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NUMERICAL APPLICATION OF GENERALIZED MONOTONE METHOD FOR POPULATION MODELS

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ABSTRACT. This paper provides a methodology to compute coupled upper and lower solutions. We will use mathematical modeling to examine population growth and decay of a single and dual animal species, of a nonlinear differential equations with initial conditions to compute solutions. In this work we provide a methodology to compute coupled lower and upper solutions on any given interval. We develop Accelerated convergence results using generalized monotone method. We have both theoretical and numerical results.

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1. Introduction

Mathematical Modeling of many nonlinear problems in science and engineering leads to the qualitative study of nonlinear differential equations with initial and boundary conditions. It is rarely possible to compute such solutions in explicit form using standard methods. In this study, we will be examining population growth and decay of a single and dual animal species and the available theories used for such studies. We study the General Logistic Equation for single species and the Volterra-Lotka Model for dual species. Such models are also useful in the infectious diseases and ecological models. They are also useful in the study of periodic solutions with impulses. See [7, 10] for details. The classical results known for the existence of solutions are the Peano's Theorem and the Picard's Theorem. Peano's Theorem assumes that the

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nonlinear function is continuous and bounded and provides only theoretical existence results, locally. Picard's Theorem assumes the nonlinear function is continuous and satisfies Lipschitz Condition and provides the existence and uniqueness of solutions locally. However, Picard's Theorem is theoretical as well as computational. It is well known that the monotone method combined with the method of upper and lower solutions provides both theoretical and computational method to compute minimal and maximal solutions. In addition, the interval of existence is guaranteed by the upper and lower solutions. In the usual monotone method, they assume that the nonlinear function is increasing or could be made increasing in the unknown variable. Monotone method has been developed when the nonlinear function is decreasing in the unknown variable. See [4,8,6] for details.

Monotone method when extended to include the situation when the nonlinear function is the sum of an increasing and decreasing function combined with coupled upper and lower solutions is known as Generalized monotone method. See [5,3] for details. In order to apply the Generalized monotone method, the major hurdle is to compute the coupled upper and lower solutions of type I to our desired interval. In this paper, we develop both theoretical and numerical methods to compute the coupled upper and lower solutions to any desired interval when we know the natural lower and upper solutions on that interval. Certainly, equilibrium solutions are natural upper and lower solutions for all time. Our results are developed for both the scalar equation and the dual system model. In addition, we can also accelerate this process by implementing the Gauss-Seidel Iteration Method, which can be used when studying scalar and vector equations. Finally, we provide numerical results as an application to our theoretical main result. We provide a scalar and a dual system model in our examples.

2. **Preliminary Results**

In this section, we recall known results related to scalar and two system first order differential equations of the following form:

$$u' = f(t,u) + g(t,u), \quad u(0) = u_0 \text{ on } [0,T] = J \tag{2.1}$$

and

$$u_i' = f_i(t,u) + g_i(t,u), \quad u_i(0) = u_{0i} \text{ on } [0,T] = J \text{ for } i = 1,2 \tag{2.2}$$

where $f, g \in C([J] \times \mathbb{R}, \mathbb{R})$, where as in (2.2) $f_i, g_i \in C([J] \times \mathbb{R}^2, \mathbb{R}^2)$

Here and throughout this paper we assume $f(t,u)$ is non-decreasing in u on J and $g(t,u)$ is non-increasing in u on J. Similarly, we also assume $f_i(t,u_1,u_2)$ is non-decreasing in u_1 and u_2, and $g_i(t,u_1,u_2)$ is non-increasing in u_1 and u_2 for $i = 1,2$ and for $t \in [0,T] = J$.

We recall the following known definitions which are needed for our main results.

Definition 2.1. The function $v_0, w_0 \in C^1([0,T],\mathbb{R})$ are called natural lower and upper solutions of (2.1) if:

$$v_0' \leq f(t,v_0) + g(t,v_0), \quad v_0(0) \leq u_0$$

and

$$w_0' \geq f(t,w_0) + g(t,w_0), \quad w_0(0) \geq u_0$$

Definition 2.2. The functions $v_0, w_0 \in C^1([J],\mathbb{R})$ are called coupled lower and upper solutions of (2.1) of Type I if:

$$v_0' \leq f(t,v_0) + g(t,w_0), \quad v_0(0) \leq u_0$$

and

$$w_0' \geq f(t,w_0) + g(t,v_0), \quad w_0(0) \geq u_0$$

The next known result is relative to natural lower and upper solutions of (2.1) . For this purpose we let $F(t,u) = f(t,u) + g(t,u)$

Theorem 2.3. *Let $v, w \in C^1(J,\mathbb{R})$ be lower and upper solutions of the first order initial value problem* (2.1) *respectively. Suppose that*

$$F(t,x) - F(t,y) \leq L(x-y) \quad \text{whenever } x \geq y, \quad \text{and} \quad L > 0$$

and where L represents a constant. Then $v(0) \leq w(0)$ implies that $v(t) \leq w(t)$, $t \in J$.

Next we state a Corollary of Theorem 2.1 which is useful in monotone method or generalized monotone method.

Corollary 2.4. *Let $p(t) \in C^1(J,\mathbb{R})$ such that $p'(t) \leq Lp(t)$, and $p(0) \leq 0$ implies $p(t) \leq 0$.*

We define the following sector Ω for convenience. That is:

$$\Omega = [(t,u) : v(t) \leq u(t) \leq w(t),\ t \in J]. \tag{2.3}$$

Theorem 2.5. *Suppose $v, w \in C^1[J,\mathbb{R}]$ are coupled lower and upper solutions of* (2.1) *such that $v(t) \leq w(t)$ on J and $F \in C(\Omega,\mathbb{R})$. Then there exists a solution $u(t)$ of* (2.1) *such that $v(t) \leq u(t) \leq w(t)$ on J, provided $v(0) \leq u_0 \leq w(0)$.*

Theorem 2.6. *Assume that*

(i) $v_0, w_0 \in C^1[J,\mathbb{R}]$ are coupled lower and upper solutions of type I with $v_0(t) \leq w_0(t)$ on J.
(ii) $f, g \in C[J \times \mathbb{R},\mathbb{R}]$, $f(t,u)$ is nondecreasing in u and $g(t,u)$ is nonincreasing in u on J.

Then there exists monotone sequences $v_n(t)$ *and* $w_n(t)$ *on* J *such that* $v_n(t) \to v(t)$ *and* $w_n(t) \to w(t)$ *uniformly and monotonically and* (v, w) *are coupled minimal and maximal solutions, respectively to equation* (2.1). *That is,* (v, w) *satisfy*

$$v' = f(t, v) + g(t, w), \quad v(0) = u_0, \quad \text{on } J, \tag{2.4}$$

$$w' = f(t, w) + g(t, v), \quad w(0) = u_0, \quad \text{on } J. \tag{2.5}$$

Here the iterative scheme is given by

$$v'_{n+1} = f(t, v_n) + g(t, w_n), \quad v_{n+1}(0) = u_0, \quad \text{on } J. \tag{2.6}$$

$$w'_{n+1} = f(t, w_n) + g(t, v_n), \quad w_{n+1}(0) = u_0, \quad \text{on } J. \tag{2.7}$$

Theorem 2.7. *Let all the hypothesis of Theorem* 2.6 *be satisfied. Further, let*

$$f(t, u_1) - f(t, u_2) \leq L_1(u_1 - u_2)$$

$$g(t, u_1) - g(t, u_2) \geq -M_1(u_1 - u_2)$$

where L_1 *and* M_1 *are constants, whenever* $v_0 \leq u_2 \leq u_1 \leq w_0$ *then* $v = w = u$ *is the unique solution of equation* (2.1)

The next result is monotone method for (2.1) where we use natural lower and upper solutions.

Theorem 2.8. *Assume that*

(i) $v_0, w_0 \in C^1(J, \mathbb{R})$ *are natural lower and upper solutions with* $v_0(t) \leq w_0(t)$ *on* J.
(ii) $f, g \in C(J \times \mathbb{R}, \mathbb{R}), f(t, u)$ *is nondecreasing in* u *and* $g(t, u)$ *is nonincreasing in* u *on* J.

Then there exists monotone sequences $v_n(t)$ *and* $w_n(t)$ *on* J *such that* $v_n(t) \to v(t)$ *and* $w_n(t) \to w(t)$ *uniformly and monotonically and* (v, w) *are coupled minimal and maximal solutions, respectively to equation* (2.1). *That is,* (v, w) *satisfy*

$$v' = f(t, v) + g(t, w), \quad v(0) = u_0, \quad \text{on } J, \tag{2.8}$$

$$w' = f(t, w) + g(t, v), \quad w(0) = u_0, \quad \text{on } J, \tag{2.9}$$

provided $v_0 \leq v_1$ *and* $w_1 \leq w_0$ *on* J.

The above theorem uses v_0, w_0 as natural lower and upper solutions. Then v_1, w_1 will be coupled lower and upper solutions only on some interval $[0, t_1)$ not necessarily on $[0, T]$. This is the motivation for our main result relative to equation (2.1).

The next result is existence of the solution to two systems of differential equations with initial conditions. The two system of equations with initial conditions is a generalization of the Volterra-Lotka population model of two species. The following definition is needed relative to the system (2.2).

Definition 2.9. Let v_i, w_i for $i = 1,2$ be $C(J, \mathbb{R})$. Then v_i and w_i are called coupled lower and upper solutions of (2.2) if they satisfy the following inequalities:

$$v_i' \leq f_i(t, v_1, v_2) + g_i(t, w_1, w_2), \quad v_i(0) \leq u_{0i} \tag{2.10}$$

$$w_i' \geq f_i(t, w_1, w_2) + g_i(t, v_1, v_2), \quad w_i(0) \geq u_{0i} \tag{2.11}$$

The next result is the Existence Theorem for the solutions of the system (2.2).

Theorem 2.10. *Let $f_i, g_i \in C(J \times \mathbb{R}^2, \mathbb{R}^2)$ such that $f_i(t,u)$ is nondecreasing in u and $g_i(t,u)$ is nonincreasing in u for $t \in J$, and for each $i = 1,2$. Let $v_0, w_0 \in C^1(J, \mathbb{R}^2]$ be coupled lower and upper solutions of (2.2), such that $v_{0,i}(t) \leq w_{0,i}(t)$ for $i = 1,2$ on J. Then, there exists monotone sequences $\{v_{n,i}\}$ and $\{w_{n,i}\}$ which converges uniformly and monotonically to coupled minimal and maximal solutions of (2.2) such that $v_{n,i} \to v_i$ and $w_{n,i} \to w_i$ as $n \to \infty$, provided $v_{0,i}(0) \leq u_i(0) \leq w_{0,i}(0)$ for $i = 1,2$. Further, if u is any solution of (2.2) such that $v_{0,i} \leq u_i \leq w_{0,i}$, then $v \leq u \leq w$ on J.*

Proof. See [3] for details. □

The following result is a comparison theorem related to coupled lower and upper solutions.

Theorem 2.11. *Let (v_{01}, v_{02}) and (w_{01}, w_{02}) be coupled lower and upper solutions of (2.2). Further let*

(i) $f_i(t,u)$ is nondecreasing in u_i components and $g_i(t,u)$ is nonincreasing in u_i components for $i = 1,2$;
(ii) $f_i(t,u)$ and $g_i(t,u)$ satisfy the one sided Lipschitz condition of the form,

$$f_i(t,u) - f_i(t,\overline{u}) \leq L_i \sum_{j=1}^{2} (u_j - \overline{u_j}), \quad L_i > 0, \quad i = 1,2$$

and

$$g_i(t,u) - g_i(t,\overline{u}) \geq -M_i \sum_{j=1}^{2} (u_j - \overline{u_j}), \quad M_i > 0, \quad i = 1,2$$

whenever $u_i \geq \overline{u_i}$ for $i = 1,2$.

Then $v_i(t) = w_i(t) = u_i(t)$ for $i = 1,2$, where $u_i(t)$ is the unique solution of (2.2).

Proof. See [3] for details. □

The following Corollary is useful in the generalized monotone method.

Corollary 2.12. *Let*

$$p_i'(t) \leq \sum_{j=1}^{2} (L_{ij} + M_{ij}) p_j, \qquad \textit{for } i = 1, 2$$

Then we have $p_i(t) \leq 0$ *for* $i = 1, 2$ *on* $J = [0, T]$*, whenever* $p_i(0) \leq 0$ *for* $i = 1, 2$*.*

3. Main Results

The generalized monotone method is well known for scalars and system of first order differential equations with initial conditions using coupled lower and upper solutions of type I as described in our preliminaries. It is easy to observe that coupled lower and upper solutions of type I implies that they are also natural lower and upper solution. However, the converse is not true. In theory, we know that the existence of natural lower and upper solutions, where the lower solution is less than or equal to the upper solution, we have a solution of (2.1) such that $v_0 \leq u \leq w_0$ on J whenever $v_0(0) \leq u_0 \leq w_0(0)$.In the generalized monotone method, if we use natural lower and upper solution we need an extra assumption, that is $v_0(t) \leq v_1(t)$ and $w_1(t) \leq w_0(t)$ on J. Note that the sequences are developed as in theorem 2.6.

Consider the example

$$u' = 2u - 3u^2, \quad u(0) = \frac{1}{2}, \quad t \in [0, T], \quad T \geq 1.$$

Then $v_0(t) = 0$ and $w_0(t) = \frac{2}{3}$ are natural lower and upper solutions respectively. Then using the iterations in Theorem 2.6 we get

$$v_1(t) = \frac{1}{2} - \frac{4t}{3} \quad \text{and} \quad w_1(t) = \frac{4t}{3} + \frac{1}{2}.$$

It is easy to observe $v_1(t) \geq v_0(t)$ and $w_1(t) \leq w_0(t)$ on $[0, \frac{3}{8}]$. In order to apply Theorem 2.6, we need

$$v_1(t) \geq v_0(t) \quad \text{and} \quad w_1(t) \leq w_0(t) \quad \text{on} \quad [0, T].$$

This is the motivation for our main result. Our aim is to develop a method to construct coupled lower and upper solutions on the interval $J = [0, T]$, so that we can apply Theorem 2.6 to compute the coupled minimal and maximal solutions for equation (2.1).If f and g satisfies one sided Lipschitz Condition, we can also prove that the coupled minimal and maximal solutions of (2.1) will converge to the unique solution of (2.1).

Theorem 3.1. *Assume that*

(i) $v_0, w_0 \in C[J, \mathbb{R}]$ *are natural lower and upper solutions of* (2.1) *such that* $v_0(t) \leq w_0(t)$ *on* J*.*

(ii) $f, g \in C[J \times \mathbb{R}, \mathbb{R}]$, $f(t,u)$ *is nondecreasing and* $g(t,u)$ *is nonincreasing in* u *on* J. *Then there exists monotone sequences* $\{v_n(t)\}$ *and* $\{w_n(t)\}$ *on* J *such that* $v_n(t) \to v(t)$ *and* $w_n(t) \to w(t)$ *uniformly and monotonically and* (v, w) *are coupled lower and upper solutions of* (2.1) *such that* $v \le w$ *on* J. *The iterative scheme is given by*

$$v'_{n+1} = f(t, v_n) + g(t, w_n), \quad on \quad [0, t_n], \quad v_{n+1}(0) = u_0$$
$$w'_{n+1} = f(t, w_n) + g(t, v_n), \quad on \quad [0, \overline{t_n}\,], \quad w_{n+1}(0) = u_0,$$

where $v_n(t) \ge v_0(t)$ *on* $[0, t_n)$ *and* $w_n(t) \le w_0(t)$ *on* $[0, \overline{t_n})$. *Also define* $v_{n+1}(t)$, $w_{n+1}(t)$ *on* $[t_n, T]$ *and* $[\overline{t_n}, T]$ *respectively as the solution of*

$$v'_{n+1} = f(t, v_0) + g(t, w_0), \quad v_{n+1}(t_n) = \lim_{h \to 0} v_{n=1}(t_n - h)$$
$$w'_{n+1} = f(t, w_0) + g(t, v_0), \quad w_{n+1}(\overline{t_n}) = \lim_{h \to 0} w_{n+1}(\,\bar{t}_n - h).$$

Proof. From Theorem 2.6 we have $v_0(t) \le v_1(t)$ on $[0, t_1]$ and $w_1(t) \le w_0(t)$ on $[0, \overline{t_1}\,]$. If $t_1 \ge T$, and $\overline{t_1} \ge T$ there is nothing to prove since one can use Theorem 2.6 to compute coupled minimal and maximal solutions. If not, certainly $t_1 < T$ and $\overline{t_1} < T$. Then redefine $v_1(t)$ and $w_1(t)$ as follows

$$v'_1(t) = f(t, v_0) + g(t, w_0), \;\; v_1(0) = u_0 \quad \text{on } [0, t_1],$$
$$w'_1(t) = f(t, w_0) + g(t, v_0), \;\; w_1(0) = u_0 \quad \text{on } [0, \overline{t_1}\,],$$

and

$$v_1(t) = v_0(t) \quad \text{on } [t_1, T],$$
$$w_1(t) = w_0(t) \quad \text{on } [\,\overline{t_1}, T],$$

such that $v_1(t_1) = v_0(t_1)$ and $w_1(\overline{t_1}) = w_0(\overline{t_1}\,)$.

Proceeding in this manner, we get

$$v'_n = f(t, v_{n-1}) + g(t, w_{n-1}), \quad v_n(0) = u_0 \quad \text{on } [0, t_{n-1})$$
$$v_n(t) = v_0(t) \quad \text{on } [t_{n-1}, T], \quad \text{such that } v_n(t_{n-1}) = v_0(t_{n-1})$$

Similarly,

$$w'_n = f(t, w_{n-1}) + g(t, v_{n-1}), \quad w_n(0) = u_0 \quad \text{on } [0, \bar{t}_{n-1})$$
$$w_n(t) = w_0(t) \quad on \quad [\,\bar{t}_{n-1}, T], \quad \text{such that } w_n(\,\bar{t}_{n-1}) = w_0(\,\bar{t}_{n-1}).$$

Now let v_n, w_n intersect v_0 and w_0 at $t_n, \overline{t_n}$ respectively. If $t_n \ge T$, and $\overline{t_n} \ge T$ we can stop the process. Certainly $v_n \le w_n$ and v_n and w_n are coupled minimum and maximum solutions of (2.1) respectively.

Now we can show that the sequence $\{v_n(t)\}$ and $\{w_n(t)\}$ constructed above are equicontinuous and uniformly bounded on J. Hence by Ascoli Arzela's theorem, a subsequence converges uniformly and monotonically. Since the sequences are monotone, the entire sequence converges uniformly and monotonically to u, and w respectively.

It is easy to observe:

$$v_n' = f(t, v_{n-1}) + g(t, w_{n-1}), \quad v_n(0) = u_0 \quad \text{on } [0, t_{n-1})$$

$$v_n(t) = v_0(t) \quad \text{on } [t_{n-1}, T], \quad \text{such that } v_n(t_{n-1}) = v_0(t_{n-1})$$

and

$$w_n' = f(t, w_{n-1}) + g(t, v_{n-1}), \quad w_n(0) = u_0 \quad \text{on } [0, \bar{t}_{n-1})$$

$$w_n(t) = w_0(t) \quad \text{on } [\,\bar{t}_{n-1}, T], \quad \text{such that } w_n(\,\bar{t}_{n-1}) = w_0(\,\bar{t}_{n-1})$$

for all $n \geq 1$.

As $n \to \infty$, $t_n, \overline{t_n} \to T$, $v_n(t) \to v(t)$, and $w_n(t) \to w(t)$, uniformly and monotonically.

Further,

$$v' = f(t, v) + g(t, w), \quad v(0) = u \quad \text{on } J$$

and

$$w' = f(t, w) + g(t, v), \quad w(0) = u \quad \text{on } J.$$

Hence v, w are coupled lower and upper solutions of (2.1) on J. This concludes the proof. □

Remark 3.2. Now that Theorem 3.1 provides coupled lower and upper solutions of (2.1) we can develop sequences $\{v_n\}$ and $\{w_n\}$ using Theorem 2.6. These sequences converge uniformly and monotonically to coupled minimal and maximal solutions. Further if uniqueness condition is satisfied, the sequences converge to the unique solution of (2.1). However, in generalized monotone method even for scalar equations like (2.1), we can apply Gauss Seidel method such that sequences converge faster. This is precisely the next result.

Theorem 3.3. *Let all the hypothesis of Theorem* 2.6 *hold. Then there exist monotone sequences* v_n *and* w_n *where the iterative scheme is given by*

$$v_{n+1}^{*\prime} = f(t, v_n^*) + g(t, w_n^*), \quad v_{n+1}^*(0) = u_0 \tag{3.1}$$

$$w_{n+1}^{*\prime} = f(t, w_n^*) + g(t, v_{n+1}^*), \quad w_{n+1}^*(0) = u_0 \tag{3.2}$$

where $v_0^* = v_1$ *and* w_0^* *is the solution of* ${w_0}^{*\prime} = f(t, w_0) + g(t, v_1)$, $w_0^*(0) = u_0$.
OR

$$v_{n+1}^{*\prime} = f(t, v_n^*) + g(t, w_{n+1}^*), \quad v_{n+1}^*(0) = u_0 \tag{3.3}$$

$$w_{n+1}^{*\prime} = f(t, w_n^*) + g(t, v_n^*), \quad w_{n+1}^*(0) = u_0 \tag{3.4}$$

where ${w_0}^* = w_1$ *and* v_0^* *is the solution of* $v_0^{*\prime} = f(t, v_0) + g(t, w_1)$, $v_0^*(0) = u_0$.

Proof. We provide a brief proof. One can easily see that $v_0(t) \leq v_1(t)$ on J. Now it is enough if we prove that $w_0^* \leq w_1$. Let $p(t) = w_0^* - w_1, \quad p(0) = 0$.

$$\begin{aligned} p'(t) &= {w_0}^{*\prime} - w_1' \\ &= f(t, w_0) + g(t, v_1) - (f(t, w_0) + g(t, v_0)) \\ &= g(t, v_1) - g(t, v_0) \\ &\leq 0, \end{aligned}$$

since $v_1(t) \geq v_0(t)$ on J.

This implies $p(t) \leq 0$ on J .That is $w_0^* \leq w_1$ on J. Continuing the process, we can show that that the sequences $\{v_n^*\}$ and $\{w_n^*\}$ converges faster than the sequence $\{v_n\}$ and $\{w_n\}$ computed using Theorem 2.6. □

Theorem 3.4. *Assume that*

(i) $v_{0i}, w_{0i} \in C[J, \mathbb{R}^2]$ *for* $i = 1, 2$ *are natural lower and upper solutions of systems* (2.2) *such that* $v_{0i}(t) \leq w_{0i}(t)$ *on* J.

(ii) $f_i, g_i \in C[J \times \mathbb{R}^2, \mathbb{R}^2]$ *such that* $f_i(t, u)$ *is nondecreasing in* u *and* $g_i(t, u)$ *is nonincreasing in* u *for* $t \in J$, *and for each* $i = 1, 2$. *Then there exists monotone sequences* $\{v_{n,i}(t)\}$ *and* $\{w_{n,i}(t)\}$ *on* J *such that* $v_{n,i}(t) \to v(t)$ *and* $w_{n,i}(t) \to w(t)$ *uniformly and monotonically and* (v, w) *are coupled lower and upper solutions of* (2.2) *such that* $v \leq w$ *on* J. *The iterative scheme for two system is given by*

$$\begin{aligned} v_{n,1}' &= f_1(t_1, v_{n-1,1}, v_{n-1,2}) + g_1(t_1, w_{n-1,1}, w_{n-1,2}),\ v_{n,1} = u_{01}\ \text{on}\ [0, t_{n-1,1}] \\ v_{n,2}' &= f_2(t_1, v_{n-1,1}, v_{n-1,2}) + g_2(t_1, w_{n-1,1}, w_{n-1,2}),\ v_{n,2} = u_{02}\ \text{on}\ [0, t_{n-1,2}] \\ w_{n,1}' &= f_1(\overline{t_1}, w_{n-1,1}, w_{n-1,2}) + g_1(\overline{t_1}, v_{n-1,1}, v_{n-1,2}),\ w_{n,1} = u_{01}\ \text{on}[0, \bar{t}_{n-1,1}] \\ w_{n,2}' &= f_2(\overline{t_1}, w_{n-1,1}, w_{n-1,2}) + g_2(\overline{t_1}, v_{n-1,1}, v_{n-1,2}),\ w_{n,2} = u_{02}\ \text{on}[0, \bar{t}_{n-1,2}] \end{aligned}$$

and

$$\begin{aligned} v_{n,1}(t) &= v_{0,1}(t) \quad \text{on}\ [t_{n-1,1}, T] \\ v_{n,2}(t) &= v_{0,2}(t) \quad \text{on}\ [t_{n-1,2}, T] \\ w_{n,1}(t) &= w_{0,1}(t) \quad \text{on}\ [\bar{t}_{n-1,1}, T] \\ w_{n,2}(t) &= w_{0,2}(t) \quad \text{on}\ [\bar{t}_{n-1,2}, T] \end{aligned}$$

Remark 3.5. Note that Theorem 3.4 provides a method to compute coupled lower and upper solutions of (2.2) on the desired interval $[0, T]$. We can develop an accelerated convergence result for the system (2.2) similar to Theorem 3.4. This is precisely our next result.

Theorem 3.6. *Let all the hypothesis of Theorem* 2.10 *hold. Then there exists sequences* $\{v_{n,i}^*\}, \{w_{n,i}^*\}$ *for* $i = 1, 2$, *on* $[0, T]$, *such that it converges uniformly and* *monotonically to coupled minimal and maximal solutions of* (2.2). *These sequences converge at a much faster pace than the sequences of Theorem* 2.10. *The sequences* $\{v^*_{n,i}\}$, *and* $\{w^*_{n,i}\}$, *are developed as follows: where the iterative scheme is given by*

$$\begin{aligned}
{v^*}'_{n+1,1} &= f_1(t, v^*_{n,1}, v^*_{n,2}) + g_1(t, w^*_{n,1}, w^*_{n,2}), \quad v_{n,1}(0) = u_{0,1} \\
{v^*}'_{n+1,2} &= f_2(t, v^*_{n+1,1}, v^*_{n,2}) + g_2(t, w^*_{n,1}, w^*_{n,2}), \quad v_{n,2}(0) = u_{0,2} \\
{w^*}'_{n+1,1} &= f_1(t, w^*_{n,1}, w^*_{n,2}) + g_1(t, v^*_{n+1,1}, v^*_{n+1,2}), \quad w_{n,1}(0) = u_{0,1} \\
{w^*}'_{n+1,2} &= f_2(t, w^*_{n+1,1}, w^*_{n,2}) + g_2(t, v^*_{n+1,1}, v^*_{n+1,2}), \quad w_{n,2}(0) = u_{0,2}
\end{aligned}$$

Proof. Let $v_{1,1} = v^*_{0,1}$, then

$$\begin{aligned}
{v^*}'_{0,2} &= f_2(t, v^*_{0,1}, v_{0,2}) + g_2(t, w_{0,1}, w_{0,2}) \\
{w^*}'_{0,1} &= f_1(t, w_{0,1}, w_{0,2}) + g_1(t, v^*_{0,1}, v^*_{0,2}) \\
{w^*}'_{0,2} &= f_2(t, w^*_{0,1}, w_{0,2}) + g_2(t, v^*_{0,1}, v^*_{0,2})
\end{aligned}$$

We will prove that $v^*_{0,2} \geq v_{1,2}$ on J. For that purpose, set $p(t) = v^*_{0,2} - v_{1,2}$, $p(0) = 0$

$$\begin{aligned}
p'(t) &= {v^*}'_{0,2} - {v^*}'_{1,2} \\
&= f_2(t, v^*_{0,1}, v_{0,2}) + g_2(t, w_{0,1}, w_{0,2}) - (f_2(t, v_{0,1}, v_{0,2}) + g_2(t, w_{0,1}, w_{0,2})) \\
&= f_2(t, v_{1,1}, v_{0,2}) - f_2(t, v_{0,1}, v_{0,2}) \\
&\geq 0 \text{ since } v_{1,1} \geq v_{0,1}
\end{aligned}$$

This proves $v^*_{0,2} \geq v_{1,2}$. Similarly, we can prove $w^*_{0,1} \geq w_{1,1}$ using the information $v_{0,1} \leq v_{1,1} = v^*_{0,1}$ and $v_{0,2} \leq v_{1,2} \leq v_{0,2}$. Continuing this process we can show the sequences $\{v^*_{n,i}\}$ and $\{w^*_{n,i}\}$ converges faster than the sequence $\{v_{n,i}\}$ and $\{w_{n,i}\}$ computed using Theorem 2.10. □

4. **Numerical Results**

In this section, we provide several numerical examples justifying our results of section 3. Initially we take a simple logistic equation and apply Theorem 2.6 of the preliminary. In order to apply Theorem 2.6, we assume that v_1 and w_1 should satisfy $v_0 \leq v_1, \quad w_1 \leq w_0$ on $[0, T]$. However, we have not used the best estimate of v_0 and w_0 on the entire interval. Consider the example

$$u' = u - u^2, \quad u_0(0) = \frac{1}{2} \tag{4.1}$$

It is easy to observe, $v_0(t) \equiv 0$ and $w_0(t) \equiv 1$, are lower and upper solutions of equation (4.1). The following graph is an application of Theorem 2.6.

From the graph $v_0 \leq v_1, w_1 \leq w_0$ on $[0, 0.5]$. Theorem 2.6 only guarantees that $v_1 \leq v_2$ and $w_2 \leq w_1$ on $[0, 0.5]$. However in this example $v_n(t)$ and $w_n(t)$ are computed till $n = 7$. This estimate of $v_n(t)$ and $w_n(t)$ are not useful since they will not

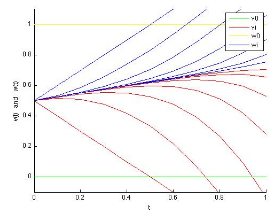

FIGURE 1. Coupled Lower and Upper Solutions of (4.1)

be coupled lower and upper solutions beyond 0.5. In order to make our method more accurate, we apply Theorem 3.1 to special logistic equation. Note that Theorem 3.1 provides a method to compute coupled lower and upper solutions. This is precisely what we have done for equation (4.1) in the following graphs.

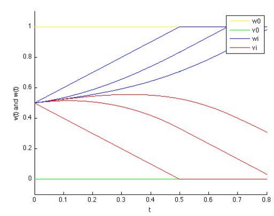

FIGURE 2. Coupled Lower and Upper Solutions of (4.1) using Theorem 3.1

In the next graph, we use the coupled and lower solutions of Figure 2 and apply Theorem 2.6.

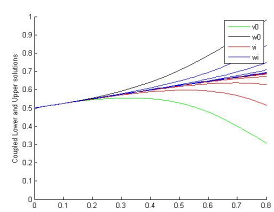

FIGURE 3. 4 iterations of (4.1) using Theorem 2.6

In the next graph, we use the coupled and lower solutions of Figure 2 and apply Theorem 3.3. This provides accelerated convergence compared with the result of Figure 3.

Remark 4.1. Notice Figure 3 took four iterations to approximate the unique solution and Figure 4 only took three iterations and provided a better approximate unique solution of 4.1.

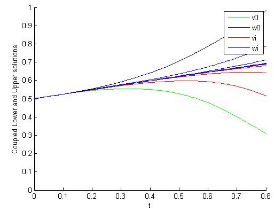

FIGURE 4. 3 iterations of (4.1) using Theorem 3.3

Next we consider the prey-predator model

$$\left.\begin{aligned} u_1' &= u_1(5 - 2u_1 - 3u_2) = 0, \quad u_0 = \frac{1}{2} \\ u_2' &= u_2(-2 + u_1 + u_2) = 0, \quad u_0 = \frac{1}{2} \end{aligned}\right\} \tag{4.2}$$

It is easy to observe $(v_{01}, v_{02}) = (0, 0)$ and $(w_{01}, w_{02}) = (1, 1)$ are the equilibrium solutions. Hence they are also natural lower and upper solutions. Using $w_{01} \equiv 1$ and $w_{02} \equiv 1$ and $v_{01} \equiv 0$ and $v_{02} \equiv 0$, we obtained our graph representing coupled lower and upper solutions of (4.2) below.

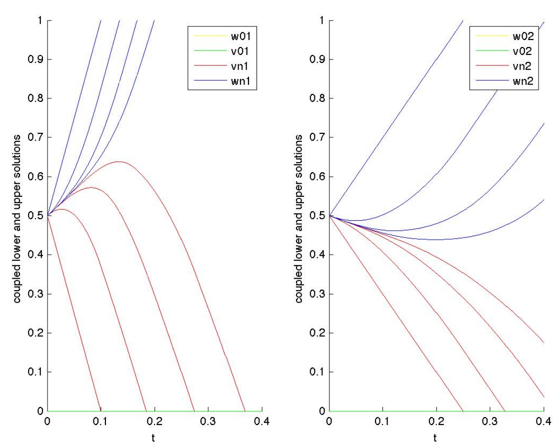

FIGURE 5. Coupled Lower and Upper Solutions of (4.2) using Theorem 3.4

Next, using the coupled lower and upper solutions from Figure 5, and Theorem 2.10 we obtain:

In the next graph we use the coupled lower and upper solutions of Figure 5 and apply Theorem 3.6.

Observe that Figure 7 converges faster than that of Figure 6.

5. **Conclusion**

In this paper, we developed both theoretical and numerical methods to compute the coupled upper and lower solutions to any desired interval when we know the

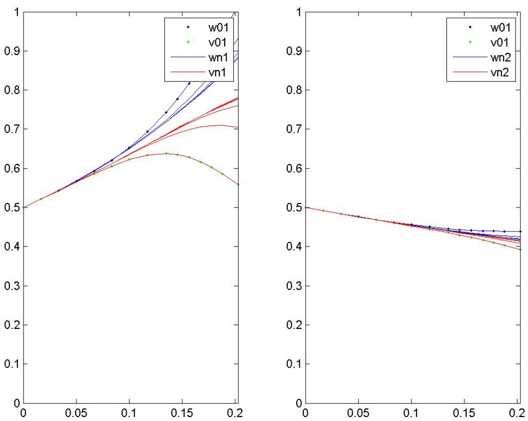

FIGURE 6. Coupled Lower and Upper Solutions of (4.2) using Theorem 2.10

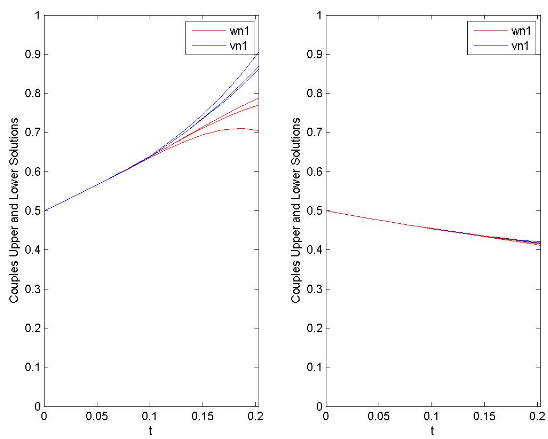

FIGURE 7. Coupled Lower and Upper Solutions of (4.2) using Theorem 3.6.

natural lower and upper solutions on that interval. Our results are developed for both the scalar equation and the dual system model.In addition, we can also accelerate this process by implementing the Gauss-Seidel Iteration Method. We also provided several numerical results as an application to our theoretical main results.

REFERENCES

[1] B. Adams, N. Davis, P. Epps, F. Miller, and D. Mullen,*Existence of Solution for Impulsive Hybrid Differential Equations*, (2011).

[2] Cronin, J. (1994). *Differential Equations: Introduction and Qualitative Theory*. New York: Marcel Dekker, Inc.

[3] D. Stutson and A. S. Vatsala, *Generalized Monotone Method for Caputo Fractional Differential Systems Via Coupled Lower and Upper Solutions*, (2011)

[4] G. S. Ladde, V. Lakshmikantham and A. S. Vatsala, *Monotone Iterative Techniques for Nonlinear Differential Equations*, Pitman, (1985).

[5] I. H. West and A. S. Vatsala, *Generalized Monotone Iterative Method for Initial Value Problems*, Appl. Math Lett **17** (2004), 1231-1237.

[6] M. Sokol and A. S. Vatsala, *A unified exhaustive study of monotone iterative method for initial value problems*, Nonlinear Studies 8 (4), 429-438, (2001).

[7] Robert D. Holt and John Pickering,*Infectious Disease and Species Coexistence: A Model of Lotka-Volterra Form*, (1985).

[8] T. G. Bhaskar and F. A. McRae,*Monotone Iterative Technique for nonlinear problems involving the difference of two monotone functions*, Applied Mathematics and Computation 133, 187-192, (2002).

[9] V. Lakshmikantham, D. D. Bainov, and P. S. Simeonov, Theory of Impulsive Differential Equations, World Scientific, Singapore, (1989).

[10] Zhen Jin, Ma Zhien, and Han Maoan,*The existence of periodic solutions of the n-species Lotka-Volterra competition systems with impulsive*, (2004).