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SORTING: A MATHEMATICAL MAPPING

SRIMANTA PAL

Electronics and Communication Sciences Unit, Indian Statistical Institute
203 B T Road, Kolkata 700 108, India
srimanta@isical.ac.in,srimanta59@gmail.com

ABSTRACT. Sorting can be viewed as a mapping in mathematics. The computational model for the ith sorted element y_i in the sorted sequence $\{y_1, y_2, \ldots, y_m\}$ is a mathematical function of the elements in an unsorted sequence $\{x_1, x_2, \ldots, x_n\}$ where $m \leq n$ and m is the number of distinct elements in the unsorted sequence $\{x_1, x_2, \ldots, x_n\}$.

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1. Introduction

Sorting is an attentive and most important problem among all computational tasks in mathematics, statistics and computer science. It is frequently occurring problem in various applications such as collecting related things, finding duplicates, and enhance the traditional searching techniques in the field of business, computer, science, and technology. Sorting is a technique that applied on an unsorted set of elements and transformed to a sequence of sorted elements with respect to some given ordering. In general the sorting techniques can be grouped into two major classes (Mehlhorn 1984):

1. *Linear ordering/Comparison-based class*: They make only use of the fact that the universe is linearly order. The sorting algorithm belongs to this class is known as *general sorting algorithms*.
2. *Lexicographical ordering class*: The algorithms in this class only work for keys in a restricted domain.

The traditional sorting techniques are designed in an algorithmic approach such as bubble sort, heap sort, quick sort, merge sort, etc. (Knuth 1973, Mehlhorn 1984). The time-complexity of these algorithms varies from $O(n^2)$ to $O(1)$ (Knuth 1973, Mehlhorn 1984, Jang & Prasana 1995). For example, the performance of bubble sort algorithm is $O(n^2)$, quick sort algorithm is $O(n \log n)$ (Knuth 1973, Mehlhorn 1984) but of some parallel sorting algorithms (Akl 1989) are $O(1)$.

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In this paper we revisit the sorting problem from different angles and this is not like a traditional algorithmic approach. Here we model the sorting problem as a mathematical function of the given unsorted distinct elements. That means we design a mathematical relationship between the element y_i in the sorted sequence $\{y_1, y_2, \ldots, y_n\}$ and the elements $x_1, x_2, \ldots, x_n$ in the unsorted sequence $\{x_1, x_2, \ldots, x_n\}$. This mathematical relationship is established based on linear ordering.

This paper is organized into six sections. In Section 2 we describe a mathematical model for the sorting problem and stated a theorem. In Section 3 we describe a set of definitions and terminology for the proof of the theorem. In Section 4 we proved a set of lemmas and illustrations. In Section 5 we discuss the computational complexity of the proposed model. Finally, Section 6 concludes the outcome of this model.

2. Modeling of Sorting Problem

Suppose $X = \{x_1, x_2, \ldots, x_n\}$ be an unsorted sequence and $Y = \{y_1, y_2, \ldots, y_n\}$ be a sorted sequence of X. The sorted sequence Y is obtained from X by some mathematical transformation as

$$\{x_1, x_2, \ldots, x_n\} \stackrel{\text{Sorting}}{\longrightarrow} \{y_1, y_2, \ldots, y_n\}.$$

For example, a sorted sequence $Y = \{7, 5, 4, 3\}$ is obtained from an unsorted sequence $X = \{4, 3, 7, 5\}$. Here the first element 4 in the unsorted sequence $\{4, 3, 7, 5\}$ is mapped to the third element in the sorted sequence $\{7, 5, 4, 3\}$. Similarly the second element 3 in X is mapped to the fourth element in Y, and so on. This mapping is shown in Fig. 1.

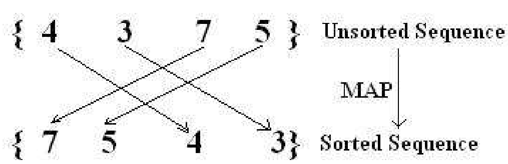

FIGURE 1. A map of unsorted sequence to sorted sequence.

Now we shall design a mathematical function for this transformation. Here the ith sorted element y_i in $Y = \{y_1, y_2, \ldots, y_n\}$ is a function of the elements $x_1, x_2, \ldots, x_n$ in the unsorted sequence $X = \{x_1, x_2, \ldots, x_n\}$. That is, $y_i = F_i(x_1, x_2, \ldots, x_n)$ for $i \in \{1, 2, \ldots, m\}$ where $m (\leq n)$ is the number of distinct elements in the sequence $X = \{x_1, x_2, \ldots, x_n\}$ and F_i is the function of $x_1, x_2, \ldots, x_n$. The mathematical function for the sorting problem is stated in Theorem 2.1.

Theorem 2.1. *An element y_i of a sorted sequence $\{y_1, y_2, \dots, y_n\}$ has a mathematical relationship to the elements of the unsorted sequence $\{x_1, x_2, \dots, x_n\}$. This relationship can be written as:*

$$y_i = \sum_{q=1}^{n} \left[\left\{ \frac{\sum_{j=1}^{n} \left\{ x_j H^{(2)} \left(\sum_{k=1}^{n} H^{(1)}(x_j, x_k), q \right) \right\}}{\sum_{j=1}^{n} H^{(2)} \left(\sum_{k=1}^{n} H^{(1)}(x_j, x_k), q \right)} \right\} \times \; H^{(4)}_{q,j} \left(r^{(1)}_q \right) \right]$$

for $i = 1, 2, \dots, m$, $m(\leq n)$ is the number of distinct elements in the sequence $X = \{x_1, x_2, \dots, x_n\}$. where

$$\begin{aligned}
y_i &\in \{x_1, x_2, \dots, x_n\} \\
f^{(1)}_u &= \sum_{j=1}^{n} H^{(2)} \left(\sum_{k=1}^{n} H^{(1)}(x_j, x_k), u \right), \quad u = 1, 2, \dots, n \\
r_i &= \sum_{j=1}^{n} H^{(1)}(x_j, x_i) \\
r^{(1)}_i &= \begin{cases} \dfrac{\sum_{j=1}^{n} r_j H^{(2)}(r_j, i)}{\sum_{j=1}^{n} H^{(2)}(r_j, i)}, & \sum_{j=1}^{n} H^{(2)}(r_j, i) \neq 0 \\ 0, & \text{otherwise} \end{cases}
\end{aligned}$$

$$H^{(1)}(x_j, x_i) = \begin{cases} 1, & x_j \geq x_i \\ 0, & \text{otherwise} \end{cases}$$

$$H^{(2)}(x, i) = \begin{cases} 1, & x = i \\ 0, & \text{otherwise} \end{cases}$$

$$H^{(3)}\left(f^{(1)}_i\right) = \begin{cases} 1, & f^{(1)}_i = 0 \\ 0, & \text{otherwise} \end{cases}$$

$$H^{(4)}_{q,i}\left(r^{(1)}_q\right) = \begin{cases} 1, & r^{(1)}_q = i \\ 0, & \text{otherwise} \end{cases}$$

$$m = n - \sum_{u=1}^{n} H^{(3)}\left(f^{(1)}_u\right).$$

We prove Theorem 2.1 after describing a set of terminology in the following section.

3. Some Related Definitions and Terminology

In this section we define a set of related definitions and terminology used in the proof of Theorem 2.1.

Definition 3.1. (Unsorted sequence) A sequence $\{x_1, x_2, \ldots, x_n\}$ is said to be *unsorted sequence* when the logical relationship between any two elements x_i and x_j in the sequence are not known for all $i, j \in \{1, 2, \ldots, n\}$ and $i \neq j$.

Example 3.2. An unsorted sequence is $\{4, 3, 7, 5\}$.

Definition 3.3. (Sorted sequence) A sequence $\{x_1, x_2, \ldots, x_n\}$ is said to be *sorted sequence* when the logical relationship between any two elements x_i and x_j in the sequence are known for all $i, j \in \{1, 2, \ldots, n\}$ and $i \neq j$.

Example 3.4. A sorted sequence is {3, 4, 5, 7} or {7, 5, 4, 3}.

Definition 3.5. (**Descending order**) A sorted sequence $\{x_1, x_2, \ldots, x_n\}$ is said to be in *descending order* if $x_i \geq x_j$ for $i < j$, $i, j \in \{1, 2, \ldots, n\}$.

Example 3.6. Descending ordered sorted sequence is {7, 5, 4, 3}.

Definition 3.7. (**Ascending order**) A sorted sequence $\{x_1, x_2, \ldots, x_n\}$ is said to be in *ascending order* if $x_i \leq x_j$ for $i < j$, $i, j \in \{1, 2, \ldots, n\}$.

Example 3.8. Ascending ordered sorted sequence is {3, 4, 5, 7}.

Definition 3.9. (**Pseudo-ranking**) A sequence $\{x_1, x_2, \ldots, x_n\}$ is equivalent to the sequence $\{r_1, r_2, \ldots, r_n\}$ where $r_i \in \{1, 2, \ldots, n\}$ for all $i \in \{1, 2, \ldots, n\}$. This r_i is said to be the *pseudo rank* of the element x_i in the sequence $\{x_1, x_2, \ldots, x_n\}$. The combined sequence is $\left\{ \begin{pmatrix} x_1 \\ r_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ r_2 \end{pmatrix}, \ldots, \begin{pmatrix} x_n \\ r_n \end{pmatrix} \right\}$. This combined sequence is termed as *pseudo ranked sequence.*

Example 3.10. The sequence $\{97, 53, 86, 75, 64, 75, 97, 53, 75, 53\}$ is equivalent to $\{2, 10, 3, 6, 7, 6, 2, 10, 6, 10\}$. Therefore the new sequence $\{2, 10, 3, 6, 7, 6, 2, 10, 6, 10\}$ is representing the ranks sequence of $\{97, 53, 86, 75, 64, 75, 97, 53, 75, 53\}$, i.e., the pseudo rank of the element 75 in the sequence is 6. Therefore the pseudo ranked sequence can be represented by

$$\left\{ \begin{pmatrix} 97 \\ 2 \end{pmatrix}, \begin{pmatrix} 53 \\ 10 \end{pmatrix}, \begin{pmatrix} 86 \\ 3 \end{pmatrix}, \begin{pmatrix} 75 \\ 6 \end{pmatrix}, \begin{pmatrix} 64 \\ 7 \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} 75 \\ 6 \end{pmatrix}, \begin{pmatrix} 97 \\ 2 \end{pmatrix}, \begin{pmatrix} 53 \\ 10 \end{pmatrix}, \begin{pmatrix} 75 \\ 6 \end{pmatrix}, \begin{pmatrix} 53 \\ 10 \end{pmatrix} \right\}.$$

Definition 3.11. (Pure-ranking) A sequence $\{x_1, x_2, \ldots, x_n\}$ is equivalent to $\{r_1, r_2, \ldots, r_n\}$ where $r_i \in \{1, 2, \ldots, m\}$ for all $i \in \{1, 2, \ldots, n\}$ and $m(\leq n)$ is the number of distinct elements in the sequence $\{x_1, x_2, \ldots, x_n\}$. Then this r_i is said to be the *pure rank* of the element x_i in the sequence $\{x_1, x_2, \ldots, x_n\}$. The combined sequence is $\left\{\begin{pmatrix} x_1 \\ r_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ r_2 \end{pmatrix}, \ldots, \begin{pmatrix} x_n \\ r_n \end{pmatrix}\right\}$. This combined sequence is termed as *pure ranked sequence.*

Example 3.12. The sequence $\{97, 53, 86, 75, 64, 75, 97, 53, 75, 53\}$ is equivalent to $\{1, 5, 2, 3, 4, 3, 1, 5, 3, 5\}$. Therefore the new sequence $\{1, 5, 2, 3, 4, 3, 1, 5, 3, 5\}$ is representing the pure rank sequence of $\{97, 53, 86, 75, 64, 75, 97, 53, 75, 53\}$, i.e., the pure rank of the element 75 in the sequence is 3. Therefore its pure ranked sequence is represented by

$$\left\{\begin{pmatrix} 97 \\ 1 \end{pmatrix}, \begin{pmatrix} 53 \\ 5 \end{pmatrix}, \begin{pmatrix} 86 \\ 2 \end{pmatrix}, \begin{pmatrix} 75 \\ 3 \end{pmatrix}, \begin{pmatrix} 64 \\ 4 \end{pmatrix}, \begin{pmatrix} 75 \\ 3 \end{pmatrix}, \begin{pmatrix} 97 \\ 1 \end{pmatrix}, \begin{pmatrix} 53 \\ 5 \end{pmatrix}, \begin{pmatrix} 75 \\ 3 \end{pmatrix}, \begin{pmatrix} 53 \\ 5 \end{pmatrix}\right\}.$$

Definition 3.13. (Frequency ranked sequencing) A (pure or pseudo) rank sequence

$$\left\{\begin{pmatrix} x_1 \\ r_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ r_2 \end{pmatrix}, \ldots, \begin{pmatrix} x_n \\ r_n \end{pmatrix}\right\}$$

of the given sequence $\{x_1, x_2, \ldots, x_n\}$ is transformed to a frequency ranked sequence,

$$\left\{\begin{pmatrix} x_1 \\ r_1 \\ f_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ r_2 \\ f_2 \end{pmatrix}, \ldots, \begin{pmatrix} x_m \\ r_m \\ f_m \end{pmatrix}\right\}$$

where f_i is the frequency of x_i for $i \in \{1, 2, \ldots, m\}$ and $m(\leq n)$ is the number of distinct elements in the sequence $\{x_1, x_2, \ldots, x_n\}$.

Example 3.14. Consider a pure ranked sequence

$$\left\{\begin{pmatrix} 97 \\ 1 \end{pmatrix}, \begin{pmatrix} 53 \\ 5 \end{pmatrix}, \begin{pmatrix} 86 \\ 2 \end{pmatrix}, \begin{pmatrix} 75 \\ 3 \end{pmatrix}, \begin{pmatrix} 64 \\ 4 \end{pmatrix}, \begin{pmatrix} 75 \\ 3 \end{pmatrix}, \begin{pmatrix} 97 \\ 1 \end{pmatrix}, \begin{pmatrix} 53 \\ 5 \end{pmatrix}, \begin{pmatrix} 75 \\ 3 \end{pmatrix}, \begin{pmatrix} 53 \\ 5 \end{pmatrix}\right\}$$

of the given sequence $\{97, 53, 86, 75, 64, 75, 97, 53, 75, 53\}$. Now we can transform this pure ranked sequence to frequency ranked sequence as

$$\left\{\begin{pmatrix} 97 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 86 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 75 \\ 3 \\ 3 \end{pmatrix}, \begin{pmatrix} 64 \\ 4 \\ 1 \end{pmatrix}, \begin{pmatrix} 53 \\ 5 \\ 3 \end{pmatrix}\right\}.$$

where rank (pure) and frequency of the element 53 in the given sequence $\{97, 53, 86, 75, 64, 75, 97, 53, 75, 53\}$ are 5 and 3 respectively.

4. **Proof of Theorem 2.1**

Now we prove the Theorem 2.1. In this proof we need a set of results those are discussed in the form of following lemmas.

Lemma 4.1. *An unsorted sequence $X = \{x_1, x_2, \ldots, x_n\}$ can be transformed to a pseudo ranked sequence*

$$\left\{ \begin{pmatrix} x_1 \\ r_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ r_2 \end{pmatrix}, \ldots, \begin{pmatrix} x_n \\ r_n \end{pmatrix} \right\}$$

where

$$r_i = \sum_{j=1}^{n} H^{(1)}(x_j, x_i),$$

$$H^{(1)}(x_j, x_i) = \begin{cases} 1, & x_j \geq x_i \\ 0, & \text{otherwise} \end{cases} \qquad i \in \{1, 2, \ldots, n\}$$

then r_i is the pseudo rank of the element x_i in the sequence $X = \{x_1, x_2, \ldots, x_n\}$ and $1 \leq r_i \leq n$.

Proof. The sum $r_i = \sum_{j=1}^{n} H^{(1)}(x_j, x_i)$ is an integer. Since $x_i \in \{x_1, x_2, \ldots, x_n\}$ then each term $H^{(1)}(x_j, x_i)$ in the sum is contributing either 0 or 1. Again the minimum value of r_i is 1 since $H^{(1)}(x_i, x_i) = 1$. In the sum, each term $H^{(1)}(x_j, x_i)$ contributes 1 when x_i is the minimum in the sequence. Therefore the sum r_i is n. So each r_i must lies between 1 and n, i.e., $1 \leq r_i \leq n$. Note that, if $x_i \notin \{x_1, x_2, \ldots, x_n\}$ then $r_i = 0$.

Case I: If two distinct elements x_i, x_j such that $x_i, x_j \in \{x_1, x_2, \ldots, x_n\}$ and $x_i \neq x_j$ then $r_i \neq r_j$ for all $i, j \in \{1, 2, \ldots, n\}$ and $i \neq j$.

In this case, the element x_i partitions the given sequence $X = \{x_1, x_2, \ldots, x_n\}$ into two parts. One part contains the elements $< x_i$ and other part contains the elements $\geq x_i$. The number of elements in the second part is r_i. Similarly we get r_j. Since $x_i \neq x_j$ then the number of elements in the second parts in both the cases is not same, i.e., $r_i \neq r_j$. Again if $x_i < x_j$ then $r_i > r_j$. Therefore r_i is distinctly representing the x_i by an integer. So r_i is the rank of x_i.

Case II: If two elements x_i, x_j such that $x_i, x_j \in \{x_1, x_2, \ldots, x_n\}$ and $x_i = x_j$ then $r_i = r_j$ for all $i, j \in \{1, 2, \ldots, n\}$ and $i \neq j$.

In this case, the rank of x_i and x_j are same. That means same integer will assign for r_i and r_j. Suppose $x_k \in X$ and it is minimum then $r_k = n$ where $n = |X|$.

Therefore all integer in between 1 and n is not used for ranking the elements of X. Hence present ranks are pseudo ranks.

The pseudo ranked sequence can be written as $\left\{\begin{pmatrix} x_1 \\ r_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ r_2 \end{pmatrix}, \ldots, \begin{pmatrix} x_n \\ r_n \end{pmatrix}\right\}$ where the elements in the sequence $\{x_1, x_2, \ldots, x_n\}$ are not in sorted order. □

Example 4.2. Consider an unsorted sequence $X = \{10, 13, 17, 5, 2, 5, 2, 12, 17, 2, 5, 17\}$. Then we compute $H^{(1)}$ where, $H^{(1)}_{ji} = H^{(1)}(x_j, x_i)$.

$$H^{(1)} = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Therefore the pseudo ranked sequence is

$$\left\{\begin{pmatrix} 10 \\ 6 \end{pmatrix}, \begin{pmatrix} 13 \\ 4 \end{pmatrix}, \begin{pmatrix} 17 \\ 3 \end{pmatrix}, \begin{pmatrix} 5 \\ 9 \end{pmatrix}, \begin{pmatrix} 2 \\ 12 \end{pmatrix}, \begin{pmatrix} 5 \\ 9 \end{pmatrix},\right.$$

$$\left.\begin{pmatrix} 2 \\ 12 \end{pmatrix}, \begin{pmatrix} 12 \\ 5 \end{pmatrix}, \begin{pmatrix} 17 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 12 \end{pmatrix}, \begin{pmatrix} 5 \\ 9 \end{pmatrix}, \begin{pmatrix} 17 \\ 3 \end{pmatrix}\right\}.$$

Example 4.3. Consider an unsorted sequence $X = \{4, 3, 7, 5\}$ then we can compute

$$H^{(1)} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{pmatrix}.$$

The pseudo ranked sequence is $\left\{\begin{pmatrix} 4 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 7 \\ 1 \end{pmatrix}, \begin{pmatrix} 5 \\ 2 \end{pmatrix}\right\}$.

Note that, this is a pure ranked sequence since there is no duplication in elements in the given sequence X.

Lemma 4.4. *A pseudo ranked sequence*

$$\left\{\begin{pmatrix} x_1 \\ r_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ r_2 \end{pmatrix}, \ldots, \begin{pmatrix} x_n \\ r_n \end{pmatrix}\right\}$$

is obtained from the unsorted input sequence $X = \{x_1, x_2, \ldots, x_n\}$*. This pseudo ranked sequence is then transformed to frequency ranked sequence*

$$\left\{\begin{pmatrix} x_1^{(1)} \\ r_1^{(1)} \\ f_1^{(1)} \end{pmatrix}, \begin{pmatrix} x_2^{(1)} \\ r_2^{(1)} \\ f_2^{(1)} \end{pmatrix}, \ldots, \begin{pmatrix} x_n^{(1)} \\ r_n^{(1)} \\ f_n^{(1)} \end{pmatrix}\right\}$$

where

$$x_i^{(1)} = \begin{cases} \dfrac{\displaystyle\sum_{j=1}^{n} x_j H^{(2)}(r_j, i)}{\displaystyle\sum_{j=1}^{n} H^{(2)}(r_j, i)}, & \sum_{j=1}^{n} H^{(2)}(r_j, i) \neq 0 \\ 0, & \textit{otherwise} \end{cases},$$

$$x_i^{(1)} \in \{0, x_1, x_2, \ldots, x_n\},$$

$$r_i^{(1)} = \begin{cases} \dfrac{\displaystyle\sum_{j=1}^{n} r_j H^{(2)}(r_j, i)}{\displaystyle\sum_{j=1}^{n} H^{(2)}(r_j, i)}, & \sum_{j=1}^{n} H^{(2)}(r_j, i) \neq 0 \\ 0, & \textit{otherwise} \end{cases}$$

$$r_i^{(1)} \in \{0, r_1, r_2, \ldots, r_n\},$$

$$f_i^{(1)} = \sum_{j=1}^{n} H^{(2)}(r_j, i), \quad 0 \leq f_i^{(1)} \leq n,$$

$$H^{(2)}(r_j, i) = \begin{cases} 1, & r_j = i \\ 0, & \textit{otherwise} \end{cases}$$

for $i, j \in \{1, 2, \ldots, n\}$*. This transformed sequence must have the following property:*

1. *The elements in the sequence* $\left\{x_1^{(1)}, x_2^{(1)}, \ldots, x_n^{(1)}\right\}$ *are in sorted order, that is,* $x_i^{(1)} > x_j^{(1)}$*,* $i < j$ *when* $r_i^{(1)}, r_j^{(1)} \neq 0$ *and* $i, j \in \{1, 2, \ldots, n\}$*.*
2. *The nonzero elements of the sequence* $\left\{r_1^{(1)}, r_2^{(1)}, \ldots, r_n^{(1)}\right\}$ *are in sorted order, that is,* $r_i^{(1)} < r_j^{(1)}$*,* $i < j$ *when* $r_i^{(1)}, r_j^{(1)} \neq 0$ *and* $i, j \in \{1, 2, \ldots, n\}$*.*

Note that, unused elements and ranks in the sequence are considered as zeros.

Proof. The objective of this lemma is to combine all the ith ranked elements in the sequence $\{x_1, x_2, \ldots, x_n\}$ at the ith position of the new sequence $\left\{x_1^{(1)}, x_2^{(1)}, \ldots, x_n^{(1)}\right\}$. And also the rank of $x_i^{(1)}$ in the sequence $\left\{x_1^{(1)}, x_2^{(1)}, \ldots, x_n^{(1)}\right\}$ is $r_i^{(1)} = i$.

Here the function $H^{(2)}(r_j, i)$ is marking all the ith ranked elements in the sequence $\{x_1, x_2, \ldots, x_n\}$ and also counting them by $f_i^{(1)} = \sum_{j=1}^{n} H^{(2)}(r_j, i)$. Suppose p number of elements are having same rank (say, ith rank) and value of the element x_i is v (say) then $\sum_{j=1}^{n} x_j H^{(2)}(r_j, i) = pv$ since $p = f_i^{(1)}$ and $v = x_i$. Therefore $\frac{pv}{p} = v$ $(p \neq 0)$, i.e., the new sequence $\left\{x_1^{(1)}, x_2^{(1)}, \ldots, x_n^{(1)}\right\}$ is related to

$$x_i^{(1)} = \frac{\sum_{j=1}^{n} x_j H^{(2)}(r_j, i)}{\sum_{j=1}^{n} H^{(2)}(r_j, i)},$$

provided $\sum_{j=1}^{n} H^{(2)}(r_j, i) \neq 0$. Similarly

$$r_i^{(1)} = \frac{\sum_{j=1}^{n} r_j H^{(2)}(r_j, i)}{\sum_{j=1}^{n} H^{(2)}(r_j, i)},$$

provided $\sum_{j=1}^{n} H^{(2)}(r_j, i) \neq 0$. Now the relation $x_i^{(1)} = \frac{\sum_{j=1}^{n} x_j H^{(2)}(r_j, i)}{\sum_{j=1}^{n} H^{(2)}(r_j, i)}$ can be rewritten as

$$x_i^{(1)} = \begin{cases} cc\frac{1}{f_i^{(1)}} \sum_{j=1}^{n} x_j \delta_{ji}, & f_i^{(1)} \neq 0 \\ 0, & \text{otherwise} \end{cases}$$

$$r_i^{(1)} = \begin{cases} \frac{1}{f_i^{(1)}} \sum_{j=1}^{n} \left(\sum_{k=1}^{n} H^{(1)}(x_j, x_k) \right) \delta_{ji}, & f_i^{(1)} \neq 0 \\ 0, & \text{otherwise} \end{cases}$$

where,

$$\delta_{ji} = H^{(2)}\left(\sum_{k=1}^{n} H^{(1)}\left(x_j, x_k\right), i\right) = \begin{cases} 1, & \sum_{k=1}^{n} H^{(1)}\left(x_j, x_k\right) = i \\ 0, & \text{otherwise} \end{cases}$$

$$f_i^{(1)} = \sum_{j=1}^{n} \delta_{ji}$$

□

Example 4.5. We compute $H^{(2)}$ using the pseudo ranked sequence in Example 4.2 Now the partially sorted frequency ranked sequence can be computed using above relation as

$$\left\{\begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}17\\3\\3\end{pmatrix}, \begin{pmatrix}13\\4\\1\end{pmatrix}, \begin{pmatrix}12\\5\\1\end{pmatrix},\right.$$
$$\left.\begin{pmatrix}10\\6\\1\end{pmatrix}, \begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}5\\9\\3\end{pmatrix}, \begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}2\\12\\3\end{pmatrix}\right\}$$

where

$$H^{(2)} = \begin{pmatrix} 0&0&0&0&0&1&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&1 \\ 0&0&0&0&1&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&0&1&0&0&0 \\ 0&0&1&0&0&0&0&0&0&0&0&0 \end{pmatrix}$$

Example 4.6. We compute $H^{(2)}$ using the pseudo ranked sequence in Example 4.3

$$H^{(2)} = \begin{pmatrix} 0&0&1&0 \\ 0&0&0&1 \\ 1&0&0&0 \\ 0&1&0&0 \end{pmatrix}$$

Now the partially sorted frequency sequence is $\left\{\begin{pmatrix}7\\1\\1\end{pmatrix},\begin{pmatrix}5\\2\\1\end{pmatrix},\begin{pmatrix}4\\3\\1\end{pmatrix},\begin{pmatrix}3\\4\\1\end{pmatrix}\right\}$.

Lemma 4.7. *The pseudo ranked sequence*

$$\left\{\begin{pmatrix}x_1^{(1)}\\r_1^{(1)}\\f_1^{(1)}\end{pmatrix},\begin{pmatrix}x_2^{(1)}\\r_2^{(1)}\\f_2^{(1)}\end{pmatrix},\ldots,\begin{pmatrix}x_n^{(1)}\\r_n^{(1)}\\f_n^{(1)}\end{pmatrix}\right\}$$

is transformed to pure ranked sequence $\left\{\begin{pmatrix}x_1^{(2)}\\r_1^{(2)}\\f_1^{(2)}\end{pmatrix},\begin{pmatrix}x_2^{(2)}\\r_2^{(2)}\\f_2^{(2)}\end{pmatrix},\ldots,\begin{pmatrix}x_n^{(2)}\\r_n^{(2)}\\f_n^{(2)}\end{pmatrix}\right\}$ *by the following transformations.*

$$x_i^{(2)} = x_i^{(1)},$$
$$r_i^{(2)} = \left(1 - H^{(3)}(f_i^{(1)})\right)\left(r_i^{(1)} - \sum_{j=1}^{i} H^{(3)}(f_i^{(1)})\right),$$
$$f_i^{(2)} = f_i^{(1)} \quad \textit{for } i = 1, 2, \ldots, n$$

where

$$H^{(3)}\left(f_i^{(1)}\right) = \begin{cases} 1, & f_i^{(1)} = 0 \\ 0, & \text{otherwise} \end{cases}$$

and the number of nonzero frequency elements is computed by the formula $m = n - \sum_{u=1}^{n} H^{(3)}(f_u^{(1)})$.

Proof. In this transformation $x_i^{(1)}$ and $f_i^{(1)}$ remains unchanged. Only we have to rescale the ranks of each element. That means the number of 0 ranked elements have to subtract from its present rank. Here the function $H^{(3)}\left(f_i^{(1)}\right)$ computes a flag that indicates $H^{(3)}\left(f_i^{(1)}\right)$ is 1 for 0 frequency. Therefore the new rank will be $r_i^{(1)} - \sum_{j=1}^{i} H^{(3)}(f_i^{(1)})$. But it finds a minor problem when $r_i^{(1)} = 0$. This can be rewritten as $\left(1 - H^{(3)}(f_i^{(1)})\right)\left(r_i^{(1)} - \sum_{j=1}^{i} H^{(3)}(f_i^{(1)})\right)$ where the factor $\left(1 - H^{(3)}(f_i^{(1)})\right)$ will take care the condition when $r_i^{(1)} = 0$. □

Example 4.8. Using the partially sorted frequency sequence in Example 4.5, we can compute

$$H^{(3)} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \end{pmatrix}.$$

Therefore the partially sorted pure ranked sequence is

$$\left\{\begin{pmatrix}0\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\0\end{pmatrix},\begin{pmatrix}17\\1\\3\end{pmatrix},\begin{pmatrix}13\\2\\1\end{pmatrix},\begin{pmatrix}12\\3\\1\end{pmatrix},\begin{pmatrix}10\\4\\1\end{pmatrix},\begin{pmatrix}0\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\0\end{pmatrix},\begin{pmatrix}5\\5\\3\end{pmatrix},\begin{pmatrix}0\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\0\end{pmatrix},\begin{pmatrix}2\\6\\3\end{pmatrix}\right\}$$

Here $m = n - \sum_{u=1}^{n} H^{(3)}(f_u^{(1)}) = 12 - 6 = 6$.

Example 4.9. Using the partially sorted frequency sequence in Example 4.6, we can compute

$$H^{(3)} = \begin{pmatrix} 0 & 0 & 0 & 0 \end{pmatrix}.$$

Therefore the partially sorted pure ranked sequence is

$$\left\{\begin{pmatrix}7\\1\\1\end{pmatrix},\begin{pmatrix}5\\2\\1\end{pmatrix},\begin{pmatrix}4\\3\\1\end{pmatrix},\begin{pmatrix}3\\4\\1\end{pmatrix}\right\},$$

Here $m = n - \sum_{u=1}^{n} H^{(3)}(f_u^{(1)}) = 4 - 0 = 4$. Now we can rewrite $r_i^{(2)}$ as $r_i^{(2)} = \left(1 - H^{(3)}(f_i^{(1)})\right)\left(r_i^{(1)} - \sum_{j=1}^{i} H^{(3)}(f_i^{(1)})\right)$ where,

$$r_i^{(1)} = \begin{cases} \dfrac{1}{f_i^{(1)}} \displaystyle\sum_{j=1}^{n} \left(\sum_{k=1}^{n} H^{(1)}(x_j, x_k) \right) \delta_{ji}, & f_i^{(1)} \neq 0 \\ 0, & \text{otherwise} \end{cases}$$

$$\delta_{ji} = H^{(2)}\left(\sum_{k=1}^{n} H^{(1)}(x_j, x_k), i \right) = \begin{cases} 1, & \displaystyle\sum_{k=1}^{n} H^{(1)}(x_j, x_k) = i \\ 0, & \text{otherwise} \end{cases}$$

$$f_i^{(1)} = \sum_{j=1}^{n} \delta_{ji}$$

Lemma 4.10. *The pseudo sorted sequence is transformed to pure sorted sequence, i.e.,*

$$\left\{\begin{pmatrix}x_1^{(2)}\\r_1^{(2)}\\f_1^{(2)}\end{pmatrix},\begin{pmatrix}x_2^{(2)}\\r_2^{(2)}\\f_2^{(2)}\end{pmatrix},\ldots,\begin{pmatrix}x_n^{(2)}\\r_n^{(2)}\\f_n^{(2)}\end{pmatrix}\right\} \to \left\{\begin{pmatrix}x_1^{(3)}\\r_1^{(3)}\\f_1^{(3)}\end{pmatrix},\begin{pmatrix}x_2^{(3)}\\r_2^{(3)}\\f_2^{(3)}\end{pmatrix},\ldots,\begin{pmatrix}x_m^{(3)}\\r_m^{(3)}\\f_m^{(3)}\end{pmatrix}\right\}$$

where $x_i^{(3)} = \sum_{q=1}^{n} x_q^{(2)} H_{q,i}^{(4)}\left(r_q^{(2)}\right),\ r_i^{(3)} = \sum_{q=1}^{n} r_q^{(2)} H_{q,i}^{(4)}\left(r_q^{(2)}\right),\ f_i^{(3)} = \sum_{q=1}^{n} f_q^{(2)} H_{q,i}^{(4)}\left(r_q^{(2)}\right),$ *and* $H_{q,i}^{(4)}\left(r_q^{(2)}\right) = \begin{cases} 1, & r_q^{(2)} = i \\ 0, & \textit{otherwise} \end{cases}$

Proof. The task of this lemma is to shift the elements to their appropriate rank and index is same. The function $H_{q,i}^{(4)}\left(r_q^{(2)}\right)$ computes the ith entry that will shift to the qth entry by the following transformation $x_i^{(3)} = \sum_{q=1}^{n} x_q^{(2)} H_{q,i}^{(4)}\left(r_q^{(2)}\right),\ r_i^{(3)} = \sum_{q=1}^{n} r_q^{(2)} H_{q,i}^{(4)}\left(r_q^{(2)}\right),\ f_i^{(3)} = \sum_{q=1}^{n} f_q^{(2)} H_{q,i}^{(4)}\left(r_q^{(2)}\right),\ H_{q,i}^{(4)}\left(r_q^{(2)}\right) = \begin{cases} 1, & r_q^{(2)} = i \\ 0, & \text{otherwise} \end{cases}$ for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$. ☐

Example 4.11. The partially sorted pure ranked sequence from Example 4.8 we obtained $m = 6$. Then we can compute $H^{(4)} = \begin{pmatrix} 0&0&1&0&0&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&1 \end{pmatrix}$

Therefore the sorted sequence is

$$\left\{ \begin{pmatrix} 17 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 13 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 12 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 10 \\ 4 \\ 1 \end{pmatrix}, \begin{pmatrix} 5 \\ 5 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 6 \\ 3 \end{pmatrix} \right\}.$$

Example 4.12. The partially sorted pure ranked sequence from Example 4.9 we obtained $m = 4$. Then we can compute $H^{(4)} = \begin{pmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{pmatrix}$. Now the sorted sequence is $\left\{ \begin{pmatrix} 7 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 5 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} \right\}$. Now we can rewrite $x_i^{(3)}$ as

$$x_i^{(3)} = \sum_{q=1}^{n} \left[\left\{ \frac{\sum_{j=1}^{n} \left\{ x_j H^{(2)} \left(\sum_{k=1}^{n} H^{(1)}(x_j, x_k), q \right) \right\}}{\sum_{j=1}^{n} H^{(2)} \left(\sum_{k=1}^{n} H^{(1)}(x_j, x_k), q \right)} \right\} \times H_{q,j}^{(4)}\left(r_q^{(2)}\right) \right]$$

for $i = 1, 2, \ldots, m$.

Hence Lemmas 4.1–4.10 follow the Theorem 2.1.

5. Computational Complexity

The time complexity for the computation of the expression for y_i depends on the machine and its architecture. In case of sequential computation its time complexity is not satisfactory with respect to the algorithmic approach but for other computing environment such as parallel (Jang & Prasana 1995) or neurocomputing paradigm (Chen & Hsieh 1993, Takefuji & Lee 1990, Yao 1995) is very good and it comes down to $O(1)$. It is also interesting that the time complexity for the computation does not depends on the distribution of the elements in the unsorted sequence.

6. Conclusion

Sorting sorts an unsorted sequence by the traditional algorithmic method. In this paper we have designed an exact mathematical relationship between the sorted elements y_i in the sorted sequence $\{y_1, y_2, \ldots, y_n\}$ with the elements $x_1, x_2, \ldots, x_n$ in the unsorted sequence $\{x_1, x_2, \ldots, x_n\}$. If we need only the ith sorted element then formula for $y_i = F_i\,(x_1, x_2, \ldots, x_n)$ for $i = 1, 2, \ldots, m$ where $m \leq n$ is sufficient without sorting the whole sequence. So it directly computes the ith order statistics. This mathematical relationship is established based on linear ordering.

This design can be extended to lexicographical ordering or any other user defined ordering by the proper choice of the function $H^{(1)}\,(x_j, x_i)$.

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