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CALCULATING ZEROS OF THE SECOND KIND (h, q)-BERNOULLI POLYNOMIALS

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ABSTRACT. In [11], we observed the behavior of real roots of the second kind Bernoulli polynomials. The main purpose of this paper is to investigate the zeros of the second kind (h, q)-Bernoulli polynomials $B_{n,q}^{(h)}(x)$ for $-1 < q < 0$. Furthermore, we give a table for the solutions of the second kind (h, q)-Bernoulli polynomials $B_{n,q}^{(h)}(x)$ for $-1 < q < 0$.

AMS (MOS) Subject Classification. 11B68, 11S40, 11S80

1. Introduction

Recently, several mathematicians have studied the Bernoulli numbers and polynomials, the Euler numbers and polynomials, the q-Bernoulli numbers and polynomials, and the second kind q-Bernoulli numbers and polynomials (see [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11]). These numbers and polynomials possess many interesting properties and arises in many areas of mathematics and physics. In the 21st century, the computing environment will make more and more rapid progress. The importance of numerical simulation and analysis in mathematics is steadily increasing. Using computer, a realistic study for these numbers and polynomials is very interesting. It is the aim of this paper to observe an interesting phenomenon of 'scattering' of the zeros of the second kind (h, q)-Bernoulli polynomials $B_{n,q}^{(h)}(x)$ for $-1 < q < 0$ in the complex plane. The main motivation of this paper is summarized as follows: We introduce the second kind (h, q)-Bernoulli numbers $B_{n,q}^{(h)}$ and polynomials $B_{n,q}^{(h)}(x)$. In Section 2, using a numerical investigation, we observe the beautiful zeros of the second kind q-Bernoulli polynomials $B_{n,q}^{(h)}(x)$ for $-1 < q < 0$. Finally, we investigate the structure of the roots of the second kind (h, q)-Bernoulli polynomials $B_{n,q}^{(h)}(x)$ for $-1 < q < 0$. In Section 3, we shall discuss the more general open problems and observations. Throughout this paper, we always make use of the following notations: $\mathbb{N} = \{1, 2, 3, \dots\}$ denotes the set of natural numbers, $\mathbb{Z}$ denotes the set of integers, $\mathbb{R}$ denotes the set of real numbers, and $\mathbb{C}$ denotes the set of complex numbers.

First, we introduce the second kind (h, q)-Bernoulli numbers and Bernoulli polynomials. The second kind Bernoulli numbers B_n are defined by means of the following

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generating function:

$$F(t)=\frac{2te^t}{e^{2t}-1}=\sum_{n=0}^{\infty}B_n\frac{t^n}{n!},\quad (|t|<\pi). \tag{1.1}$$

The second kind Bernoulli polynomials $B_n(x)$ of degree n in x, are defined by means of the following generating function:

$$F(x,t)=\frac{2te^t}{e^{2t}-1}e^{xt}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!}. \tag{1.2}$$

Let q be a complex number with $|q|<1$ and $h\in\mathbb{Z}$. By the meaning of (1.1), let us define the second kind (h,q)-Bernoulli numbers $B_{n,q}^{(h)}$ as follows:

$$F_q^{(h)}(t)=\frac{2te^t}{q^he^{2t}-1}=\sum_{n=0}^{\infty}B_{n,q}^{(h)}\frac{t^n}{n!}. \tag{1.3}$$

By the meaning of (1.2), let us define the second kind (h,q)-Bernoulli polynomials $B_{n,q}^{(h)}(x)$ as follows:

$$F_q^{(h)}(x,t)=\frac{2te^t}{q^he^{2t}-1}e^{xt}=\sum_{n=0}^{\infty}B_{n,q}^{(h)}(x)\frac{t^n}{n!}. \tag{1.4}$$

Taking $h=0$ in (1.4), we have $B_{n,q}^{(0)}(x)=B_n(x)$.

From (1.3), we have

$$\frac{2te^t}{q^he^{2t}-1}=\sum_{n=0}^{\infty}B_{n,q}^{(h)}\frac{t^n}{n!}=e^{B_q^{(h)}t},$$

which yields

$$2t=q^he^{(B_q^{(h)}+1)t}-e^{(B_q^{(h)}-1)t}.$$

Using Taylor's expansion of exponential function in the above, we obtain

$$2t=\sum_{n=0}^{\infty}\left(q^h(B_q^{(h)}+1)^n-(B_q^{(h)}-1)^n\right)\frac{t^n}{n!}.$$

By comparing the coefficients, we have the following theorem.

Theorem 1. *The second kind (h,q)-Bernoulli numbers $B_{n,q}^{(h)}$ are defined respectively by*

$$q^h(B_q^{(h)}+1)^n-(B_q^{(h)}-1)^n=\begin{cases}2, & \text{if } n=1,\\ 0, & \text{if } n\neq 1.\end{cases}$$

We obtain the first value of the second kind (h,q)-Bernoulli numbers $B_{n,q}^{(h)}$:

$$B_{1,q} = \frac{2}{-1+q^h},$$
$$B_{2,q} = -\frac{8q^h}{(-1+q^h)^2} + \frac{4}{-1+q^h},$$
$$B_{3,q} = \frac{48q^{2h}}{(-1+q^h)^3} - \frac{48q^h}{(-1+q^h)^2} + \frac{6}{-1+q^h},$$
$$B_{4,q} = -\frac{384q^{3h}}{(-1+q^h)^4} + \frac{576q^{2h}}{(-1+q^h)^3} - \frac{208q^h}{(-1+q^h)^2} + \frac{8}{-1+q^h},$$
$$B_{4,q} = \frac{3840q^{4h}}{(-1+q^h)^5} - \frac{7680q^{3h}}{(-1+q^h)^4} + \frac{4640q^{2h}}{(-1+q^h)^3} - \frac{800q^h}{(-1+q^h)^2} + \frac{10}{-1+q^h}, \cdots.$$

By (1.4), we obtain

$$\begin{aligned}
\sum_{l=0}^{\infty} B_{l,q}^{(h)}(x)\frac{t^l}{l!} &= \frac{2te^t}{q^h e^{2t}-1}e^{xt} \\
&= \sum_{n=0}^{\infty} B_{n,q}^{(h)}\frac{t^n}{n!}\sum_{m=0}^{\infty} x^m \frac{t^m}{m!} \\
&= \sum_{l=0}^{\infty}\left(\sum_{n=0}^{l} B_{n,q}^{(h)}\frac{t^n}{n!}x^{l-n}\frac{t^{l-n}}{(l-n)!}\right) \\
&= \sum_{l=0}^{\infty}\left(\sum_{n=0}^{l}\binom{l}{n} B_{n,q}^{(h)}x^{l-n}\right)\frac{t^l}{l!}.
\end{aligned}$$

By comparing coefficients $\frac{t^l}{l!}$, we have the following theorem.

Theorem 2. *For any positive integer n, we have*

$$B_{n,q}^{(h)}(x) = \sum_{k=0}^{n}\binom{n}{k} B_{k,q}^{(h)}x^{n-k}.$$

By using computer, the second kind (h,q)-Bernoulli polynomials $B_{n,q}^{(h)}(x)$ can be determined explicitly. We obtain the first value of the second kind (h,q)-Bernoulli numbers $B_{n,q}^{(h)}(x)$:

$$B_{1,q}(x) = \frac{2}{-1+q^h},$$
$$B_{2,q}(x) = \frac{4x}{-1+q^h} - \frac{8q^h}{(-1+q^h)^2} + \frac{4}{-1+q^h},$$
$$B_{3,q}(x) = \frac{6x^2}{-1+q^h} - \frac{24q^h x}{(-1+q^h)^2} + \frac{12x}{-1+q^h} + \frac{48q^{2h}}{(-1+q^h)^3} - \frac{48q^h}{(-1+q^h)^2} + \frac{6}{-1+q^h}, \cdots.$$

Let m be a positive integer. We compute the following sum

$$
\begin{aligned}
\sum_{n=0}^{\infty} B_{n,q}^{(h)}(x)\frac{t^n}{n!} &= \frac{2te^t}{q^h e^{2t}-1}e^{xt} \\
&= \sum_{a=0}^{m-1} q^{ha}\frac{2te^{mt}}{e^{2mt}-1}e^{(2a+x+1-m)t} \\
&= \frac{1}{m}\sum_{a=0}^{m-1} q^{ha}\frac{2(mt)}{e^{mt}-e^{-mt}}e^{\left(\frac{2a+x+1-m}{m}\right)mt} \qquad (1.5)\\
&= \sum_{a=0}^{m-1}\left(\frac{1}{m}\sum_{n=0}^{\infty} q^{ha}B_{n,q^m}\left(\frac{2a+x+1-m}{m}\right)\frac{(mt)^n}{n!}\right) \\
&= \sum_{n=0}^{\infty}\left(m^{n-1}\sum_{a=0}^{m-1} q^{ha}B_{n,q^m}\left(\frac{2a+x+1-m}{m}\right)\right)\frac{t^n}{n!}.
\end{aligned}
$$

Comparing the coefficient of $\frac{t^n}{n!}$ on both sides of (1.5), we obtain the following multiplication theorem.

Theorem 3. *For any positive integer m, we obtain*

$$
B_{n,q}^{(h)}(x) = m^{n-1}\sum_{i=0}^{m-1} q^{hi}B_{n,q^m}^{(h)}\left(\frac{2i+x+1-m}{m}\right) \quad \textit{for } n \geq 0.
$$

Since

$$
\begin{aligned}
\sum_{l=0}^{\infty} B_{l,q}^{(h)}(x+y)\frac{t^l}{l!} &= \frac{2te^t}{q^h e^{2t}-1}e^{(x+y)t} \\
&= \sum_{n=0}^{\infty} B_{n,q}^{(h)}(x)\frac{t^n}{n!}\sum_{m=0}^{\infty} y^m\frac{t^m}{m!} \\
&= \sum_{l=0}^{\infty}\left(\sum_{n=0}^{l} B_{n,q}^{(h)}(x)\frac{t^n}{n!}y^{l-n}\frac{t^{l-n}}{(l-n)!}\right) \\
&= \sum_{l=0}^{\infty}\left(\sum_{n=0}^{l}\binom{l}{n}B_{n,q}^{(h)}(x)y^{l-n}\right)\frac{t^l}{l!},
\end{aligned}
$$

we have the following additional theorem.

Theorem 4. *The second kind (h,q)-Bernoulli polynomials $B_{n,q}^{(h)}(x)$ satisfies the following relation:*

$$
B_{l,q}^{(h)}(x+y) = \sum_{n=0}^{l}\binom{l}{n}B_{n,q}^{(h)}(x)y^{l-n}. \qquad (1.6)
$$

By (1.4), we have

$$\begin{aligned}\sum_{n=0}^{\infty} B_{n,q^{-1}}^{(h)}(-x)\frac{(-t)^n}{n!} &= \frac{-2te^{-t}}{q^{-h}e^{-2t}-1}e^{(-x)(-t)} \\ &= \frac{2q^h te^t}{q^h e^{2t}-1}e^{xt} \\ &= \sum_{n=0}^{\infty} B_{n,q}^{(h)}(x)\frac{t^n}{n!}.\end{aligned}$$

By comparing coefficients $\frac{t^n}{n!}$, we have the following theorem.

Theorem 5. *For $n \in \mathbb{N}$, we have*

$$B_{n,q}^{(h)}(x) = (-1)^n q^{-h} B_{n,q^{-1}}^{(h)}(-x).$$

From (1.4), we have

(1.7)

$$\begin{aligned}\sum_{n=0}^{\infty}\left(q^h B_{n,q}^{(h)}(x+2) - B_{n,q}^{(h)}(x)\right)\frac{t^n}{n!} &= -2t\sum_{n=0}^{\infty} q^{h(n+1)}e^{(2n+1+x+2)t} + 2t\sum_{n=0}^{\infty} q^{hn}e^{(2n+1+x)t} \\ &= 2te^{(x+1)t} \\ &= 2t\sum_{n=0}^{\infty}(x+1)^n\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} 2n(x+1)^{n-1}\frac{t^n}{n!}.\end{aligned}$$

Comparing the coefficient of $\frac{t^n}{n!}$ on both sides of (1.7), we get the following theorem.

Theorem 6. *For any positive integer n, we have*

$$q^h B_{n,q}^{(h)}(x+2) - B_{n,q}^{(h)}(x) = 2n(x+1)^{n-1}. \tag{1.8}$$

2. Zeros of the second kind (h,q)-Bernoulli polynomials

In this section, we observed the behavior of real roots of the second kind (h,q)-Bernoulli polynomials $B_{n,q}^{(h)}(x)$ for $-1 < q < 0$. We assume that $q \in \mathbb{C}$, with $-1 < q < 0$ and h is an odd positive integer. We display the shapes of the second kind (h,q)-Bernoulli polynomials $B_{n,q}^{(h)}(x)$ and we investigate the zeros of the second kind (h,q)-Bernoulli polynomials $B_{n,q}^{(h)}(x)$ for $-1 < q < 0$. For $n = 1, \ldots, 10$, we can draw a plot of the second kind (h,q)-Bernoulli polynomials $B_{n,q}^{(h)}(x)$, respectively. This shows the ten plots combined into one. We display the shape of $B_{n,q}^{(h)}(x), -10 \leq x \leq 10$, $q = -1/2$ (Figure 1).

We investigate the beautiful zeros of the second kind (h,q)-Bernoulli polynomials $B_{n,q}^{(h)}(x)$ by using a computer. We plot the zeros of the second kind (h,q)-Bernoulli

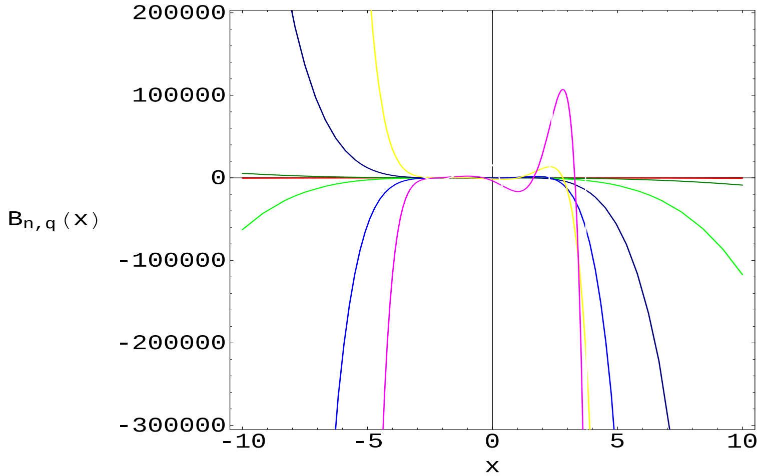

FIGURE 1. Curve of $B_{n,q}^{(h)}(x)$

polynomials $B_{n,q}^{(h)}(x)$ for $n = 15, 20, 25, 30$ and $x \in \mathbb{C}$ (Figure 2). In Figure 2 (top-left), we choose $n = 15$, $h = 3$ and $q = -1/2$. In Figure 2 (top-right), we choose $n = 20$, $h = 3$ and $q = -1/2$. In Figure 2 (bottom-left), we choose $n = 25$, $h = 3$ and $q = -1/2$. In Figure 2 (bottom-right), we choose $n = 30$, $h = 3$ and $q = -1/2$. Stacks of zeros of $B_{n,q}^{(h)}(x)$ for $1 \le n \le 30$ from a 3-D structure are presented (Figure 2).

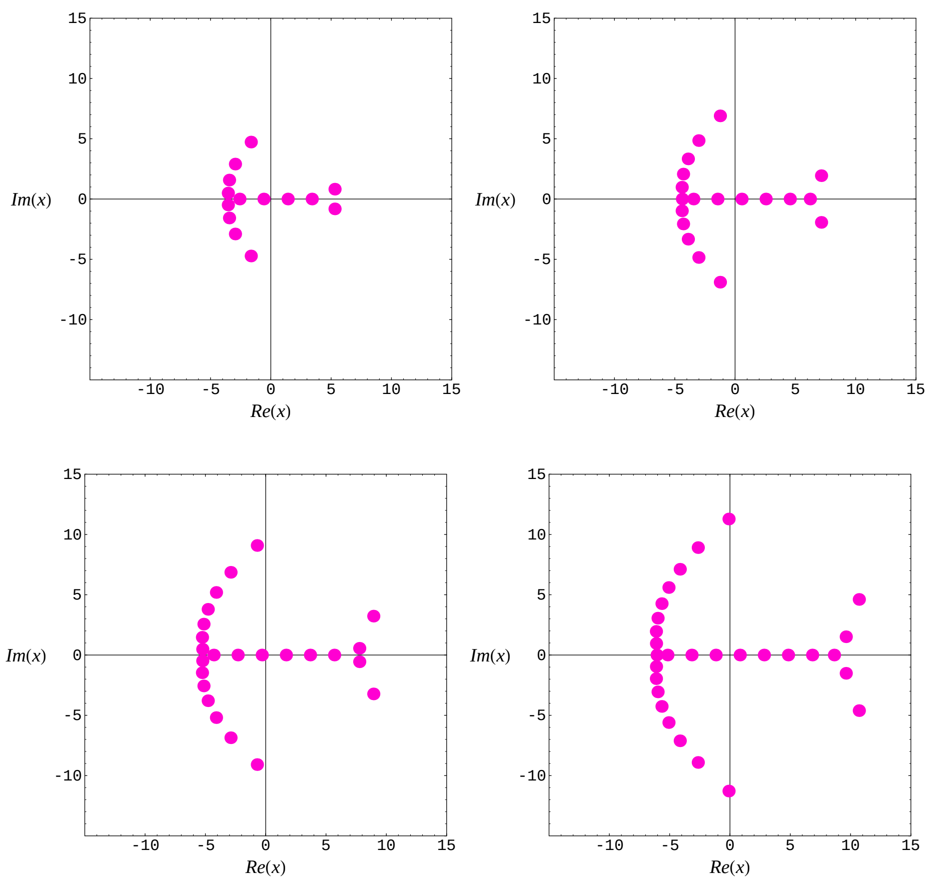

FIGURE 2. Zeros of $B_{n,q}^{(h)}(x)$ for $n = 15, 20, 25, 30$

Our numerical results for approximate solutions of real zeros of $B^{(h)}_{n,q}(x)$ are displayed (Tables 1, 2).

Table 1. Numbers of real and complex zeros of $B^{(5)}_{n,q}(x)$

	$q = -1/3$		$q = -1/2$	
degree n	real zeros	complex zeros	real zeros	complex zeros
2	1	0	1	0
3	2	0	2	0
4	1	2	1	2
5	2	2	2	2
6	1	4	3	2
7	2	4	2	4
8	3	4	3	4
9	2	6	4	4
10	3	6	3	6
11	2	8	4	6
12	3	8	3	8

We plot the zeros of the second kind (h, q)-Bernoulli polynomials $B^{(h)}_{n,q}(x)$ for $n = 30$, $q = -1/2$, $h = 5, 7, 9, 10$ and $x \in \mathbb{C}$ (Figure 3). In Figure 3 (top-left), we choose $n = 30, q = -1/2$ and $h = 5$. In Figure 3 (top-right), we choose $n = 30, q = -1/2$ and $h = 7$. In Figure 3 (bottom-left), we choose $n = 30, q = -1/2$ and $h = 9$. In Figure 3 (bottom-right), we choose $n = 30, q = -1/2$ and $h = 11$. We plot the zeros of the second kind (h, q)-Bernoulli polynomials $B^{(h)}_{n,q}(x)$ for $n = 30$, $q = -1/10, -3/10, -7/10, -9/10$ and $x \in \mathbb{C}$ (Figure 4). In Figure 4 (top-left), we choose $n = 30, h = 5$ and $q = -1/10$. In Figure 4 (top-right), we choose $n = 30, h = 5$ and $q = -3/10$. In Figure 4 (bottom-left), we choosen$n = 30, h = 5$ and $q = -7/10$. In Figure 4 (bottom-right), we choose $n = 30, h = 5$ and $q = -9/10$. We plot the zeros of $\lim_{q\to-1} B^{(h)}_{n,q}(x) = -G_n(x)$ for $n = 30$ (Figures 5, 6, see [10]).

Stacks of zeros of $B^{(h)}_{n,q}(x)$ for $1 \leq n \leq 30, h = 5$ and $q = -1/2$ from a 3-D structure are presented (Figure 7). We observe a remarkably regular structure of the complex roots of the second kind (h, q)-Bernoulli polynomials $B^{(h)}_{n,q}(x)$. We hope to verify a remarkably regular structure of the complex roots of the second kind (h, q)-Bernoulli polynomials $B^{(h)}_{n,q}(x)$ (Table 1). This numerical investigation is especially exciting because we can obtain an interesting phenomenon of scattering of the zeros of the second kind (h, q)-Bernoulli polynomials $B^{(h)}_{n,q}(x)$. These results are used not only in pure mathematics and applied mathematics, but also used in mathematical physics and other areas. Next, we calculated an approximate solution satisfying the second kind (h, q)-Bernoulli polynomials $B^{(h)}_{n,q}(x)$. The results are given in Table 2.

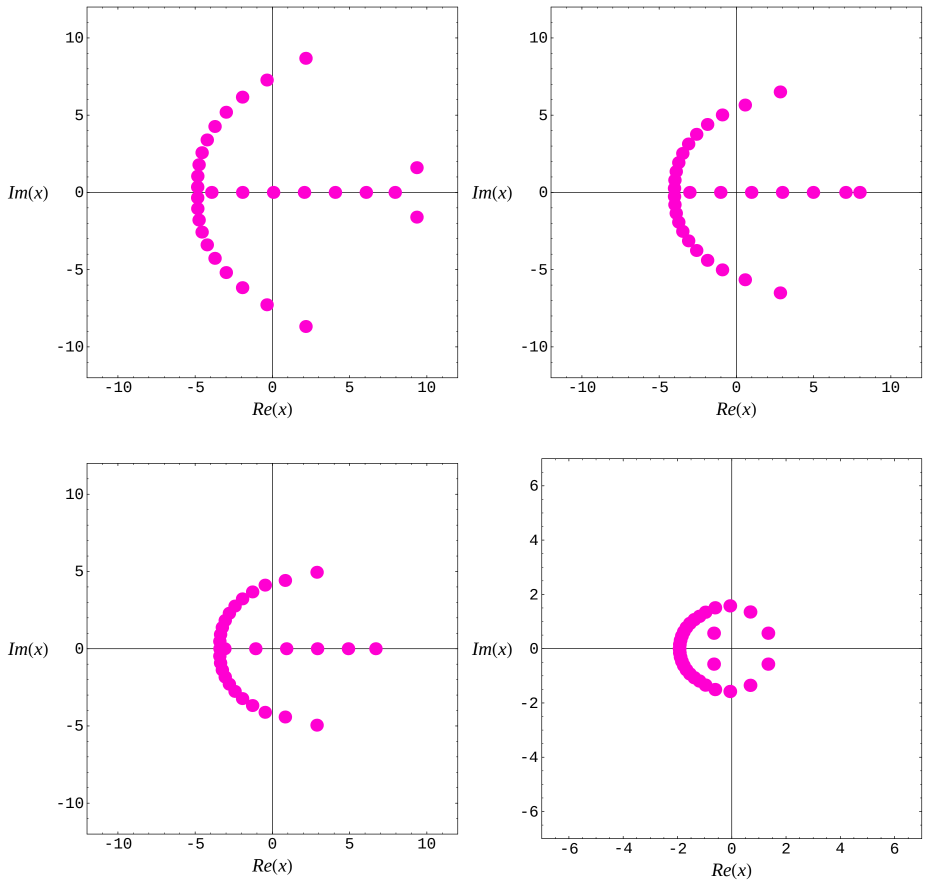

FIGURE 3. Zeros of $B_{n,q}^{(h)}(x)$ for $h = 5, 7, 9, 11$

Table 2. Approximate solutions of $B_{n,q}^{(5)}(x) = 0, q = -1/3, x \in \mathbb{R}$

degree n	x
2	-0.991803279
3	-1.11957752, -0.86402904
4	-0.62220737
5	-1.3955514, -0.33032250
6	-0.015949330
7	-1.506094, 0.309195809
8	-1.62572, -1.363323, 1.747718579
9	-1.028061, 0.971939482
10	-1.87787, -0.6951874, 1.304810878

Figure 8 shows the distribution of real zeros of $B_{n,q}^{(h)}(x)$ for $1 \leq n \leq 30$. In Figure 8 (top-left), we choose $h = 5$ and $q = -1/10$. In Figure 8 (top-right), we choose $h = 5$ and $q = -5/10$. In Figure 8 (bottom-left), we choose$h = 5$ and $q = -9/10$. In Figure 8 (bottom-right), we choose $h = 5$ and $q = -999/1000$. Figure 9 presents

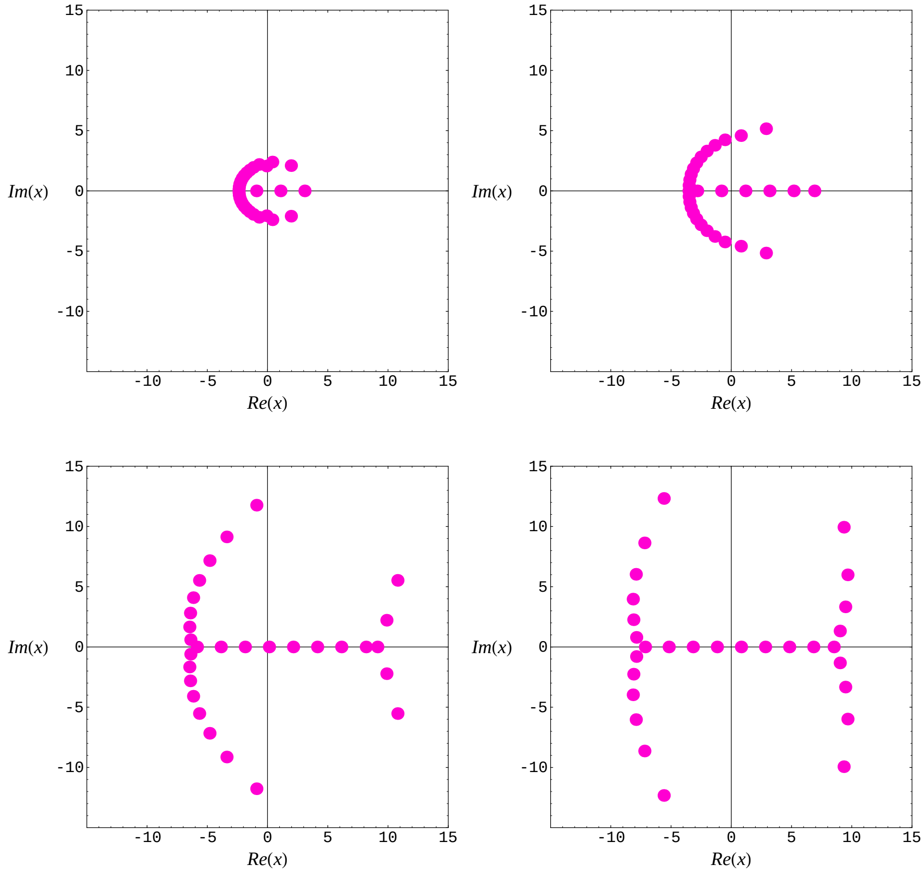

FIGURE 4. Zeros of $B_{n,q}^{(h)}(x)$

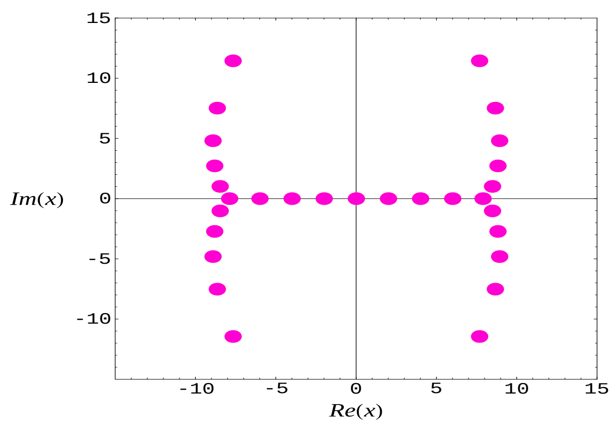

FIGURE 5. Zeros of $\lim_{q\to -1} B_{n,q}^{(h)}(x)$

the distribution of real zeros of the second kind Genocchi polynomials $G_n(x)$ for $1 \leq n \leq 30$.

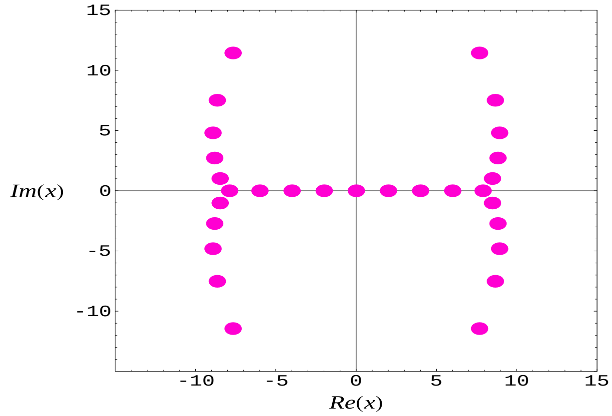

FIGURE 6. Zeros of $G_n(x)$

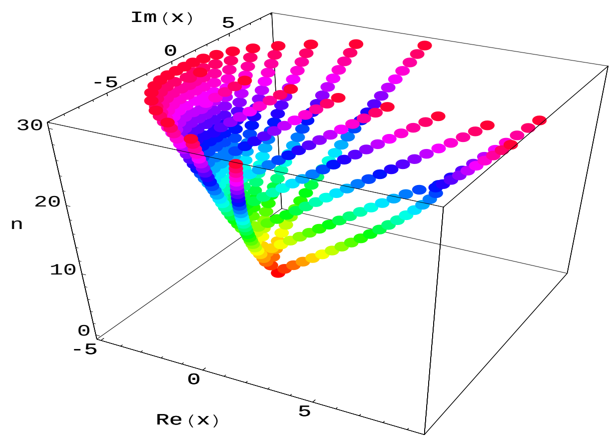

FIGURE 7. Stacks of zeros of $B_{n,q}^{(h)}(x)$ for $1 \leq n \leq 30$

3. Directions for Further Research

In [10], we observed the behavior of complex roots of the second kind Genocchi polynomials $G_n(x)$, using numerical investigation. Prove that $G_n(x), x \in \mathbb{C}$, has $Re(x) = 0$ reflection symmetry in addition to the usual $Im(x) = 0$ reflection symmetry analytic complex functions. The obvious corollary is that the second kind zeros of $G_n(x)$ will also inherit these symmetries.

$$\text{If } G_n(x_0) = 0, \text{ then } G_n(x_0^*) = 0 \tag{3.1}$$

$*$ denotes complex conjugation. Prove that $G_n(x) = 0$ has $n-1$ distinct solutions. If $G_{2n}(x)$ has $Re(x) = 0$ and $Im(x) = 0$ reflection symmetries, and $2n-1$ non-degenerate zeros, then $2n-1$ of the distinct zeros will satisfy (3.1). If the remaining one zero is to satisfy (3.1) too, it must reflect into itself, and therefore it must lie at

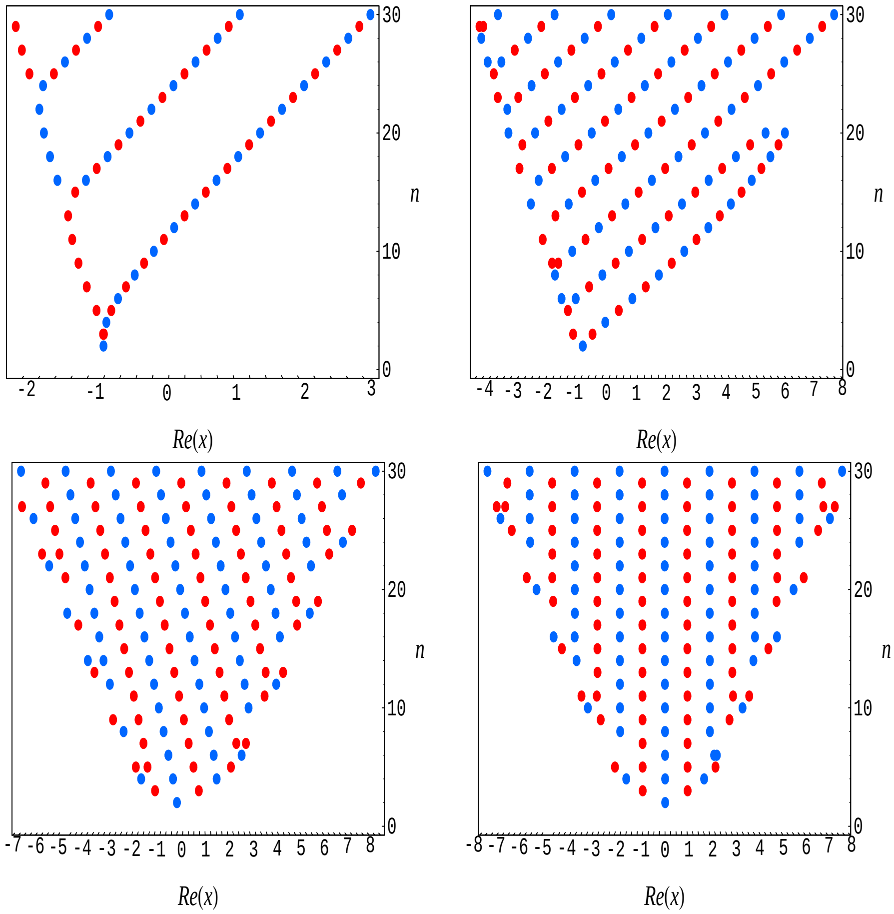

FIGURE 8. Plot of real zeros of $B_{n,q}^{(h)}(x)$ for $1 \leq n \leq 30$

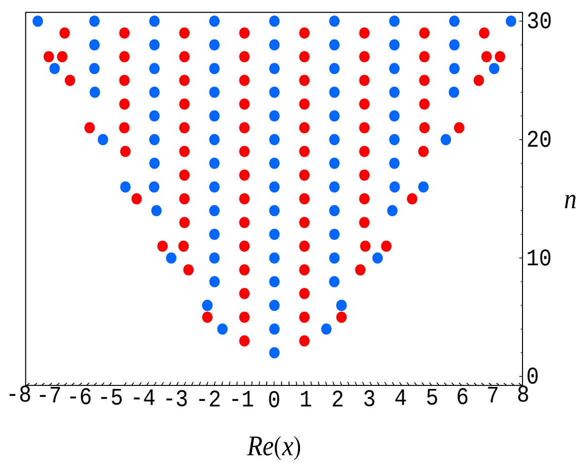

FIGURE 9. Plot of real zeros of $G_n(x)$ for $1 \leq n \leq 30$

0, the center of the structure of the zeros, ie.,

$$G_n(0) = 0 \quad \forall \text{ even } n.$$

We obtain $G_n(1) = G_n(-1) = 0 \quad \forall$ odd $n \geq 3$. Finally, we shall consider the more general problems. Prove that $B_{n,q}^{(h)}(x) = 0$ has $n-1$ distinct solutions, ie., all the zeros are non-degenerate. Find the numbers of complex zeros $C_{B_{n,q}^{(h)}(x)}$ of $B_{n,q}^{(h)}(x), Im(x) \neq 0$. Since $n-1$ is the degree of the polynomial $B_{n,q}^{(h)}(x)$, the number of real zeros $R_{B_{n,q}^{(h)}(x)}$ lying on the real plane $Im(x) = 0$ is then $R_{B_{n,q}^{(h)}(x)} = n - 1 - C_{B_{n,q}^{(h)}(x)}$, where $C_{B_{n,q}^{(h)}(x)}$ denotes complex zeros. See Table 1 for tabulated values of $R_{B_{n,q}^{(h)}(x)}$ and $C_{B_{n,q}^{(h)}(x)}$. Find the equation of envelope curves bounding the real zeros lying on the plane. We prove that $B_{n,q}^{(h)}(x), x \in \mathbb{C}$, has $Im(x) = 0$ reflection symmetry analytic complex functions. If $B_{n,q}^{(h)}(x) = 0$, then $B_{n,q}^{(h)}(x^*) = 0$, where $*$ denotes complex conjugate (see Figures 2, 3, 4, 6). Observe that the structure of the zeros of the Genocchi polynomials $G_n(x)$ resembles the structure of the zeros of the second kind (h,q)-Bernoulli polynomials $B_{n,q}^{(h)}(x)$ as $q \to -1$ (see Figures 4, 5, 7, 8, 9). In order to study the second kind (h,q)-Bernoulli polynomials $B_{n,q}^{(h)}(x)$, we must understand the structure of the second kind (h,q)-Bernoulli polynomials $B_{n,q}^{(h)}(x)$. Therefore, using computer, a realistic study for the second kind (h,q)-Bernoulli polynomials $B_{n,q}^{(h)}(x)$ plays an important part. The author has no doubt that investigation along this line will lead to a new approach employing numerical method in the field of research of the second kind (h,q)-Bernoulli polynomials $B_{n,q}^{(h)}(x)$ to appear in mathematics and physics. For related topics the interested reader is referred to [4, 5, 6, 7, 8, 9, 10, 11].

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