\} \\ &= \frac{1}{2}+\frac{\delta}{4(\frac{\delta}{2}+\vartheta)}-\frac{1}{2}e^{-\delta T}-\frac{\delta}{4(\vartheta-\frac{\delta}{2})}e^{-\delta T}+\frac{\delta^2}{4(\vartheta+\frac{\delta}{2})(\vartheta-\frac{\delta}{2})}e^{-(\frac{\delta}{2}+\vartheta)T} \end{aligned}$$

$$\begin{aligned} \frac{1}{\vartheta}E[e^{-\vartheta\tau_{\mu-1}}\mathbb{1}_{\mu<\nu}] &= \frac{1}{2\vartheta}+\frac{\delta}{4\vartheta(\frac{\delta}{2}+\vartheta)}-\frac{1}{2\vartheta}e^{-\delta T}-\frac{\delta}{4\vartheta(\vartheta-\frac{\delta}{2})}e^{-\delta T} \\ &\quad+\frac{\delta^2}{4\vartheta(\vartheta+\frac{\delta}{2})(\vartheta-\frac{\delta}{2})}e^{-(\frac{\delta}{2}+\vartheta)T}. \end{aligned} \tag{4.1}$$

Applying partial fraction decomposition in (4.1) we have

$$\begin{aligned} \frac{1}{\vartheta}E[e^{-\vartheta\tau_{\mu-1}}\mathbb{1}_{\mu<\nu}] &= \frac{1}{2\vartheta}+\frac{1}{2\vartheta}-\frac{1}{2(\frac{\delta}{2}+\vartheta)}-\frac{1}{2\vartheta}e^{-\delta T}+\frac{1}{2\vartheta}e^{-\delta T}-\frac{1}{2(\vartheta-\frac{\delta}{2})}e^{-\delta T} \\ &\quad -\frac{1}{\vartheta}e^{-(\frac{\delta}{2}+\vartheta)T}+\frac{1}{2(\vartheta+\frac{\delta}{2})}e^{-(\frac{\delta}{2}+\vartheta)T}+\frac{1}{2(\vartheta-\frac{\delta}{2})}e^{-(\frac{\delta}{2}+\vartheta)T} \end{aligned} \tag{4.2}$$

which have (4.1) reduce to

$$\begin{aligned} \frac{1}{\vartheta}E[e^{-\vartheta\tau_{\mu-1}}\mathbb{1}_{\mu<\nu}] &= \frac{1}{\vartheta}-\frac{1}{2(\frac{\delta}{2}+\vartheta)}-\frac{1}{2(\vartheta-\frac{\delta}{2})}e^{-\delta T}-e^{-\frac{\delta}{2}T}\frac{1}{\vartheta}e^{-\vartheta T} \\ &\quad + e^{-\frac{\delta}{2}T}\frac{1}{2(\vartheta+\frac{\delta}{2})}e^{-\vartheta T}+e^{-\frac{\delta}{2}T}\frac{1}{2(\vartheta-\frac{\delta}{2})}e^{-\vartheta T} \end{aligned}$$

Applying the Laplace inverse transform to $\frac{1}{\vartheta}E[e^{-\vartheta\tau_{\mu-1}}\mathbb{1}_{\mu<\nu}]$ above we have

$$\begin{aligned} F_{\tau_{\mu-1}}(x) &= \mathcal{L}_\theta^{-1}\Big\{\frac{1}{\vartheta}E[e^{-\vartheta\tau_{\mu-1}}\mathbb{1}_{\mu<\nu}]\Big\}(x) \\ &= 1-\frac{1}{2}e^{-\frac{\delta}{2}x}-\frac{1}{2}e^{+\frac{\delta}{2}x}e^{-\delta T}-e^{-\frac{\delta}{2}T}\mathbb{1}_{(T,\infty)}(x) \\ &\quad +\frac{1}{2}e^{-\frac{\delta}{2}T}e^{-\frac{\delta}{2}(x-T)}\mathbb{1}_{(T,\infty)}(x)+\frac{1}{2}e^{-\frac{\delta}{2}T}e^{+\frac{\delta}{2}(x-T)}\mathbb{1}_{(T,\infty)}(x). \end{aligned}$$

After simplifications, we arrive at

$$\begin{aligned} F_{\tau_{\mu-1}}(x) &= 1-\frac{1}{2}e^{-\frac{\delta}{2}x}-\frac{1}{2}e^{+\frac{\delta}{2}x}e^{-\delta T}-e^{-\frac{\delta}{2}T}+e^{-\frac{\delta}{2}T}\mathbb{1}_{[0,T]}(x) \\ &\quad -\frac{1}{2}e^{-\frac{\delta}{2}T}\left[e^{-\frac{\delta}{2}(x-T)}+e^{+\frac{\delta}{2}(x-T)}\right]\mathbb{1}_{[0,T]}(x) \\ &\quad +\frac{1}{2}e^{-\frac{\delta}{2}T}\left[e^{-\frac{\delta}{2}(x-T)}+e^{+\frac{\delta}{2}(x-T)}\right] \end{aligned} \tag{4.3}$$

The PDF $F_{\tau_{\mu-1}}(x)$ can also be written as

$$F_{\tau_{\mu-1}}(x)=\begin{cases}1-\frac{1}{2}e^{-\frac{\delta}{2}x}-\frac{1}{2}e^{+\frac{\delta}{2}x}e^{-\delta T}\& \text{ if } 0\le x\le T \\ 1-e^{-\frac{\delta}{2}T} & \text{if } x>T\end{cases} \tag{4.4}$$

Notice here that $F_{\tau_{\mu-1}}(0)=\frac{1}{2}-\frac{1}{2}e^{-\delta T}$, which is greater than zero. Furthermore,

$$F_{\tau_{\mu-1}}(T)=1-e^{-\frac{\delta}{2}T}$$

which makes $F_{\tau_{\mu-1}}(x)$ continuous.

In summary,

Theorem 4.1. *Under the condition of* (2.1)–(2.23)*, the marginal PDF for* $\tau_{\mu-1}$ *of the highest value* $B_{\mu-1}$ *of* $\mathcal{B}$ *prior to the first drop and before the expiration time* T *satisfies formulas* (4.4)*. Furthermore, as* $x\to\infty$*,* $F_{\tau_{\mu-1}}(x)$ *approaches* $1-e^{-\frac{\delta}{2}T}$*.*

5. **Numerical Illustrations**

Example 5.1. In this example we plot the graph of $F_{\tau_{\mu-1}}(x)$ from (4.4). This is the probability that the highest value of the stock (under the initial value Y_0) will take place in x units of time or earlier, prior to its first drop and prior to time T. The underlying financial instrument is observed over time intervals which are exponentially distributed with parameter δ. Furthermore, the successive observations of increments are Laplace distributed with parameter λ. As we can see from (4.4), $F_{\tau_{\mu-1}}(x)$ does not depend on λ and Y_0.

Figure 2 below shows the graph of $F_{\tau_{\mu-1}}(x)$ for $T=10$ and $\delta=1$:

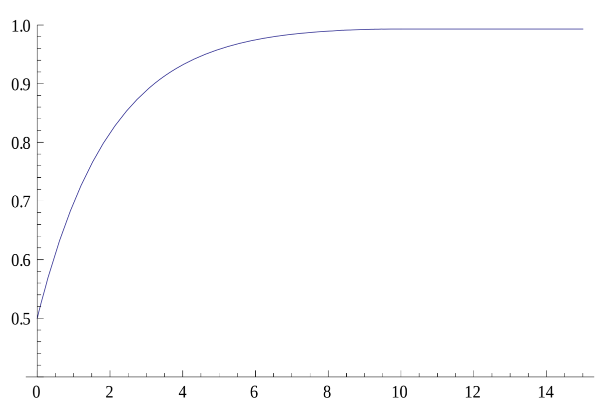

Figure 2

Figure 3 below shows the graph of $F_{\tau_{\mu-1}}(x)$ for $T = 10$ and $\delta = 0.25$

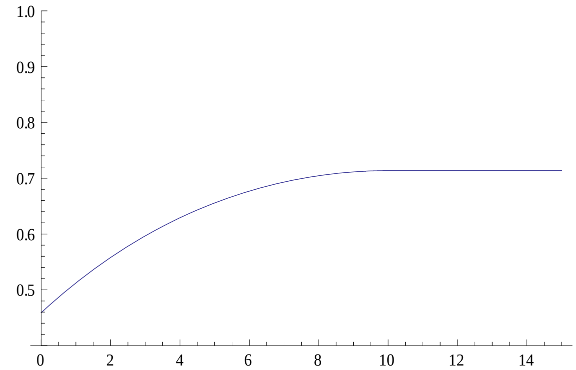

Figure 3

Example 5.2. In this example we plot the graph of $F_{B_{\mu-1}}(y)$ from (3.11). This is the probability that the highest stock value (the initial value Y_0) observed will be below y, before its first drop that occurs prior to expiration T. Figure 4 below shows the graph of $F_{B_{\mu-1}}(y)$ for, $Y_0 = 20$, $T = 10$, $\lambda = 0.2$ and $\delta = 1$

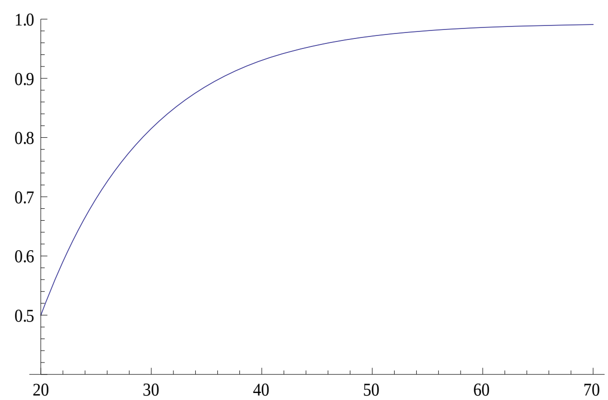

Figure 4

Figure 5 below shows the graph of $F_{B_{\mu-1}}(y)$ for $Y_0 = 20$, $T = 10$, $\lambda = 0.2$ and $\delta = 0.25$.

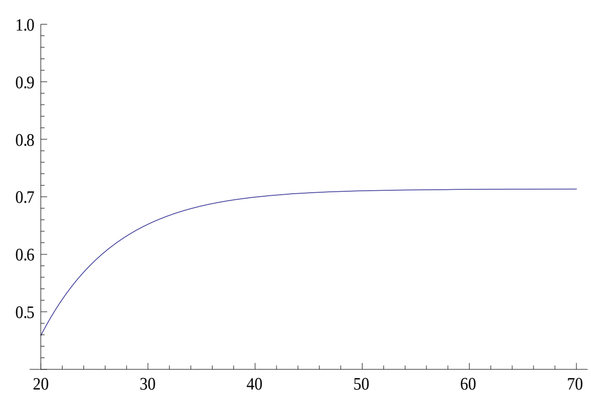

Figure 5

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