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# RANDOM ALGEBRAIC POLYNOMIALS WITH INCREASING VARIANCE OF THE COEFFICIENTS

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**ABSTRACT.** Let $Q_n(x) = \sum_{k=0}^{n} A_k x^k$ be a random algebraic polynomial in which the coefficients $A_0, A_1, A_2, \dots A_n$ form a sequence of independent normally distributed random variables with mean zero. In this paper we study the case where the variances of the coefficients $A_k$ are increasing in $k$, say $Var(A_k) = e^{k(2n-k)/(n\sqrt{n})}$, $k = 0, \dots n$. We show that the asymptotic behavior of the expected number of real zeros of $Q_n(x)$, in compare to the case of identically distributed coefficients, will increase to order $n^{1/4}$.

**Key words and Phrases:** random algebraic polynomial, number of real zeros, expected density, Gaussian coefficients

**AMS subject classifications:** Primary 60H42, Secondary 60G99

## 1. Introduction

Let $A_0, A_1, A_2, \dots$ be a sequence of independent normally distributed random variables. Also let $N_n(a, b)$ denote the number of real zeros of the polynomial

$$P_n(x) = \sum_{k=0}^{n} A_k x^k$$

in the interval $(a, b)$.

In the literature of random polynomials the coefficients are usually considered as i.i.d random variables and in all cases the expected densities have two poles on $-1$ and 1, when the coefficients have mean zero. In the cases where the coefficients have non-zero mean, then the expected density have only one pole at $-1$. One of the challenging problems in the study of random algebraic polynomials is to find which character of the coefficients could causes number of real zeros to increase.

Edelman and Kostlan [2], considered the case where the coefficients $A_k$, $k = 1, 2, \dots n$, are mean zero independent normally distributed with $Var(A_k) = \binom{n}{k}$, where the middle coefficients $A_k$ with $k$ close to $n/2$, have substantially larger variances.

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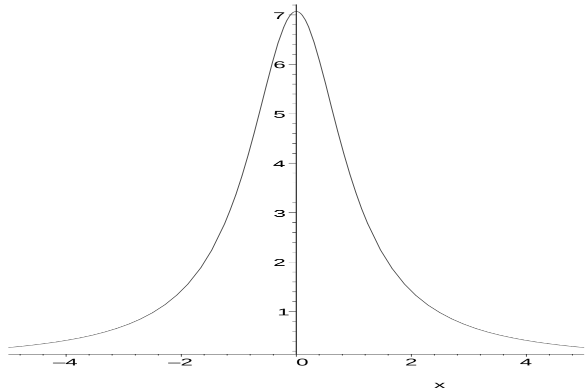


FIGURE 1. Expected density of $P_n(x)$

They showed that the expected number of real zeros of $P_n(x)$ is of order $\sqrt{n}$. They didn't report on the distribution of zeros. We find that in such a case zeros are concentrated around zero, and the expected density has only one pole at zero. The expected density is plotted in Figure 1.

In this paper we consider the case that the variance of the coefficients $A_k$ are increasing in $k$, say $Var(A_k) = e^{k(2n-k)/n\sqrt{n}}$, $k = 0, \ldots n$. By considering such a changes in the variances of the coefficients, we find that the expected number of real zeros increases from $\log(n)$, which is for polynomials with identically distributed coefficients, to the order of $n^{1/4}$.

There is a rich literature on the theory of the expected number of real zeros of random algebraic polynomials. This area of research was elaborated by the fundamental work of M. Kac [5]. The works of Logan and Shepp [6] Ibragimov and Maslova [4], K. Farahmand [3] and M. Sambandham [10, 11] are other fundamental contributions to the subject. There has been recent interest in cases where the coefficients form certain random processes, Rezakhah and Soltani [9]; Rezakhah and Shemehsavar [7, 8].

For the convenience from now on we consider the polynomial as

$$Q_n(x) = \sum_{k=0}^{n} e^{\frac{k(2n-k)}{2n\sqrt{n}}} A_k x^k \tag{1.1}$$

where the coefficients $A_k$ are i.i.d with standard normal distribution. Since the distribution of the $A_k$ and $-A_k$ ($k = 0, \ldots, n$ ) are the same, thus the transformation $Q_(x) \to Q_n(-x)$ preserve the coefficients distributions. So the expected density in general is symmetric about the origin, and

$$EN_n(0,1) = EN_n(-1,0), \qquad EN_n(-\infty,-1) = EN_n(1,\infty).$$

Hence $EN_n(-\infty,\infty) = 2EN_n(0,\infty)$.

## 2. **Expected density of real zeros**

In this section we present an explicit formula for the expected density of the number of real zeros of $Q_n(x)$ given by (1.1), expected density in short. The expected density is a nonnegative function $f_n$ on the set of real numbers for which

$$EN_n(a,b) = \int_a^b f_n(x)dx. \tag{2.1}$$

The Kac-Rice formula (see Logan and Shepp [6]) reveals that

$$f_n(x) = \int_{-\infty}^{\infty} |t|\, p(x,0,t)dt, \tag{2.2}$$

where $p(x,s,t)$ is the joint density of $(Q_n(x), Q_n'(x))$ at $(s,t)$.

By Cramer and Lead better [1, p. 285] we have that the expected density is given by

$$f_n(x) = \frac{1}{\pi}\frac{\left(A^2(x)G^2(x) - F^2(x)\right)^{1/2}}{A^2(x)}, \tag{2.3}$$

in which

$$\begin{aligned} A^2(x) &= \mathrm{Var}(Q_n(x)) = \sum_{k=0}^{n} a_k^2(x) e^{k(2n-k)/(n\sqrt{n})}, \\ G^2(x) &= \mathrm{Var}(Q_n'(x)) = \sum_{k=0}^{n} b_k^2(x) e^{k(2n-k)/(n\sqrt{n})} \\ F(x) &= \mathrm{Cov}(Q_n(x), Q_n'(x)) = \sum_{k=0}^{n} a_k(x) b_k(x) e^{k(2n-k)/(n\sqrt{n})}, \end{aligned} \tag{2.4}$$

where $a_k(x) = x^k$, and $b_k(x) = kx^{k-1}$, $k = 0, 1, \ldots, n$.

## 3. **Asymptotic behaviour of** $EN$

In this section we obtain the asymptotic behaviour of the expected number of real zeros of $Q_n(x)$ given by (1.1). We prove the following theorem

**Theorem 3.1.** *Let $Q_n(x)$ be the random algebraic polynomial given by* (1.1) *for which $A_k$, $k = 0, \ldots, n$ are i.i.d random variables with standard normal distribution, then the expected number of real zeros of $Q_n(x)$ satisfies:*

$$EN(1,\infty) = \frac{\sqrt{\pi-2}}{2\sqrt{2\pi}}\, n^{1/4} - 0.142024 + 0.015162\, n^{-1/4} + O(n^{-1/2}). \tag{3.1}$$

$$EN(0,1) = \frac{\sqrt{\pi-2}}{2\sqrt{2\pi}} n^{1/4} + 0.142024 + 0.015162\, n^{-1/4} + O(n^{-1/2}). \tag{3.2}$$

*and*

$$EN(-\infty,\infty) = \frac{2\sqrt{\pi-2}}{\sqrt{2\pi}} n^{1/4} + 0.060646\, n^{-1/4} + O(n^{-1/2}) \tag{3.3}$$

*Proof.* We begin by computing the asymptotic behaviour of the expected number of real zeros of $Q_n(x)$, given by (1.1), for the interval $(1,\infty)$.

For $1<x<\infty$, using the change of variable $x=1+\frac{t}{n}$, by (2.1) we have that $EN(1,\infty)=\frac{1}{n}\int_0^\infty f_n\big(1+\frac{t}{n}\big)dt$. As the calculation of the asymptotic behaviour of $f_n\big(1+\frac{t}{n}\big)$ is somehow complicated, using (2.3) and (2.4) we arrange this job by calculating the asymptotic behaviour of $A^2\big(1+\frac{t}{n}\big)$, $G^2\big(1+\frac{t}{n}\big)$, and $F\big(1+\frac{t}{n}\big)$, first. Now by the equality $\big(1+\frac{t}{n}\big)^n=e^t\left(1-\frac{t^2}{2n}\right)+O\left(n^{-2}\right)$, we have that

$$\begin{aligned} n^{-1}A^2\left(1+\frac{t}{n}\right) &= \sum_{k=0}^{n} e^{\frac{k(2n-k)}{n\sqrt{n}}}\left(\frac{n+t}{n}\right)^{2k} n^{-1} \simeq \int_0^1 e^{\frac{x(2n-nx)}{\sqrt{n}}}\left(\frac{n+t}{n}\right)^{2nx} dx \\ &= e^{n^{1/2}}\Bigg[\frac{\sqrt{\pi}}{2}e^{2t}n^{-1/4}-e^{2t}tn^{-1/2}+\frac{\sqrt{\pi}}{2}e^{2t}t^2n^{-3/4} \\ &\quad -\frac{2\,e^{2t}\,t^3}{3\,n}+e^{2t}t^2\frac{\sqrt{\pi}}{4}(t^2-2)n^{-5/4}+O(n^{-3/2})\Bigg] \end{aligned} \tag{3.4}$$

Similarly we have that

$$\begin{aligned} n^{-1}G^2\left(1+\frac{t}{n}\right) &= \sum_{k=0}^{n} e^{\frac{k(2n-k)}{n\sqrt{n}}}\left(\frac{n+t}{n}\right)^{2k-2} k^2n^{-1} \\ &\simeq \int_0^1 e^{\frac{x(2n-nx)}{\sqrt{n}}}\left(\frac{n+t}{n}\right)^{2nx-2} n^2x^2dx \\ &= e^{n^{1/2}}\Bigg[\frac{\sqrt{\pi}}{2}e^{2t}n^{7/4}-e^{2t}n^{3/2}-e^{2t}n^{3/2}t+\frac{\sqrt{\pi}}{2}e^{2t}n^{5/4}t^2 \\ &\quad +\sqrt{\pi}e^{2t}n^{5/4}t+\frac{\sqrt{\pi}}{4}e^{2t}n^{5/4}-e^{2t}tn-\frac{2}{3}e^{2t}t^3n-2e^{2t}t^2n \\ &\quad +e^{2t}t\frac{\sqrt{\pi}}{4}(4t^2+t^3+t-4)n^{3/4}+O\left(n^{1/2}\right)\Bigg] \end{aligned} \tag{3.5}$$

Finally we calculate the asymptotic behaviour of $F(\cdot)$ as

$$\begin{aligned} n^{-1}F\left(1+\frac{t}{n}\right) &= \sum_{k=0}^{n} e^{\frac{k(2n-k)}{n\sqrt{n}}}\left(\frac{n+t}{n}\right)^{2k-1} kn^{-1} \\ &\simeq \int_0^1 e^{\frac{x(2n-nx)}{\sqrt{n}}}\left(\frac{n+t}{n}\right)^{2nx-1} nxdx \\ &= e^{n^{1/2}}\Bigg[\frac{\sqrt{\pi}}{2}e^{2t}n^{3/4}-(t+0.5)e^{2t}\sqrt{n}+\frac{\sqrt{\pi}}{2}e^{2t}t(1+t)n^{1/4} \\ &\quad -\frac{2}{3}\,e^{2t}t^3-e^{2t}t^2+e^{2t}t\frac{\sqrt{\pi}}{4}(t^3-2t+2t^2-2)n^{-1/4}+O\left(n^{-1/2}\right)\Bigg] \end{aligned} \tag{3.6}$$

Using (2.3), (3.5),(3.6) and (3.7) we have that

$$n^{-1}f_n\left(1+\frac{t}{n}\right)=\frac{\sqrt{\pi-2}}{\sqrt{2\pi}n^{1/4}}-\frac{0.180831t}{\sqrt{n}}+\frac{0.028956t^2}{n^{3/4}}+O(n^{-1}) \tag{3.7}$$

We know that the expected number of real zeros of the polynomial (1.1) is atmost $n$, therefore the integral of $n^{-1}f_n(1+t/n)$, which is the expected number of real zeros of the polynomial (1.1) in the interval $(1,\infty)$, is also finite. On the other hand we see that (3.8) is not term by term integrable, so we use the following equalities

$$\frac{a}{n^{1/4}} = \frac{a}{n^{1/4}(1+t^2/n)} + O\left(n^{-5/4}\right), \tag{3.8}$$

$$bt^k = \frac{bt^k}{1+t^{2(k+1)}/n} + O\left(n^{-1}\right),$$

Using (3.8)–(3.9) and equalities

$$\frac{1}{\pi n^{1/4}} \int_0^\infty \frac{\sqrt{\pi-2}}{\sqrt{2\pi}(1+t^2/n)}\,dt = \frac{\sqrt{\pi-2}}{2\sqrt{2\pi}}\,n^{1/4}, \tag{3.9}$$

$$\int_0^\infty \frac{bt^k}{1+t^{2(k+1)}/n}\,dt = \frac{b\pi}{2(k+1)}\sqrt{n}$$

we evaluate the asymptotic behaviour of the expected number of real zeros of $Q_n(x)$ as

$$\begin{aligned} EN(1,\infty) &= \frac{1}{n\pi}\int_0^\infty f_n\left(1+\frac{t}{n}\right)dt \\ &= \frac{\sqrt{\pi-2}}{2\sqrt{2\pi}}\,n^{1/4} - 0.142024 + 0.015162n^{-1/4} + O(n^{-1/2}). \end{aligned} \tag{3.10}$$

This is by the fact that error terms in (3.9) are polynomials in $t$, so for integration in (3.11), (3.10) implies that orders in $n$ comes out to be $\sqrt{n}$ times of the corresponding order of the integrand. Thus we arrive at the first assertion of the theorem, i.e (3.2).

Now we are to evaluate the asymptotic behaviour of the expected number of real zeros of $Q_n(x)$ for the interval $(0,1)$. Using the change of variable $x = 1 - \frac{t}{n+t}$, by (2.1) we have that $EN(0,1) = \int_0^\infty \frac{n}{(n+t)^2} f_n\left(1-\frac{t}{n+t}\right)dt$. For the calculation of the asymptotic behaviour of $f_n\big(1-\frac{t}{n+t}\big)$, using (2.3) and (2.4) we calculate the asymptotic behaviour of $A^2\big(1-\frac{t}{n+t}\big)$, $G^2\big(1-\frac{t}{n+t}\big)$, and $F\big(1-\frac{t}{n+t}\big)$, first. Thus we have that

$$\begin{aligned} n^{-1}A^2\left(1-\frac{t}{n+t}\right) &= \sum_{k=0}^{n} e^{\frac{k(2n-k)}{n\sqrt{n}}}\left(\frac{n}{n+t}\right)^{2k} n^{-1} \simeq \int_0^1 e^{\frac{x(2n-nx)}{\sqrt{n}}}\left(\frac{n}{n+t}\right)^{2nx} dx \\ &= e^{n^{1/2}}\Bigg[\frac{\sqrt{\pi}}{2}e^{-2t}n^{-1/4} + e^{-2t}tn^{-1/2} + \frac{\sqrt{\pi}}{2}e^{-2t}t^2n^{-3/4} + \frac{2e^{-2t}t^3}{3n} \\ &\quad + e^{-2t}t^2\frac{\sqrt{\pi}}{4}(t^2+2)n^{-5/4} + O(n^{-3/2})\Bigg] \end{aligned} \tag{3.11}$$

Similarly we have that

$$n^{-1}G^2\left(1-\frac{t}{n+t}\right) = \sum_{k=0}^{n} e^{\frac{k(2n-k)}{n\sqrt{n}}}\left(\frac{n}{n+t}\right)^{2k-2} k^2 n^{-1}$$

$$\simeq \int_0^1 e^{\frac{x(2n-nx)}{\sqrt{n}}}\left(\frac{n}{n+t}\right)^{2\,nx-2} n^2x^2dx \tag{3.12}$$

$$= e^{n^{1/2}}\Bigg[\frac{\sqrt{\pi}}{2}e^{-2t}n^{7/4} - e^{-2t}(1-t)n^{3/2} + \frac{e^{-2t}\sqrt{\pi}}{4}(1-4t+2t^2)n^{5/4} + \frac{e^{-2t}t}{3}(3-6t+2t^2)n + \frac{e^{-2t}t\sqrt{\pi}}{4}(t^3-4t^2+5t+4)n^{3/4} + O(n^{1/2})\Bigg]$$

Finally we calculate the asymptotic behaviour of $F(\cdot)$ as

$$n^{-1}F\left(1-\frac{t}{n+t}\right) = \sum_{k=0}^{n} e^{\frac{k(2n-k)}{n\sqrt{n}}}\left(\frac{n}{n+t}\right)^{2\,k-1} kn^{-1}$$

$$\simeq \int_0^1 e^{\frac{x(2n-nx)}{\sqrt{n}}}\left(\frac{n}{n+t}\right)^{2\,nx-1} nxdx$$

$$= e^{n^{1/2}}\Bigg[\frac{\sqrt{\pi}}{2}e^{-2t}n^{3/4} + \frac{e^{-2t}}{2}(2t-1)\sqrt{n} + \frac{\sqrt{\pi}e^{-2t}}{2}(t^2-t)n^{1/4} + \frac{e^{-2t}}{3}(2t^3-3t^2) + \frac{e^{-2t}t\sqrt{\pi}}{4}(t^3-2t^2+2t+2)n^{-1/4} + O(n^{-1/2})\Bigg]$$

Using (2.3), (3.12), (3.13) and (3.14) we have that

$$\frac{n}{(n+t)^2}f_n\left(1-\frac{t}{n+t}\right) = \frac{\sqrt{\pi-2}}{\sqrt{2\pi}n^{1/4}} + \frac{0.180831t}{\sqrt{n}} + \frac{0.028956t^2}{n^{3/4}} + O(n^{-1}) \tag{3.13}$$

We know that the expected number of real zeros is atmost $n$, so the expected density $\frac{n}{(n+t)^2}f_n(1-\frac{t}{n+t})$ is integrable. On the other hand we see that (3.15) is not term by term integrable, so using (3.9), (3.10) and (3.15) we evaluate the asymptotic behaviour of the expected number of real zeros of $Q_n(x)$ as

$$EN(0,1) = \frac{1}{\pi}\int_0^{\infty}\frac{n}{(n+t)^2}f_n(1-\frac{t}{n+t})dt \tag{3.14}$$

$$= \frac{\sqrt{\pi-2}}{2\sqrt{2\pi}}n^{1/4} + 0.142024 + 0.015162\,n^{-1/4} + O(n^{-1/2}).$$

For the error term, the same reason as for error term of (3.11) can be applied. So we arrive at the second assertion of the theorem, i.e (3.3), and the theorem is proved. □

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