Neural, Parallel, and Scientific Computations 20 (2012) 349-358

MINIMIZATION OF THE ESTIMATION ERROR BY CONTROL IN SYSTEMS WITH MULTIPLICATIVE NOISE

LEV RYASHKO

Institute of Mathematics and Computer Science, Ural Federal University, Ekaterinburg, Russia

ABSTRACT. In this paper, we consider a problem of the estimation for the stochastic system with multiplicative noise. For the current state the system with additive noise, the estimation problem is solved by Kalman-Bucy filter. For stochastic systems, this filter is widely used in the control theory for the construction of optimal regulators under the incomplete information. Many results of this theory, a separation theorem, e.g., rely on the following important property of the systems with additive noise. The fact is that in linear systems with additive noise, a choice of the control strategy does not influence on the uncertainty of the system state. It allows us to distinguish a control and observation. However, systems with multiplicative noises do not exhibit such independence. Estimation errors in these systems depend on the control, so one can regulate the accuracy of the estimation choosing this control. In this paper, we construct a regulator which provides the highest accuracy for the estimator.

AMS (MOS) Subject Classification. 37H20

1. Statement of Problem

Consider a stochastic system

$$\dot{x} = Ax + Bu + \rho(x,u)\dot{v}_1 + \dot{v}_2 \tag{1.1}$$

$$\dot{y} = Cx + \dot{w}, \quad \rho(x,u) \doteq \sqrt{x^\top Qx + u^\top Pu}, \tag{1.2}$$

where x is a n-vector of the state, u is a r-vector of the control, y is a m-vector of observable variables, $v_1(t), v_2(t)$ are n-dimensional and $w(t)$ is m-dimensional independent Wiener processes with parameters:

$$\mathrm{E}\,(v_i(t) - v_i(s)) = 0, \mathrm{E}\,(w(t) - w(s)) = 0,$$

$$\mathrm{E}\left([v_i(t) - v_i(s)][v_i(t) - v_i(s)]^\top\right) = V_i|t-s|,$$

$$\mathrm{E}\left([w(t) - w(s)][w(t) - w(s)]^\top\right) = W|t-s|.$$

Here $A_{n\times n}$, $B_{n\times r}$ and $C_{m\times n}$ are constant matrices. Quadratic $n\times n$-matrices V_1, V_2 and $Q_{n\times n}, P_{r\times r}, W_{r\times r}$ are constant, symmetric and positive definite.

Received October 23, 2012 1061-5369 $15.00 ©Dynamic Publishers, Inc.

The control u is formed by the observations of y by the following regulator:

$$u = -Kz, \tag{1.3}$$

$$\dot{z} = Az + Bu + L(\dot{y} - Cz). \tag{1.4}$$

A study of stabilizing abilities of such regulator in the systems with multiplicative noise is based on the theory of mean-square stability of systems with state-dependent noise [1]–[9].

The system (1.1), (1.2) with the regulator (1.3), (1.4) forms a closed-loop system

$$\begin{aligned} \dot{x} &= Ax - BKz + \rho(x, -BKz)\dot{v}_1 + \dot{v}_2 \\ \dot{z} &= LCx + (A - BK - LC)z + L\dot{w}. \end{aligned}$$

It is convenient to rewrite this system in a more compact form:

$$\dot{X} = \mathcal{A}(R)X + \sqrt{X^\top Q(R) X}\,\dot{\xi} + \dot{\eta}. \tag{1.5}$$

Here

$$X = \begin{bmatrix} x \\ z \end{bmatrix}, \mathcal{A}(R) = \begin{bmatrix} A & -BK \\ LC & A - BK - LC \end{bmatrix}, R = [K, L],$$

$$Q(R) = \begin{bmatrix} Q & 0 \\ 0 & K^\top P K \end{bmatrix}, \xi = \begin{bmatrix} v_1 \\ 0 \end{bmatrix}, \eta = \begin{bmatrix} v_2 \\ Lw \end{bmatrix}.$$

Excluding the additive noise $\dot{\eta}$ in (1.5), we get the following system:

$$\dot{X} = \mathcal{A}(R)X + \sqrt{X^\top Q(R) X}\,\dot{\xi}. \tag{1.6}$$

Consider a set

$$\mathcal{R} = \{R | \text{the system (1.6) is exponentially mean square stable}\}.$$

The system (1.6) is said to be stabilizable if $\mathcal{R} \neq \emptyset$. Necessary and sufficient conditions of the stabilizability can be found in [7].

If $R \in \mathcal{R}$ then the system (1.5) has a unique stationary distributed state $X(R) = [x_R, z_R]^\top$ and

$$\mathrm{E}\,(X(R)) = 0, \mathrm{E}\left(X(R)X^\top(R)\right) = \mathbb{X}(R).$$

The matrix $\mathbb{X}(R)$ of second moments satisfies the following equation

$$\mathcal{A}(R)\mathbb{X} + \mathbb{X}\mathcal{A}^\top(R) + \mathrm{tr}(Q(R)\mathbb{X})\mathbb{V}_1 + \mathbb{V}_2(R) = 0. \tag{1.7}$$

Here

$$\mathbb{V}_1 = \begin{bmatrix} V_1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad \mathbb{V}_2(R) = \begin{bmatrix} V_2 & 0 \\ 0 & LWL^\top \end{bmatrix}.$$

Now, when the regulator has been chosen, consider the estimation problem.

Using [8], [10] one can show that the system

$$\dot{\hat{x}} = A\hat{x} + Bu + F\,(\dot{y} - C\hat{x}) \tag{1.8}$$

is a linear optimal stationary filter for the system (1.1), (1.2) with the regulator (1.3), (1.4). Here

$$F = \Lambda C^\top W^{-1}, \tag{1.9}$$

and the matrix Λ can be found from the following equation:

$$A\Lambda + \Lambda A^\top - \Lambda C^\top W^{-1} C\Lambda + \lambda V_1 + V_2 = 0. \tag{1.10}$$

In (1.10), $\lambda = \mathrm{E}\left(\rho^2(x_R, u_R)\right); x_R$ and $u_R = -Kx_R$ are stationary distributed vectors of state and control, respectively, of the system (1.1), (1.2) with the regulator $R \in \mathcal{R}$. The matrix

$$\Lambda = \lim_{t\to\infty} \mathrm{E}\left([x(t) - \hat{x}(t)][x(t) - \hat{x}(t)]^\top\right) = \mathrm{E}\left([x_R - \hat{x}_R][x_R - \hat{x}_R]^\top\right)$$

characterizes an asymptotic estimation error.

In the case when the system is forced by additive noise only ($\lambda = 0$), the matrix Λ does not depend on R, so the control has no influence on the estimation error. This is a case of classical stationary Kalman-Bucy filter [11].

In the case of multiplicative noise, there is no such independence. Varying R one can change Λ, so one can control the estimation error $\hat{x}$ by the regulator (1.3), (1.4).

Consider a scalar value $g(t) = a^\top x(t)$, where a is a constant n-dimensional vector. For $g(t)$, the estimation is $\hat{g}(t) = a^\top \hat{x}(t)$. As a criterion of the accuracy of this estimation, we consider a value

$$J(R) = \mathrm{E}(g_R - \hat{g}_R)^2,$$

where $g_R = a^\top x_R, \hat{g}_R = a^\top \hat{x}_R$.

The aim of this paper is to solve the following optimization problem.

Problem 1. It is necessary to find

$$\inf_{R\in\mathcal{R}} J(R).$$

As far as

$$J(R) = \mathrm{E}\left(a^\top x_R - a^\top \hat{x}_R\right)^2 = \mathrm{E}\left(a^\top (x_R - \hat{x}_R)(x_R - \hat{x}_R)^\top a\right) = a^\top \Lambda a,$$

where the positive definite matrix Λ is a solution of the algebraic Riccaty equation (1.10), and the quadratic form $a^\top \Lambda a$ monotonically decreases as λ decreases, the Problem 1 is reduced to the following.

Problem 2. It is necessary to find

$$\inf_{R\in\mathcal{R}} \mathrm{E}\left(\rho^2(x_R, u_R)\right),$$

where x_R, u_R are stationary distributed vectors of the state and control, respectively, of the system (1.1), (1.2) with the regulator (1.3), (1.4).

The Problem 2 is a well known problem of the optimization of the standard quadratic criterion under the incomplete information.

For systems with additive noise only, this problem can be easily solved via the separation theorem [12]. For systems with multiplicative noise, there is no such effective method. It is worth noting one essential feature of the Problem 2. The minimized quantity $\mathrm{E}\,(\rho^2(x_R, u_R))$ is directly connected with the value of the intensity $\rho(x, u)$ of multiplicative noise $\rho(x, u)\dot{v}_1$ in the system (1.1). Therefore, by minimizing the criterion we at the same time minimize an influence of multiplicative noise on the system. This specific feature allows us to solve the Problem 2.

2. **Solution of Stabilization Problem**

At first, consider the Problem 2 for the case of complete information. These results we will use in what follows.

Suppose that the control u in the system (1.1) is formed by the following feedback

$$u = -Kx$$

and a closed-loop system looks as

$$\dot{x} = (A - BK)x + \rho(x, -BK)\dot{v}_1. \tag{2.1}$$

Consider a set

$$\mathcal{K} \doteq \left\{ K \,\middle|\, \text{ the system (2.1) is exponentially mean square stable } \right\}.$$

For any $K \in \mathcal{K}$, a state x_K and control $u_K = -Kx_K$ in the system (1.1) have a stationary distribution.

Theorem 1. *Let the system* (1.1) *be stabilizable under complete information* ($\mathcal{K} \neq \emptyset$)*. Then a solution of the minimization problem*

$$\inf_{K \in \mathcal{K}} \mathrm{E}\left(\rho^2(x_K, u_K)\right)$$

for the system (1.1) *can be written as*

$$\bar{K} = P^{-1}B^{\top}M,$$

where the positive definite matrix M is governed by the equation

$$A^{\top}M + MA - MBP^{-1}B^{\top}M + Q = 0. \tag{2.2}$$

One can find a proof of this Theorem in Appendix.

It is worth noting that the matrix $\bar{K}$ does not depend on the parameters V_1, V_2 of the system (1.1) noise. As for additive noise $\dot{v}_2$, this result is well known. The fact that the optimal control does not depend on the parameter V_1 of the multiplicative noise $\rho(x,u)\dot{v}_1$, is quite unexpected. In a general case of the arbitrary quadratic criterion, there is no such independence.

The regulator found in Theorem 1 is optimal for any V_1. But this fact is true only if $\mathcal{K} \neq \emptyset$ and hence if $\operatorname{tr}(MV_1) < 1$ [7].

In the case of incomplete information, the Problem 2 because of the following equalities

$$\mathrm{E}\left(\rho^2(x_R,u_R)\right) = \mathrm{E}\left(X^\top(R)Q(R)X(R)\right) = \operatorname{tr}(Q(R)\mathbb{X}(R))$$

can be reduced to the deterministic Problem 3.

Problem 3. It is necessary to find

$$I = \inf_{R\in\mathcal{R}} \operatorname{tr}(Q(R)\mathbb{X}),$$

where $\mathbb{X}$ is a solution of the system (1.7).

Assign some sequence $R_s, s = 0,1,2,\ldots$. Let us prove that $\lim\limits_{s\to\infty} R_s = \bar{R} \in \mathcal{R}$ and $\operatorname{tr}(Q(\bar{R})\mathbb{X}(\bar{R})) = I$. For the constructing R_s let us consider the following auxiliary problem with the parameter λ.

Problem 4. It is necessary to find

$$I(\lambda) = \inf_{R\in\mathcal{R}_0} \operatorname{tr}(Q(R)\mathbb{Y}(R,\lambda)),$$

where $\mathbb{Y}(R,\lambda)$ is a solution of the system

$$\mathcal{A}(R)\mathbb{Y} + \mathbb{Y}\mathcal{A}^\top(R) + \lambda\mathbb{V}_1 + \mathbb{V}_2 = 0, \lambda \geq 0. \tag{2.3}$$

Here

$$\mathcal{R}_0 = \left\{ R \,\middle|\, \text{ all eigenvalues of } \mathcal{A}(R) \text{ have negative real parts } \right\}.$$

This problem is a problem of the optimization of the stationary quadratic criterion $\mathrm{E}\left(\rho^2(x_R,u_R)\right)$ for the system

$$\begin{aligned} \dot{x} &= Ax + Bu + \sqrt{\lambda}\dot{v}_1 + \dot{v}_2 \\ \dot{y} &= Cx + \dot{w} \end{aligned}$$

with the regulator (1.3), (1.4).

Due to the separation theorem, the solution of this problem for $\mathcal{R}_0 \neq \emptyset$ can be written as

$$K = \bar{K} = P^{-1}B^\top M \tag{2.4}$$

$$L = \bar{L}(\lambda) = \Sigma C^\top W^{-1}, \tag{2.5}$$

where M is a solution of the equation (2.2), and Σ is a solution of the following equation

$$A\Sigma + \Sigma A^\top - \Sigma C^\top W^{-1} C\Sigma + \lambda V_1 + V_2 = 0. \tag{2.6}$$

So, if $\mathcal{R}_0 \neq \emptyset$, the optimal value $I(\lambda)$ of the criterion of the Problem 4 is attained for $R = \left[\bar{K}, \bar{L}(\lambda)\right] \in \mathcal{R}_0$.

Assign the sequence R_s by the following recurrent formulas:

$$\lambda_0 = 0, \lambda_{s+1} = \operatorname{tr}(Q(R_s)\mathbb{Y}_s), s = 0, 1, \ldots, \tag{2.7}$$

where R_s is a solution of the Problem 4 for $\lambda = \lambda_s$, $\mathbb{Y}_s = \mathbb{Y}(R_s, \lambda_s)$ is a solution of the system (2.3) for $R = R_s, \lambda = \lambda_s$. Note that

$$\lambda_{s+1} = \inf_{R \in \mathcal{R}_0} \operatorname{tr}\left(Q(R)\mathbb{Y}(R, \lambda_s)\right).$$

Theorem 2. *Suppose that the system* (1.1), (1.2) *is stabilizable by the regulator* (1.3), (1.4) ($\mathcal{R} \neq \emptyset$). *Then*

(a) *the sequence R_s converges;*
(b) $\lim\limits_{s\to\infty} R_s = \bar{R} \in \mathcal{R}$*;*
(c) *the regulator $\bar{R}$ solves the Problem 3.*

Here, $\bar{R} = [\bar{K}, \bar{L}], \bar{L} = \lim\limits_{s\to\infty} \bar{L}(\lambda_s)$, $I = \lim\limits_{s\to\infty} \lambda_s$, $I = \operatorname{tr}\big(Q(\bar{R})\mathbb{Y}(\bar{R})\big)$.

Proof of this theorem is in Appendix.

As far as $I = \operatorname{tr}\big(Q(\bar{R})\mathbb{Y}(\bar{R})\big) = \mathrm{E}\left(\rho^2(x_{\bar{R}}, u_{\bar{R}})\right)$ the equation (2.6) for $\lambda = I$ coincides with the equation (1.10). Hence, it holds that $\bar{L} = F$, that is the optimal filter (1.8) coincides with the dynamical system (1.4) which belongs to the optimal regulator. In this case, $\hat{x} = z$ and the optimal estimation can be got from the system (1.4). Here, the system (1.8) becomes redundant.

3. Algorithm of Construction of Optimal Regulator

Now we will write the above theoretical results in the form adapted for the following analysis.

One can show that the matrix $\mathbb{X}$ which is a solution of the system (1.7) for $R = \bar{R}$, consists of the matrices Λ_1 and Λ_2:

$$\mathbb{X} = \begin{bmatrix} \Lambda_1 + \Lambda_2 & \Lambda_2 \\ \Lambda_2 & \Lambda_2 \end{bmatrix}.$$

These matrices satisfy to the system

$$A\Lambda_1 + \Lambda_1 A^\top - \Lambda_1 C^\top W^{-1} C \Lambda_1 + \operatorname{tr}\left[Q(\Lambda_1 + \Lambda_2) + \bar{K}^\top P \bar{K} \Lambda_2\right] V_1 + V_2 = 0, \tag{3.1}$$

$$(A - B\bar{K})\Lambda_2 + \Lambda_2 (A - B\bar{K})^\top + \Lambda_1 C^\top W^{-1} C \Lambda_1 = 0. \tag{3.2}$$

Here $\bar{K}$ can be found from (2.4). The other parameters of the optimal regulator $\bar{R}$ are calculated by the formula:

$$\bar{L} = \Lambda_1 C^\top W^{-1}.$$

Now a method of successive approximations can be written as follows.

Algorithm of the construction of regulator $\bar{R}$.

1. To find $\bar{K}$ using (2.4), (2.2);
2. To find sequences Λ_1^s and Λ_2^s from the recurrent formulas

$$\begin{aligned} &A\Lambda_1^{s+1} + \Lambda_1^{s+1}A^\top - \Lambda_1^{s+1}C^\top W^{-1}C\Lambda_1^{s+1} + \lambda_s V_1 + V_2 = 0, \\ &\lambda_s = \operatorname{tr}\left[Q(\Lambda_1^s + \Lambda_2^s) + \bar{K}^\top P\bar{K}\Lambda_2^s\right], \end{aligned} \tag{3.3}$$

$$(A - B\bar{K})\Lambda_2^{s+1} + \Lambda_2^{s+1}(A - B\bar{K})^\top + \Lambda_1^{s+1}C^\top W^{-1}C\Lambda_1^{s+1} = 0 \tag{3.4}$$

with the initial data $\Lambda_1^0 = 0, \Lambda_2^0 = 0$.

3. To find $\Lambda_1 = \lim\limits_{s\to\infty} \Lambda_1^s$ and calculate $\bar{L} = \bar{\Lambda}_1 C^\top W^{-1}$.

Values $\bar{K}$ and $\bar{L}$ are the parameters of the optimal regulator $\bar{R}$.

This algorithm is reduced to the solution of the Riccaty equation (2.2) and repeated solution of the pair of equations (3.3), (3.4). First of them is a Riccaty equation, second one is a Lyapunov equation. These equations are well studied and can be solved by standard numerical procedures.

Appendix

Proof of the Theorem 1. Using [13], one can prove the fact that the feedback coefficient $\bar{K}$ of the optimal control $u = -\bar{K}x$ can be found as

$$\bar{K} = (P + \operatorname{tr}(DV_1)P)^{-1} B^\top D,$$

where D is a solution of the equation

$$A^\top D + DA + \operatorname{tr}(DV_1)Q + Q - DB\,(P + \operatorname{tr}(DV_1)P)^{-1} B^\top D = 0.$$

The matrix $M = D/(1 + \operatorname{tr}(DV_1))$ satisfies the equation (2.2), and $\bar{K} = P^{-1}B^\top M$. Theorem 1 is proved.

For the proof of Theorem 2, consider two Lemmas.

Lemma 3. *For the sequence λ_s $(s = 0, 1, \dots)$ in (2.7) it holds that $\lambda_s \le \lambda_{s+1} \le I$.*

Proof. For any $R \in \mathcal{R}_0$ it holds that

$$0 = \lambda_0 \le I \le \operatorname{tr}(Q(R)\mathbb{X}(R)).$$

From these inequalities it follows that

$$0 \prec \mathbb{Y}(R, \lambda_0) \prec \mathbb{Y}(R, \operatorname{tr}(Q(R)\mathbb{X}(R))) = \mathbb{X}(R),$$

$$\lambda_0 \le \operatorname{tr}(Q(R)\mathbb{Y}(R, \lambda_0)) \le \operatorname{tr}(Q(R)\mathbb{X}(R)).$$

Here $\mathbb{X} \prec \mathbb{Y}$ means that $\mathbb{Y} - \mathbb{X}$ is non-negative definite.

Using $\mathcal{R} \in \mathcal{R}_0$, we get

$$\lambda_0 \leq \inf_{\mathcal{R}_0} \operatorname{tr}(Q(R)\mathbb{Y}(R,\lambda_0)) \leq \inf_{R} \operatorname{tr}(Q(R)\mathbb{X}(R)).$$

So, it holds that $\lambda_0 \leq \lambda_1 \leq I$. By induction, it can be proved that $\lambda_s \leq \lambda_{s+1} \leq I$. □

Lemma 4. *If the system* (1.7) *for* $R \in \mathcal{R}_0$ *has a positive definite solution* $\mathbb{X}$ *then* $R \in \mathcal{R}$.

Proof. Let $R \in \mathcal{R}_0$. Then the inequality (see [7])

$$\operatorname{tr}(Q(R)\mathbb{Z}) < 1 \tag{3.5}$$

is a necessary and sufficient condition for $R \in \mathcal{R}$. Here $\mathbb{Z}$ is a solution of the system

$$\mathcal{A}(R)\mathbb{Z} + \mathbb{Z}\mathcal{A}^\top(R) + \mathbb{V}_1 = 0.$$

Consider also a system

$$\mathcal{A}(R)\mathbb{Y} + \mathbb{Y}\mathcal{A}^\top(R) + \mathbb{V}_2(R) = 0$$

with the solution $\mathbb{Y}$. Suppose that the system (1.7) for $R \in \mathcal{R}_0$ has a solution $\mathbb{X} \succ 0$. Then it holds that $\operatorname{tr}(Q(R)\mathbb{X}) \geq 0$.

If $\operatorname{tr}(Q(R)\mathbb{X}) > 0$ then $\mathbb{Z} = (\mathbb{X} - \mathbb{Y})/\operatorname{tr}(Q(R)\mathbb{X})$ and the inequality (3.5) holds.

If $\operatorname{tr}(Q(R)\mathbb{X}) = 0$ then $\mathbb{X} = \mathbb{Y}$. For $V_2 > 0$ there exists $\mu > 0$ satisfying $\mathbb{V}_1 \prec \mu\mathbb{V}_2(R)$ that is $\mathbb{Z} \prec \mu\mathbb{Y} = \mu\mathbb{X}$. So, it holds that $\operatorname{tr}(Q(R)\mathbb{Z}) = 0 < 1$.

In two above cases, the inequality (3.5) holds, hence $R \in \mathcal{R}_0$. □

Proof of the Theorem 2. Let R_s be a solution of the Problem 4 for $\lambda = \lambda_s$. Because of (2.4)-(2.6) it holds that $R_s = [\bar{K}, \bar{L}(\lambda_s)]$. From Lemma 1 it follows that

$$\bar{\lambda} = \lim_{s\to\infty} \lambda_s \leq I$$

and so there exists a limit of R_s:

$$\bar{R} = \lim_{s\to\infty} R_s = [\bar{K}, \bar{L}(\bar{\lambda})] \in \mathcal{R}_0.$$

Theorem 2 (a) is proved.

A convergence of the sequences λ_s and R_s imply the convergence of the sequence $\mathbb{Y}_s$. Denote

$$\bar{\mathbb{Y}} = \lim_{s\to\infty} \mathbb{Y}_s = \mathbb{Y}(\bar{R}, \bar{\lambda}) \succ 0.$$

Passing to the limit in (2.3) for $\lambda = \lambda_s, R \in R_s$ and (2.7) we get $\bar{\mathbb{Y}} \succ 0$ as a solution of the following system

$$\mathcal{A}(\bar{R})\mathbb{Y} + \mathbb{Y}\mathcal{A}^\top(\bar{R}) + \bar{\lambda}\mathbb{V}_1 + \mathbb{V}_2(\bar{R}) = 0, \quad \bar{\lambda} = \operatorname{tr}(Q(\bar{R})\mathbb{Y}).$$

Thus, we have proved that for $R = \bar{R}$ the system (1.7) has a solution $\mathbb{X}(\bar{R}) = \bar{\mathbb{Y}} \succ 0$. Due to Lemma 2, this means that $\bar{R} \in \mathcal{R}$. Theorem 2 (b) is proved.

From the inequalities

$$I \leq \operatorname{tr}\left(Q(\bar{R})\mathbb{X}(\bar{R})\right) = \operatorname{tr}\left(Q(\bar{R})\bar{\mathbb{Y}}\right) = \bar{\lambda} \leq I$$

it follows that

$$\operatorname{tr}\left(Q(\bar{R})\mathbb{X}(\bar{R})\right) = \bar{\lambda} = I.$$

Theorem 2 (c) is proved.

Acknowledgments

This research was partially supported by Federal Target Programs 1.1099.2011; 14.A18.21.0364.

REFERENCES

[1] I. Ja. Kats and N. N. Krasovskii, The stability of systems with random parameters, *Prikl. Mat. Mech.*, 24:809–823, 1960.

[2] W. M. Wonham, Optimal stationary control of a linear system with state dependent noise, *SIAM J. Control Optim.*, 5:486–500, 1967.

[3] U. G. Hausmann, Optimal stationary control with state and control dependent noise, *SIAM J. Control Optim.*, 9:184–198. 1971.

[4] X. Mao, *Exponential stability of stochastic differential equations*, Marcel Dekker, 1994.

[5] H. Deng, M. Krstic and R. J. Williams, Stabilization of stochastic nonlinear systems driven by noise of unknown covariance, *IEEE Trans. Autom. Control*, 46:1237–1253, 2001.

[6] L. Hou and A. N. Michel, Moment stability of discontinuous stochastic dynamical systems, *IEEE Trans. Autom. Control*, 46:938–943, 2001.

[7] L. B. Ryashko, Stabilization of linear systems with multiplicative perturbations and incomplete information,*Journal of Applied Mathematics and Mechanics*, 45:581–587, 1982.

[8] P. J. McLane, Optimal linear filtering for linear systems with state-dependent noise, *Int. J. Control*, 1:45–51, 1969.

[9] L. Ryashko and I. Bashkirtseva, *Exponential mean square stability analysis of invariant manifolds for nonlinear SDE's*, Chapter 4 in: Stochastic Differential Equations, Ed. N. Halidias, Series: Mathematics Research Developments, pp.67–95, Nova Science Publishers, Inc., 2011.

[10] R. S. Liptser and A. N. Shiryaev, *Statistics of Random Processes*, Springer, 2001.

[11] F. Lewis, L. Xie and D. Popa, *Optimal and Robust Estimation: With an Introduction to Stochastic Control Theory*, CRC Press, 2007.

[12] W. M. Wonham, On the separation theorem of stochastic control, *SIAM. J. Contr.* 6:312–326, 1968.

[13] W. M. Wonham, On a matrix Riccaty equation of stochastic control, *SIAM. J. Contr.* 6:681–697, 1968.