Neural, Parallel, and Scientific Computations 21 (2013) 1-18

NUMERICAL METHODS FOR STOCHASTIC FRACTIONAL DIFFERENTIAL EQUATIONS

JEAN-CLAUDE PEDJEU AND G. S. LADDE

Department of Mathematical Sciences, Tennessee State University
Nashville, TN 37209, USA
Department of Mathematics and Statistics, University of South Florida
Tampa, FL 33620-5700, USA

ABSTRACT. Various dynamic processes in sciences, engineering, and finance operate under multi-time scales generated by the system parametric variations. In our earlier work, we formulated a mathematical model of such processes, and introduced a concept of a system of multi-time scale stochastic differential equations. We then investigated the fundamental properties such systems and also developed methods for finding closed-form solutions for linear or nonlinear reducible special cases. In this work, we developed a numerical algorithm that is based on the idea of numerical integration in the context of the notion of multi-time scale integration. Finally, the developed numerical scheme is applied to the mathematical model of epidemiological processes.

AMS (MOS) Subject Classification. Primary 65H20, Secondary 60H35, 60H10

1. Introduction

In this work, we develop a numerical scheme for stochastic fractional differential equations (2.1) whose fundamental properties were studied in [20]. To do so, we combine the Euler-Maruyama numerical scheme for stochastic differential equations [2, 7, 15] with the predictor-corrector numerical method for fractional differential equations [3, 4, 5, 13]. Furthermore, we prove the global convergence result for the presented numerical scheme. We then apply the developed numerical scheme to the model of epidemiological processes presented in [20].

2. Preliminaries

We recall the formulation of the stochastic fractional differential equations (SFDE):

$$dx = b(t,x)dt + \sigma_1(t,x)dB_t + \sigma_2(t,x)(dt)^\alpha, \quad x(t_0) = x_0, \tag{2.1}$$

where $0 < \alpha < 1$, $b, \sigma_1, \sigma_2 : [t_0, T] \times \mathbb{R} \to \mathbb{R}$ are continuous functions; the differentials dt, dB_t and $(dt)^\alpha$ are in the sense of Cauchy-Riemann or Lebesgue, Itô-Doob and Caputo/Riemann-Liouville fractional differentials, respectively; the differential "d"

Received January 8, 2013

(of x) is the sum of three linearly independent differential operators, d_1, d_2 , and d_3 defined by $d_1x = b(t,x)dt$, $d_2x = \sigma_1(t,x)dB_t$, and $d_3x := \sigma_2(t,x)(dt)^\alpha$, respectively. Let I_k be the integral operators corresponding differential operators d_k, $k = 1,2,3$. They are defined by

$$I_1(t) = \int_0^t b(s,x(s))ds, \tag{2.2}$$

$$I_2(t) = \int_0^t \sigma_1(s,x(s))dB(s), \tag{2.3}$$

and

$$I_3(t) = \alpha\int_0^t (t-s)^{\alpha-1}\sigma_2(s,x(s))ds, \tag{2.4}$$

for $x \in \mathrm{Dom}(I_1 \cap I_2 \cap I_3)$, where $\mathrm{Dom}(I_1 \cap I_2 \cap I_3)$ stands for the intersection of the domains of definition of the integral operators I_1, I_2, and I_3. With this, we rewrite equation (2.1) in its equivalent integral form [20] as follows:

$$x(t) - x(0) = \sum_{j=1}^{3}(I_jx)(t). \tag{2.5}$$

$$= \int_{t_0}^t b(s,x(s))ds + \int_{t_0}^t \sigma_1(s,x(s))dB_s + \alpha\int_{t_0}^t \frac{\sigma_2(s,x(s))}{(t-s)^{1-\alpha}}ds, \tag{2.6}$$

Due to the equivalency between the differential equation (2.1) and the integral equation (2.6), we recognize that a numerical scheme for (2.1) is indeed a numerical integration of (2.6). From this observation, a numerical scheme for (2.1) consists of numerical integration schemes depending on the time scales defined in Remark 3.1 in [20], $T_1(t) = t$, $T_2(t) = B(t)$, and $T_3(t) = t^\alpha$. In view of this idea, the development of numerical scheme for (2.1) depends on the numerical integration technique for (2.6). In the Throughout this section, for each time interval $[t_0, T]$ and integer $N > 1$, we assume that the partition $t_0 < t_1 < \cdots < t_N = T$ is equally spaced, that is, letting $h = (T - t_0)/N$, the times at the grid points are given as $t_k = t_0 + hk$, $k = 0,1,\ldots,N$. For the sake of simplicity, we shall use $t_0 = 0$ so that $t_k = hk$, and define $x_k = x(t_k) = x(kh)$. We also use the notations $\Delta t_k = t_{k+1} - t_k$, $\Delta B_k = B(t_{k+1}) - B(t_k)$, and $\sigma_{i,k} = \sigma_1(t_k,x_k)$, $i = 1,2$.

For this purpose, we employ the classical Euler scheme [2], the Maruyama scheme [15], a numerical approximation of the Abel-Volterra type integral equation [3, 4, 13] to approximate $I_1x(t)$, $I_2x(t)$, and $I_3x(t)$, respectively. In fact by the application of the Euler scheme [2] we have

$$I_1(t) \approx \sum_{k=0}^{N} b(t_k,x_k)\Delta t_k. \tag{2.7}$$

By applying the Maruyama scheme [15], we have

$$I_2(t) \approx \sum_{k=0}^{N} \sigma_1(t_k, x_k)\Delta B_k. \tag{2.8}$$

For $I_3x(t)$, we need to provide more details about the development of a numerical integration technique. For this purpose, we need to recall the two-point Lagrange interpolation formula [7]. For $k = 0, 1, 2, \ldots, N$, and for any $j = 0, 1, 2, \ldots, k$, the two-point Lagrange linear interpolation formula for $\sigma_2(t, x(t))$ on the interval $[t_j, t_{j+1}]$ is defined by

$$\begin{aligned} \tilde{\sigma}_{2,k+1}(t, x(t)) &= \frac{t_{j+1} - t}{t_{j+1} - t_j}\sigma_2(t_j, x(t_j)) + \frac{t - t_j}{t_{j+1} - t_j}\sigma_2(t_{j+1}, x(t_{j+1})) + e_j(t_k, t) \\ &= \frac{(t_{j+1} - t)}{h}\sigma_2(t_j, x(t_j)) + \frac{(t - t_j)}{h}\sigma_2(t_{j+1}, x(t_{j+1})) + e_j(t_k, t). \end{aligned} \tag{2.9}$$

For each $j = 0, 1, 2, \ldots, k$, setting $e_j(t) \equiv 0$, and using(2.9), we obtain

$$\begin{aligned} \int_0^{t_{k+1}} (t_{k+1} - t)^{\alpha-1}\sigma_2(t, x(t))dt &\approx \int_0^{t_{k+1}} (t_{k+1} - t)^{\alpha-1}\tilde{\sigma}_{2,k+1}(t, x(t))dt \\ &= \sum_{j=0}^{k} \int_{t_j}^{t_{k+1}} (t_{k+1} - t)^{\alpha-1}\tilde{\sigma}_{2,k+1}(t, x(t))dt \\ &= \sum_{j=0}^{k} \int_{t_j}^{t_{k+1}} (t_{k+1} - t)^{\alpha-1} \left[\frac{(t_{j+1} - t)}{h}\sigma_2(t_j, x(t_j)) \right. \\ &\qquad \left. + \frac{(t - t_j)}{h}\sigma_2(t_{j+1}, x(t_{j+1}))\right] dt \\ &= \sum_{j=0}^{k} \frac{\sigma_2(t_j, x(t_j))}{h} \int_{t_j}^{t_{k+1}} (t_{k+1} - t)^{\alpha-1}(t_{j+1} - t)dt \\ &\quad + \sum_{j=0}^{k} \frac{\sigma_2(t_{j+1}, x(t_{j+1}))}{h} \int_{t_j}^{t_{k+1}} (t_{k+1} - t)^{\alpha-1}(t - t_j)dt \end{aligned} \tag{2.10}$$

Setting $t = t_j + ph$ with $0 \leq p \leq 1$, $dt = hdp$ and recalling that $t_n = nh$ for each $n = 0, 1, 2, \ldots N$, we evaluate the integrals in the first and second terms in the right hand side of (2.10)

$$\begin{aligned} &\int_{t_j}^{t_{k+1}} (t_{k+1} - t)^{\alpha-1}(t - t_j)dt \\ &\quad = \int_0^1 ((k + 1)h - jh - ph)^{\alpha-1}(h - ph)hdp \\ &\quad = h^{\alpha+1} \int_0^1 (k + 1 - j - p)^{\alpha-1}(1 - p)dp \\ &\quad = \frac{h^{\alpha+1}}{\alpha(\alpha + 1)} \left[(k - j)^{\alpha+1} + (\alpha + j - k)(k + 1 - j)^{\alpha}\right] \end{aligned} \tag{2.11}$$

and

$$\int_{t_j}^{t_{k+1}} (t_{k+1}-t)^{\alpha-1}(t_{j+1}-t)dt$$

$$= \int_0^1 ((k+1)h - jh - ph)^{\alpha-1} phhdp$$

$$= h^{\alpha+1}\int_0^1 (k+1-j-p)^{\alpha-1} pdp$$

$$= \frac{h^{\alpha+1}}{\alpha(\alpha+1)}\left[(k+1-j)^{\alpha+1} - (\alpha+1+k-j)(k-j)^{\alpha}\right] \tag{2.12}$$

From (2.11) and (2.12), (2.10) reduces to

$$\int_0^{t_{k+1}} (t_{k+1}-t)^{\alpha-1}\sigma_2(t,x(t))dt \approx \int_0^{t_{k+1}} (t_{k+1}-t)^{\alpha-1}\tilde{\sigma}_{2,k+1}(t,x(t))dt$$

$$= \sum_{j=0}^{k} \frac{\sigma_2(t_j,x(t_j))}{h}\int_{t_j}^{t_{k+1}} (t_{k+1}-t)^{\alpha-1}(t_{j+1}-t)dt$$

$$+ \sum_{j=0}^{k} \frac{\sigma_2(t_{j+1},x(t_{j+1}))}{h}\int_{t_j}^{t_{k+1}} (t_{k+1}-t)^{\alpha-1}(t-t_j)dt$$

$$= \sum_{j=0}^{k} \frac{\sigma_2(t_j,x(t_j))}{h}\frac{h^{\alpha+1}}{\alpha(\alpha+1)}\left[(k-j)^{\alpha+1} + (\alpha+j-k)(k+1-j)^{\alpha}\right]$$

$$+ \sum_{j=0}^{k} \frac{\sigma_2(t_{j+1},x(t_{j+1}))}{h}\frac{h^{\alpha+1}}{\alpha(\alpha+1)}\left[(k+1-j)^{\alpha+1} - (\alpha+1+k-j)(k-j)^{\alpha}\right]$$

After simplifying, we rewrite

$$\int_0^{t_{k+1}} (t_{k+1}-t)^{\alpha-1}\sigma_2(t,x(t))dt \approx$$

$$= \frac{h^{\alpha}}{\alpha(\alpha+1)}\left\{\sigma_2(t_0,x(t_0))[k^{\alpha+1} + (\alpha-k)(k+1)^{\alpha}] + \sigma_2(t_{j+1},x(t_{j+1}))\right\}$$

$$+ \frac{h^{\alpha}}{\alpha(\alpha+1)}\sum_{j=1}^{k}\sigma_2(t_j,x(t_j))\left[(k-j)^{\alpha+1} + (\alpha+j-k)(k+1-j)^{\alpha}\right]$$

$$+ \sum_{l=1}^{k}\frac{h^{\alpha}\sigma_2(t_l,x(t_l))}{\alpha(\alpha+1)}\left[(k+1-l+1)^{\alpha+1} - (\alpha+1+k-l+1)(k-l+1)^{\alpha}\right]$$

$$= \frac{h^{\alpha}}{\alpha(\alpha+1)}\left\{\sigma_2(t_0,x(t_0))[k^{\alpha+1} + (\alpha-k)(k+1)^{\alpha}] + \sigma_2(t_{j+1},x(t_{j+1}))\right\}$$

$$+ \sum_{j=1}^{k}\frac{h^{\alpha}\sigma_2(t_j,x(t_j))}{\alpha(\alpha+1)}\left[(k+1-j+1)^{\alpha+1} + (k-j)^{\alpha+1} - 2(k+1-j)^{\alpha}\right]$$

(2.13)

$$= \frac{1}{\alpha} \times \sum_{j=0}^{k} a_{j,k+1}\sigma_2(t_j, x(t_j)) + \frac{h^\alpha}{\alpha(\alpha+1)} a_{k+1,k+1}\sigma_2(t_{k+1}, x(t_{k+1}))$$

where

$$a_{j,k+1} = \frac{h^\alpha}{\alpha+1} \times \begin{cases} k^{\alpha+1} + (\alpha - k)(k+1)^\alpha, & \text{if } j = 0 \\ [(k+1-j+1)^{\alpha+1} + (k-j)^{\alpha+1} & \\ \quad -2(k+1-j)^\alpha], & \text{if } 1 \leq j \leq k, \\ 1, & \text{if } j = k+1. \end{cases} \tag{2.14}$$

To reduce the error in the integration procedure for fractional integral discussed above, using a modified numerical technique, namely, a predictor-corrector method (fractional Adams-Bashforth-Moulton method) [3, 6], the term $\sigma_2(t_{k+1}, x_{k+1})$ in (2.13) is replaced with $\sigma_2(t_{k+1}, x^P_{k+1})$, where x^P_{k+1} is determined by the fractional Adams-Bashforth method

$$x^P_{k+1} = x_0 + \alpha \sum_{j=0}^{k} b_{j,k+1}\sigma_2(t_j, x_j), \tag{2.15}$$

where

$$b_{j,k+1} = \frac{h^\alpha}{\alpha}\left((k+1-j)^\alpha - (k-j)^\alpha\right). \tag{2.16}$$

Thus, we have

$$\alpha \int_0^{t_{k+1}} \frac{\sigma_2(t, x(t))}{(t_{k+1} - t)^{1-\alpha}} dt \approx \frac{h^\alpha}{\alpha+1}\sigma_2(t_{k+1}, x^P_{k+1}) + \sum_{j=0}^{k} a_{j,k+1}\sigma_2(t_j, x_j). \tag{2.17}$$

The importance of the predictor-corrector method [4] is that, for $0 < \alpha < 1$, the error behaves as

$$\max_{k=0,1,2,\ldots,N} |x(t_k) - x_k| = O(h^{1+\alpha}). \tag{2.18}$$

Now we are ready to formulate a numerical integration scheme for (2.6). From the Euler numerical integration (2.7) (for Cauchy-Riemann/Lebesgue integral), the Maruyama scheme (2.8) (for Itô-Doob integral), and the numerical approximation of the fractional integral by the predictor-corrector method (2.17), a numerical integration of the fractional stochastic integral equation (2.6) is defined by

$$\begin{aligned} x_{n+1} &= x_0 + \sum_{k=0}^{n} b(t_k, x_k)\Delta t_k + \sum_{k=0}^{n} \sigma_1(t_k, x_k)\Delta B_k + \sum_{k=0}^{n} a_{k,n+1}\sigma_2(t_k, x_k) \\ &\quad + \frac{h^\alpha}{\alpha+1}\sigma_2(t_{n+1}, x^P_{n+1}), \quad n = 0, 1, 2, \ldots, N-1, \end{aligned} \tag{2.19}$$

where $\Delta t_k = t_{k+1} - t_k$, $\Delta B_k = B(t_{k+1}) - B(t_k)$, and $a_{k,n+1}$ is defined in (2.14). From this, we conclude that the numerical integration scheme (2.19) is also a numerical scheme for the multi-time scales differential equation (2.1). In the following section, we investigate the strong convergence of the the above presented scheme.

3. **Convergence of the Numerical Scheme for SFDE**

We are now going to prove that the Euler-Maruyama type numerical approximation for the stochastic fractional differential equations discussed in the previous section converge strongly. For that, we state the following:

Theorem 3.1. *Under the assumptions of Theorem* 4.1 *in* [20]*, a strongly consistent equidistant time discrete approximation* $x_h(t)$ *(in* (2.19)*) of the solution process* $x(t)$ *of a* 1*-dimensional autonomous stochastic fractional differential equation* (2.1) *converges strongly to* $x(t)$*.*

Proof. The proof of this theorem follows from those of 1-dimensional stochastic differential equations (Ref. Kloeden and Platen [15] pp 324–326) and the predictor-corrector method [3, 6] described above.

For $0 \le t \le T$, we set

$$z(t) = \sup_{0\le s\le t} E\left(\left|x^h_{n_s} - x(s)\right|^2\right) \tag{3.1}$$

we obtain

$$\begin{aligned} z(t) = \sup_{0\le s\le t} E\Bigg\{\Bigg|&\sum_{k=0}^{n_s-1} b(t_k,x_k)\Delta t_k + \sum_{k=0}^{n_s-1}\sigma_1(t_k,x_k)\Delta B_k \\ &+\frac{h^\alpha\sigma_2(t_{n_s},x^P_{n_s})}{\alpha+1} + \sum_{k=0}^{n_s-1} a_{k,n_s}\sigma_2(t_k,x_k) - \int_0^s b(r,x(r))dr \\ &-\int_0^s \sigma_1(r,x(r))dB(r) - \alpha\int_0^s (s-r)^{\alpha-1}\sigma_2(r,x(r))dr\Bigg|^2\Bigg\}. \end{aligned}$$

By using the Schwartz inequality, we have

$$\begin{aligned} z(t) \le{}& C_0 \sup_{0\le s\le t} E\left(\left|\sum_{k=0}^{n_s-1} b(t_k,x_k)\Delta t_k - \int_0^{t_{n_s}} b(r,x(r))dr\right|^2\right) \\ &+ C_0 \sup_{0\le s\le t} E\left(\left|\sum_{k=0}^{n_s-1}\sigma_1(t_k,x_k)\Delta B_k - \int_0^{t_{n_s}}\sigma_1(r,x(r))dB(r)\right|^2\right) \\ &+ C_0 \sup_{0\le s\le t} E\left|\frac{h^\alpha\sigma_2(t_{n_s},x^P_{n_s})}{\alpha+1} + \sum_{k=0}^{n_s-1} a_{k,n_s}\sigma_2(t_k,x_k) - \alpha\int_0^{t_{n_s}}\frac{\sigma_2(r,x(r))}{(t_{n_s}-r)^{1-\alpha}}dr\right|^2 \\ &+ C_0 \sup_{0\le s\le t} E\left(\left|\int_{t_{n_s}}^s b(r,x(r))dr + \int_{t_{n_s}}^s \sigma_1(r,x(r))dB(r)\right|^2\right) \end{aligned}$$

$$+\alpha^2 C_0 \sup_{0\le s\le t} E\left(\left|\int_0^{t_{n_s}} \frac{\sigma_2(r,x(r))}{(t_{n_s}-r)^{1-\alpha}}dr - \int_0^{t_{n_s}} \frac{\sigma_2(r,x(r))}{(s-r)^{1-\alpha}}dr\right|^2\right)$$

$$+\alpha^2 C_0 \sup_{0\le s\le t} E\left(\left|\int_{t_{n_s}}^{s} (s-r)^{\alpha-1}\sigma_2(r,x(r))dr\right|^2\right) \tag{3.2}$$

We need to find estimate for each term in the right hand side of (3.2).

Using the Euler-Maruyama approximation scheme, the combined estimates for the 1st and 2nd terms in (3.2) is

$$C_0 \sup_{0\le s\le t} E\left(\left|\sum_{k=0}^{n_s-1} b(t_k,x_k)\Delta t_k - \int_0^{t_{n_s}} b(r,x(r))dr\right|^2\right)$$

$$+C_0 \sup_{0\le s\le t} E\left(\left|\sum_{k=0}^{n_s-1} \sigma_1(t_k,x_k)\Delta B_k - \int_0^{t_{n_s}} \sigma_1(r,x(r))dB(r)\right|^2\right) \le C_1 h, \tag{3.3}$$

for some positive constant C_1 depending on b, σ_1, and T.

By applying the fact that the predictor-corrector method converges with order $1+\alpha$, the 3rd term in (3.2) has estimate

$$C_0 \sup_{0\le s\le t} E\left(\left|\frac{h^\alpha \sigma_2(t_{n_s},x_{n_s}^P)}{\alpha+1} + \sum_{k=0}^{n_s-1} a_{k,n_s}\sigma_2(t_k,x_k) - \alpha\int_0^{t_{n_s}} \frac{\sigma_2(r,x(r))}{(t_{n_s}-r)^{1-\alpha}}dr\right|^2\right)$$

$$\le C_2 h^{2(1+\alpha)}, \tag{3.4}$$

for some positive constant C_2 depending on σ_2 and T.

For the fourth term in (3.2), by using the Schwartz inequality, Itô isometry [1], the growth condition and the fact that $\sup_{0\le s\le T} E|x(s)|^2 < \infty$ (Theorem 4.1 in [20]), we have

$$\sup_{0\le s\le t} E\left(\left|\int_{t_{n_s}}^{s} b(r,x(r))dr + \int_{t_{n_s}}^{s} \sigma_1(r,x(r))dB(r)\right|^2\right)$$

$$\le 2(s-t_{n_s})\int_{t_{n_s}}^{s} K^2(1+\sup_{0\le u\le T} E|x(u)|^2)dr + 2\int_{t_{n_s}}^{s} K^2(1+\sup_{0\le u\le T} E|x(u)|^2)dr$$

$$\le 2(h+1)hK^2(1+\sup_{0\le u\le T} E|x(u)|^2)$$

$$\le C_3 h, \tag{3.5}$$

for some positive constant C_3 depending on b, σ_1, and T.

Now, let's consider the fifth term in (3.2):

$$\int_0^{t_{n_s}} (t_{n_s}-r)^{\alpha-1}\sigma_2(r,x(r))dr - \int_0^{t_{n_s}} (s-r)^{\alpha-1}\sigma_2(r,x(r))dr$$

$$= \int_0^{t_{n_s}} \frac{(s-r)^{1-\alpha} - (t_{n_s}-r)^{1-\alpha}}{(t_{n_s}-r)^{1-\alpha}(s-r)^{1-\alpha}}\sigma_2(r,x(r))dr$$

$$\leq \int_0^{t_{n_s}} \frac{(s-r)^{1-\alpha} - (t_{n_s}-r)^{1-\alpha}}{(t_{n_s}-r)^{2(1-\alpha)}} \sigma_2(r,x(r))dr, \quad \text{since } t_{n_s} < s \text{ and } \alpha < 1. \tag{3.6}$$

The function $f(r) = (s-r)^{1-\alpha} - (t_{n_s}-r)^{1-\alpha}$ on the interval $[0, t_{n_s}]$, is differentiable on $(0, t_{n_s})$ and $f'(r) = (1-\alpha)[(t_{n_s}-r)^{-\alpha} - (s-r)^{-\alpha} < 0$. Thus, f is a nonnegative and decreasing function on $[0, t_{n_s}]$ with maximum value occurring at 0,

$$\begin{aligned} f(0) &= s^{1-\alpha} - t_{n_s}^{1-\alpha} \\ &\leq t_{n_s+1}^{1-\alpha} - t_{n_s}^{1-\alpha} \\ &= h^{1-\alpha}[(n_s+1)^{1-\alpha} - (n_s)^{1-\alpha}] \end{aligned}$$

Since $1-\alpha > 0$ and the function $g(x) = (x+1)^{1-\alpha} - x^{1-\alpha}$ is a nonnegative and decreasing function on $[0,\infty)$, we have

$$f(0) \leq h^{1-\alpha}[(n_s+1)^{1-\alpha} - (n_s)^{1-\alpha}] \leq h^{1-\alpha} g(0) = h^{1-\alpha} \tag{3.7}$$

On the other hand we have

$$\int_0^{t_{n_s}} \frac{1}{(t_{n_s}-r)^{2(1-\alpha)}} dr = \frac{-1}{2\alpha-1}(t_{n_s}-r)^{2\alpha-1}\Big|_0^{t_{n_s}} = \frac{t_{n_s}^{2\alpha-1}}{2\alpha-1} \leq \frac{T^{2\alpha-1}}{2\alpha-1}. \tag{3.8}$$

Now by using the linear growth condition and the fact that $\sup_{0\leq s\leq T} E|x(s)|^2 < \infty$, (see Theorem 4.1 [20]), (3.6) and (3.8), the fifth term in (3.2) reduces to

$$\alpha^2 \sup_{0\leq s\leq t} E\left(\left|\int_0^{t_{n_s}} \frac{\sigma_2(r,x(r))}{(t_{n_s}-r)^{1-\alpha}} dr - \int_0^{t_{n_s}} \frac{\sigma_2(r,x(r))}{(s-r)^{1-\alpha}} dr\right|^2\right) \leq C_4 h^{1-\alpha}, \tag{3.9}$$

where C_4 is a positive constant depending on σ_2 and T.

Finally, for the last term in (3.2), an estimate is obtained by applying the Schwartz inequality, the linear growth condition, and the fact that $\sup_{0\leq s\leq T} E|x(s)|^2 < \infty$, $t_0 \leq t \leq T$. Thus, we have

$$\begin{aligned} &\alpha^2 C_0 \sup_{0\leq s\leq t} E\left(\left|\int_{t_{n_s}}^s (s-r)^{\alpha-1}\sigma_2(r,x(r))dr\right|^2\right) \\ &\leq C_5 \int_{t_{n_s}}^s (s-r)^{2(\alpha-1)} dr (s-t_{n_s})(1+\sup_{0\leq s\leq T} E|x(s)|^2)) \\ &\leq C_5 \frac{-1}{2\alpha-1}(s-r)^{2\alpha-1}\Big|_{t_{n_s}}^s (s-t_{n_s})(1+\sup_{0\leq s\leq T} E(|x(s)|^2)) \\ &\leq C_5 \frac{(s-t_{n_s})^{2\alpha-1}}{2\alpha-1}(s-t_{n_s})(1+\sup_{0\leq s\leq T} E(|x(s)|^2)) \\ &\leq C_5 h^{2\alpha}, \end{aligned} \tag{3.10}$$

where C_5 is a some positive constant depending on σ_2 and T.

Therefore, from (3.3), (3.4), (3.5), (3.9), and (3.10), an estimate for $z(t)$ in (3.2) is given by

$$z(t) \leq C_1 h + C_2 h^{2(1+\alpha)} + C_3 h + C_4 h^{1-\alpha} + C_5 h^{2\alpha} \leq C h^{1-\alpha}. \tag{3.11}$$

Finally, using the Lyapunov inequality we can then conclude that

$$E\left|x^h(t)-x(t)\right| \le \sqrt{z(t)} \le Ch^{(1-\alpha)/2}. \tag{3.12}$$

where C is a positive constant depending on b, σ_1, σ_2, and T. □

In the next section, we apply the numerical scheme (2.19) to the approximate the solution of dynamical modeling of epidemiological processes studied in [20].

4. **Dynamical Modeling of Epidemiological Processes**

Mathematical model for the epidemic process of communicable diseases

$$dI = \theta(\widetilde{N}-I)Idt + \sigma_1 IdB(t) + \sigma_2 I(dt)^\alpha, \quad I(0) = I_0 \tag{4.1}$$

where $\theta > 0$, $\sigma_1 \neq 0$, $\sigma_2 \neq 0$. The rate functions $b(t,I) = \theta(\widetilde{N}-I)I$ (deterministic), $\sigma_1(t,I) = \sigma_1 I$ (stochastic) for some $\sigma_1 \neq 0$, and $\sigma_2(t,I) = \sigma_2 I$ (fractional) are smooth functions which guarantees the existence of solution of (4.1).

The general solution of (4.1) [20] is given by

$$\begin{aligned} I(t) &= \left[\Phi(t)C_0 + (1-n)\int_{t_0}^{t} \Phi(t,s)\left[Q(s)-(n+1)S(s)Y(s)\right]ds + \right. \\ &= \left[\Phi(t)C_0 + \theta\int_{t_0}^{t}\Phi(t,s)ds\right]^{-1}, \end{aligned} \tag{4.2}$$

where

$$\begin{aligned} \Phi(t) &= \exp\left[\int_{t_0}^{t}\left((\theta\widetilde{N}-\sigma_1^2)-\frac{1}{2}\sigma_1^2\right)ds + \int_{t_0}^{t}\sigma_1 dB(s)\right] E_\alpha\left[\Gamma(1+\alpha)^2(I^\alpha\sigma_2)(t)\right] \\ &= \exp\left[\left(\theta\widetilde{N}-\frac{3}{2}\sigma_1^2\right)t + \sigma_1 B(t)\right] E_\alpha\left[\sigma_2\Gamma(1+\alpha)t^\alpha\right]; \end{aligned}$$

and $\Phi(t,s) = \Phi(t)\Phi^{-1}(s)$ is the fundamental solution.

4.1. **Simulations study in the absence of environmental perturbations and hereditary effects.**

Let us consider the case where $\sigma_i \equiv 0$, $i = 1,2$, that is there is no environmental perturbations and hereditary effects influencing the system. Then equation (4.1) reduces to

$$dI = \theta(\widetilde{N}-I)Idt, \quad I(0) = I_0 \tag{4.3}$$

The exact solution of this equation is given by

$$I(t) = \frac{\widetilde{N}I_0}{I_0 + (\widetilde{N}-I_0)e^{-\theta\widetilde{N}(t-t_0)}} \tag{4.4}$$

The numerical scheme for (4.3) is given in (2.19). In fact it is as:

$$I_{n+1} = I_0 + \sum_{k=0}^{n} b(t_k, I_k)\Delta t_k, \quad n = 0,1,2,\ldots,N-1. \tag{4.5}$$

Moreover,

$$I_{n+1} = I_n + b(t_n, I_n)(t_{n+1} - t_n), \quad n = 0, 1, 2, \ldots, N-1. \tag{4.6}$$

All our subsequent simulation results are based on 1 unit=1000 or 100,000 or 1000,000.

If $I_0 < \widetilde{N}$, the graph of $I(t)$ in (4.4) has an S-shape with an inflection point occurring at $t = t_0 + \frac{\ln(\widetilde{N}-I_0)-\ln(I_0)}{\theta\widetilde{N}}$ (see Figure 1).

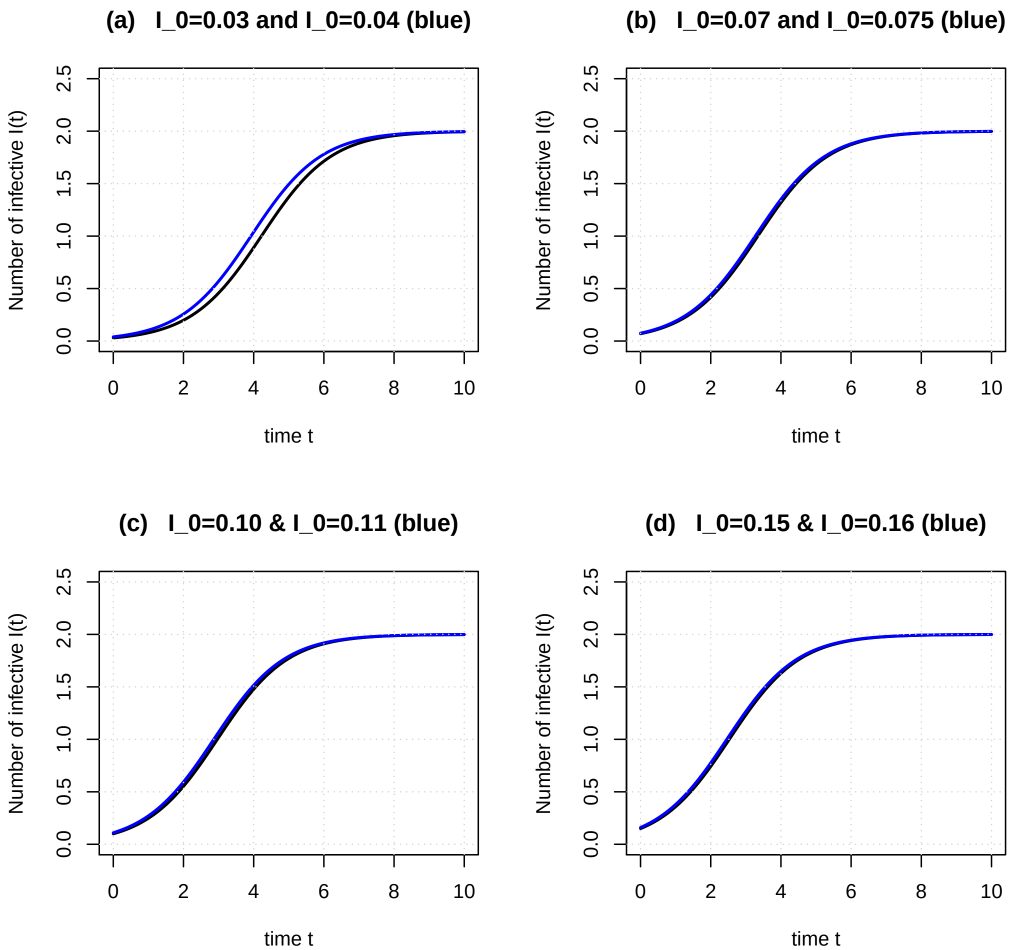

FIGURE 1. Some graphs of solution $I(t)$ in (4.4) when $\theta = 0.5$, $\widetilde{N} = 2$, $\sigma_1 = 0$, $\sigma_2 = 0$. Initial conditions: $I_0 = 0.03$ and $I_0 = 0.04$ (a), $I_0 = 0.07$ and $I_0 = 0.075$ (b), $I_0 = 0.10$ and $I_0 = 0.11$ (c), and $I_0 = 0.15$ and $I_0 = 0.16$ (d).

Observations: (1) The number of infective $I(t)$ grows faster (on an interval $[t_0, t_1]$ for some $t_1 < 10$) from the initial value I_0 and then reaches the saturation level of (population size of the species) $\widetilde{N} = 2$. This shows that the spread of the epidemic reaches the entire community rapidly (chaotic) if not contain early.

(2) The solution $I(t)$ of (4.4) is asymptotically stable as all these graphs show no distinction between two graphs with different initial conditions as the time increases.

This indicates the stable and tight community in that what happens to one individual easily affects the entire ecosystem and so is the spread of communicable diseases. If $I_0 > \widetilde{N}$, the graph of $I(t)$ falls sharply from left to right and flattens right above the saturation level $\widetilde{N}$, (see Figure 2).

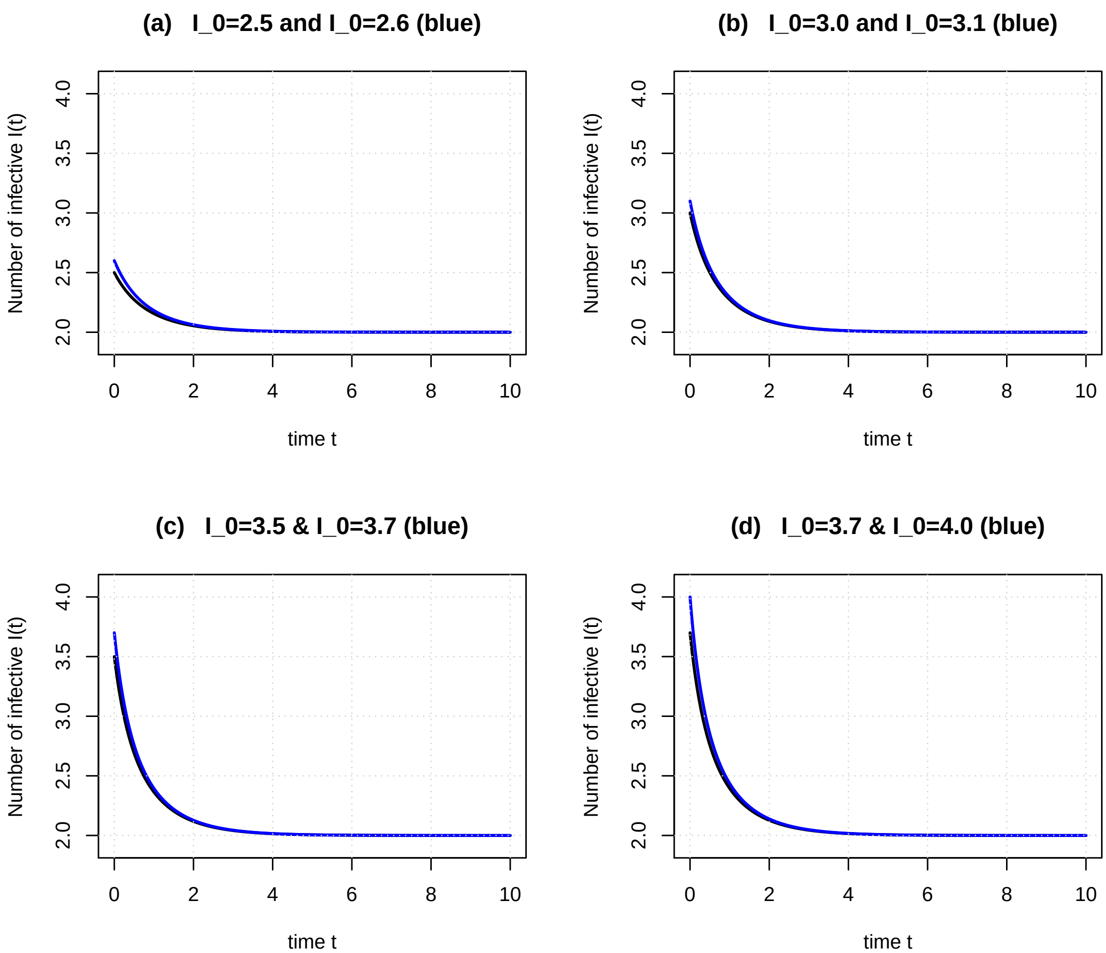

FIGURE 2. Some graphs of solution $I(t)$ in (4.4) when $\theta = 0.5$, $\widetilde{N} = 2$, $\sigma_1 = 0$, $\sigma_2 = 0$. Initial conditions: $I_0 = 2.5$ and $I_0 = 2.6$ (a), $I_0 = 3.0$ and $I_0 = 3.1$ (b), $I_0 = 3.5$ and $I_0 = 3.7$ (c), and $I_0 = 4.1$ and $I_0 = 4.2$ (d).

Observations: (1) Although taking the initial number of infective higher than the saturation level (population size) seems irrelevant, one can think of it as follows. The community is very small and an effective cure is available and this community is invaded by a large number infective individuals from a larger community or a groups of other communities/ecosystems.

(2) Again, we see that the solution $I(t)$ of (4.4) is asymptotically stable as all these graphs show no distinction between two graphs with different initial conditions as the time increases.

4.2. **Simulations study in the absence of hereditary effects (only).** Let us consider the case where $\sigma_2 \equiv 0$, that is there is no hereditary effects influencing the system. Then equation (4.1) reduces to

$$dI = \theta(\widetilde{N} - I)Idt + \sigma_1 IdB(t), \quad I(0) = I_0 \tag{4.7}$$

As shown in [20], the general solution of (4.1) is given by

$$I(t) = \left[\Phi(t)C_0 + \theta \int_{t_0}^{t} \Phi(t,s)ds\right]^{-1}, \tag{4.8}$$

where

$$\begin{aligned}\Phi(t) &= \exp\left[\int_{t_0}^{t} \left((\theta\widetilde{N} - \sigma_1^2) - \frac{1}{2}\sigma_1^2\right) ds + \int_{t_0}^{t} \sigma_1 dB(s)\right] \\ &= \exp\left[\left(\theta\widetilde{N} - \frac{3}{2}\sigma_1^2\right) t + \sigma_1 B(t)\right],\end{aligned}$$

and $\Phi(t,s) = \Phi(t)\Phi^{-1}(s)$ is the fundamental solution of (4.7).

To simulate the sample paths of $I(t)$, we utilize the numerical scheme described in (2.19), which is just the Euler-Maruyama scheme for stochastic differential equations. Thus we have, for $n = 0, 1, 2, \ldots, N-1$

$$I_{n+1} = I_0 + \sum_{k=0}^{n} b(t_k, I_k)\Delta t_k + \sum_{k=0}^{n} \sigma_1(t_k, I_k)\Delta B_k, \tag{4.9}$$

or simply

$$I_{n+1} = I_n + b(t_n, I_n)(t_{n+1} - t_n) + \sigma_1(t_n, I_n)(B(t_{n+1}) - B(t_n)). \tag{4.10}$$

In the remaining part of this section we'll only discuss the case $I_0 < \widetilde{N}$.

Observations: (1) We note that small values of σ_1 (between 0 and around 0.3), the fluctuations of $I(t)$ average about the curve of the deterministic case discussed above. This means that, on average, the environmental perturbations can be attributed to the effects of the influx of infective individuals from others neighboring communities. (2) Again, we see that the solution $I(t)$ of (4.8) is asymptotically stable as all these graphs show no distinction between two graphs with different initial conditions as the time increases.

Here we vary the diffusion coefficient σ_1 in (4.7)

Observations: We remak that as σ_1 increases, the fluctuations of $I(t)$ increases as well and still tend to average about the curve of the deterministic case discussed above. This means that the higher the rate of the environmental perturbations, the less controllable the system becomes.

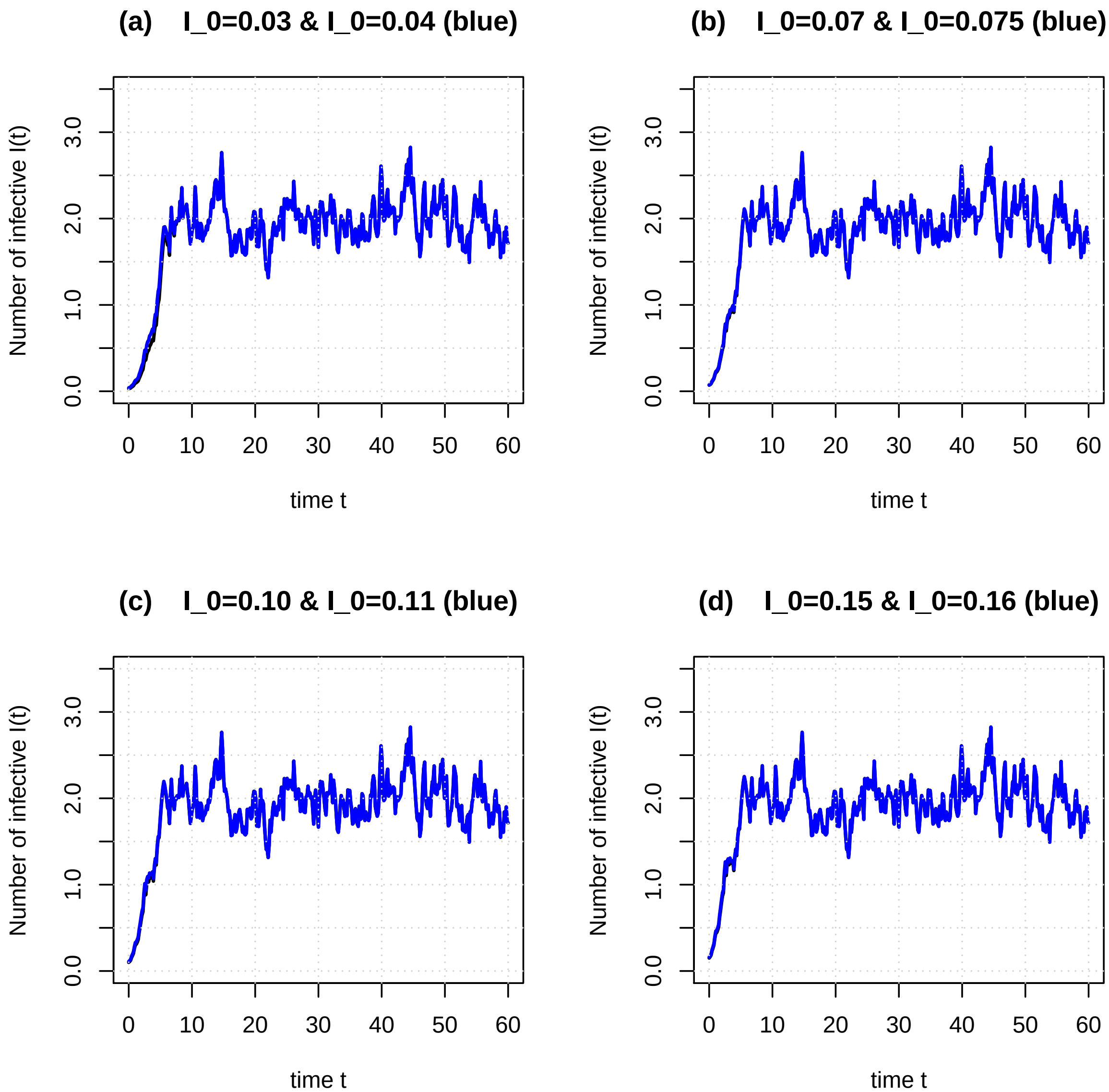

FIGURE 3. Some graphs of solution $I(t)$ of (4.7) when $\theta = 0.5$, $\widetilde{N} = 2$, $\sigma_1 = 0.2$. Initial conditions: $I_0 = 0.03$ and $I_0 = 0.04$ (a), $I_0 = 0.07$ and $I_0 = 0.075$ (b), $I_0 = 0.1$ and $I_0 = 0.11$ (c), and $I_0 = 0.15$ and $I_0 = 0.16$ (d).

4.3. **Simulations study in the absence of environmental perturbations (only).** By setting $\sigma_1 \equiv 0$, that is there is no environmental perturbations affecting the ecosystem. Then equation (4.1) reduces to

$$dI = \theta(\widetilde{N} - I)I dt + \sigma_2 I (dt)^{\alpha}, \quad I(0) = I_0 \tag{4.11}$$

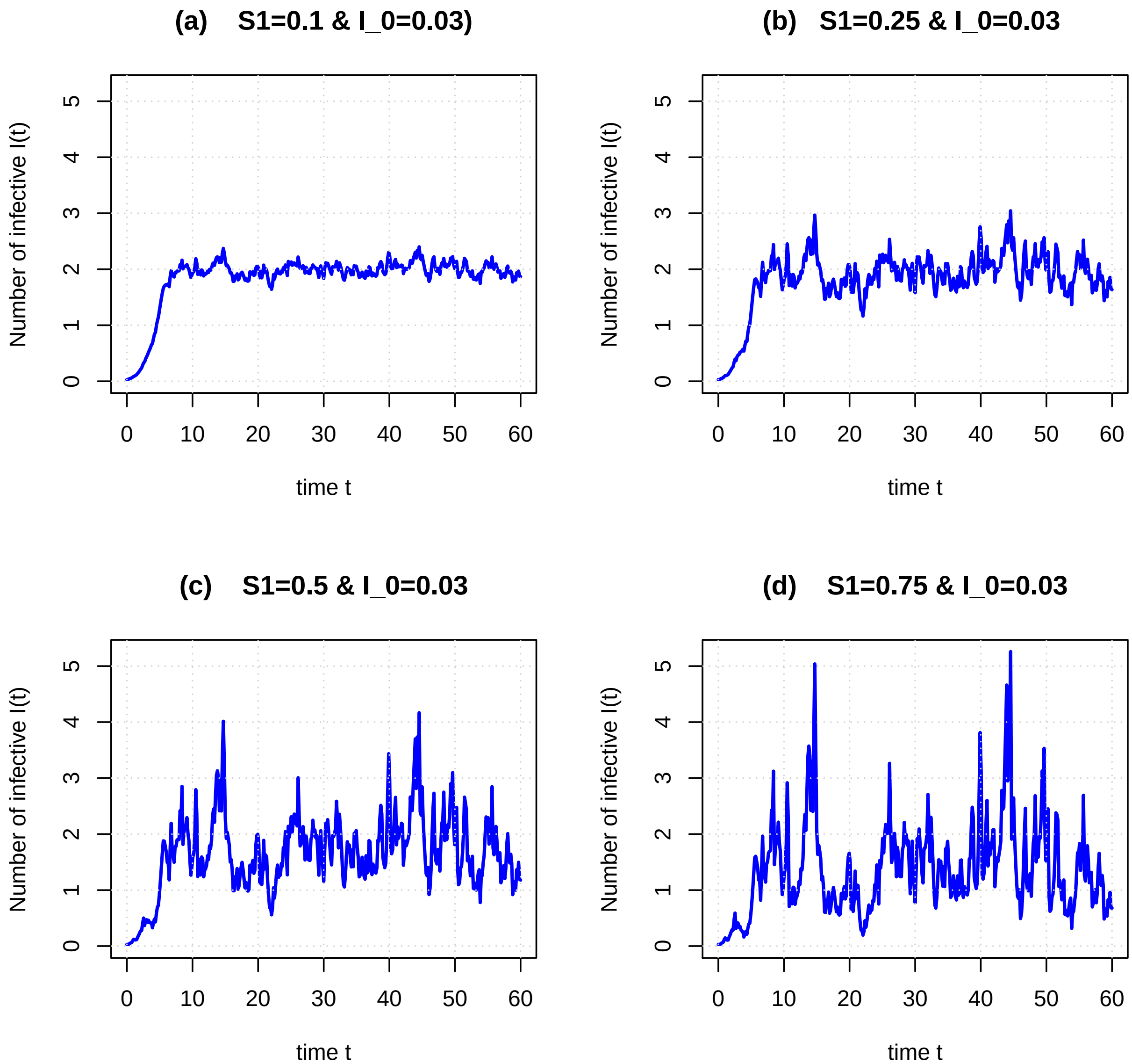

FIGURE 4. some sample paths for the solution $I(t)$ when $\theta = 0.5$, $\widetilde{N} = 2$, $\sigma_1 = 0.1$ (a), $\sigma_1 = 0.25$ (b), $\sigma_1 = 0.5$ (c), $\sigma_1 = 0.75$ (d)

The general solution of (4.11) is just simplified version of the solution of the three time-scale presented in [20],

$$I(t) = \left[\Phi(t)C_0 + \theta \int_{t_0}^{t} \Phi(t,s)ds\right]^{-1}, \tag{4.12}$$

where

$$\Phi(t) = \exp\left(\theta\widetilde{N}t\right) E_\alpha\left[\sigma_2\Gamma(1+\alpha)t^\alpha\right]$$

and $\Phi(t,s) = \Phi(t)\Phi^{-1}(s)$ is the fundamental solution of (4.11).

To graph $I(t)$, we utilize the numerical scheme described in (2.19) as follows: for $n = 0, 1, 2, \ldots, N-1$

$$x_{n+1} = x_0 + \sum_{k=0}^{n} b(t_k, x_k)\Delta t_k + \sum_{k=0}^{n} a_{k,n+1}\sigma_2(t_k, x_k) + \frac{h^\alpha}{\alpha+1}\sigma_2(t_{n+1}, x_{n+1}^P), \tag{4.13}$$

where $\Delta t_k = t_{k+1} - t_k, \Delta B_k = B(t_{k+1}) - B(t_k)$, and $a_{k,n+1}$ and x_{n+1}^P are defined in (2.14) and (2.15), respectively.

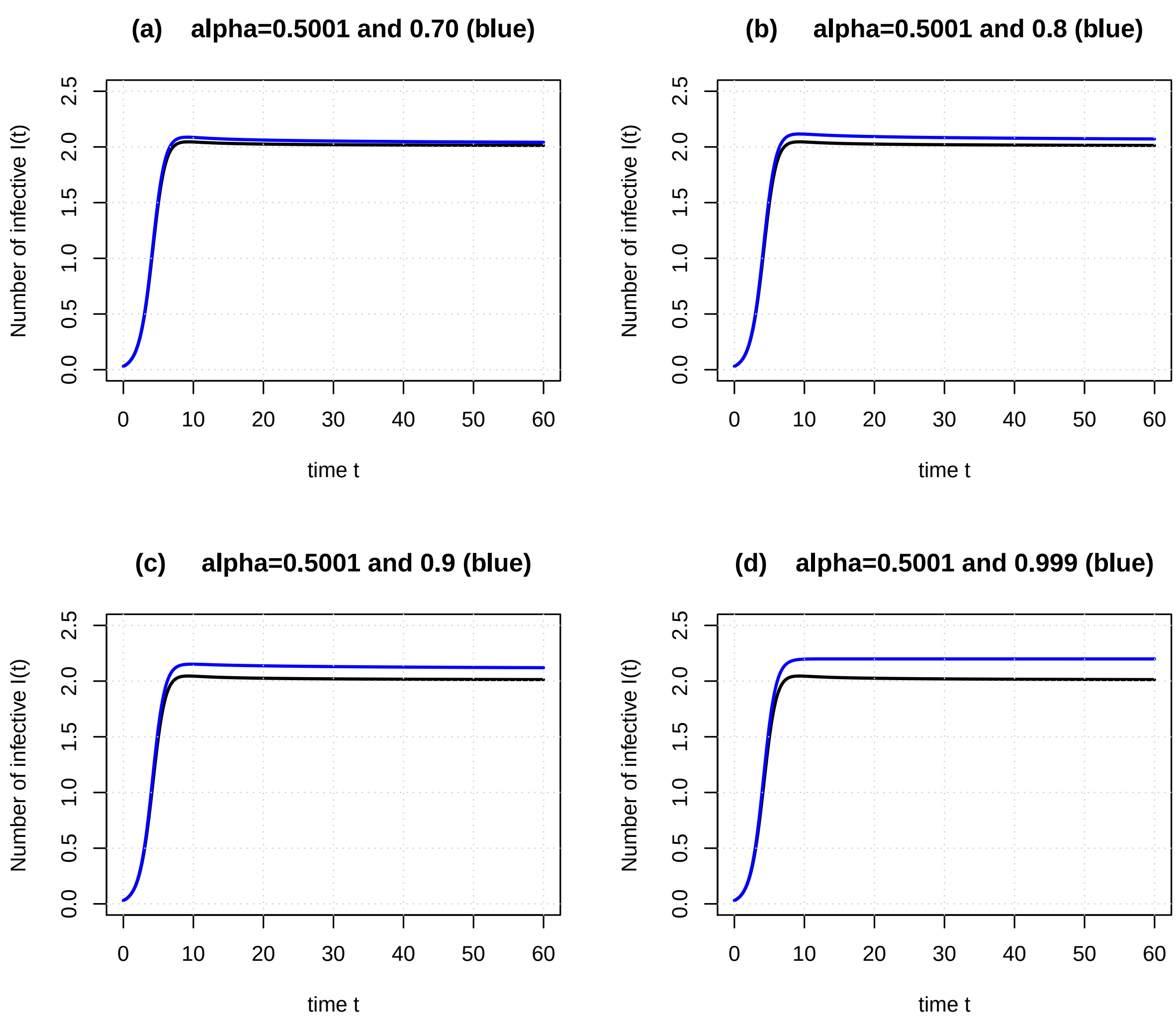

FIGURE 5. some graphs for the solution $I(t)$ when $\theta = 0.5$, $\widetilde{N} = 2$, $\sigma_2 = 0.1$, the initial condition is $I_0 = 0.03$. $\alpha = 0.5001$ and $\alpha = 0.7$ (a), $\alpha = 0.5001$ and $\alpha = 0.8$ (b), $\alpha = 0.5001$ and $\alpha = 0.9$ (c), and $\alpha = 0.5001$ and $\alpha = 0.999$ (d).

Observations: We observe that as α increases $(0.5 < \alpha < 1)$, the graph of $I(t)$ maintains an S-shape as in the deterministic $dI = \theta(\widetilde{N} - I)Idt$ discussed in subsection 4.1 above. The only difference is that here the graph of $I(t)$ grows rapidly with time and crosses the line $I = 2$ (saturation level), then reaches its maximum before starts decreasing slowly toward the saturation level. The larger the value of α, the higher the maximum value attained by $I(t)$ and the slower the rate decrease toward toward the saturation lavel. This might be explained as follows: at the time the epidemic turns chaotic, it happens that there is also a influx of infected individuals from neighboring communities, and then, part of the infected population leave the overwhelmed community of die.

4.4. **Simulations study for solution of** (4.1)**.** Taking into account of the three time scales in equation (4.1), and using the numerical scheme described in (2.19), some of the sample paths of the solution process $I(t)$ are presented

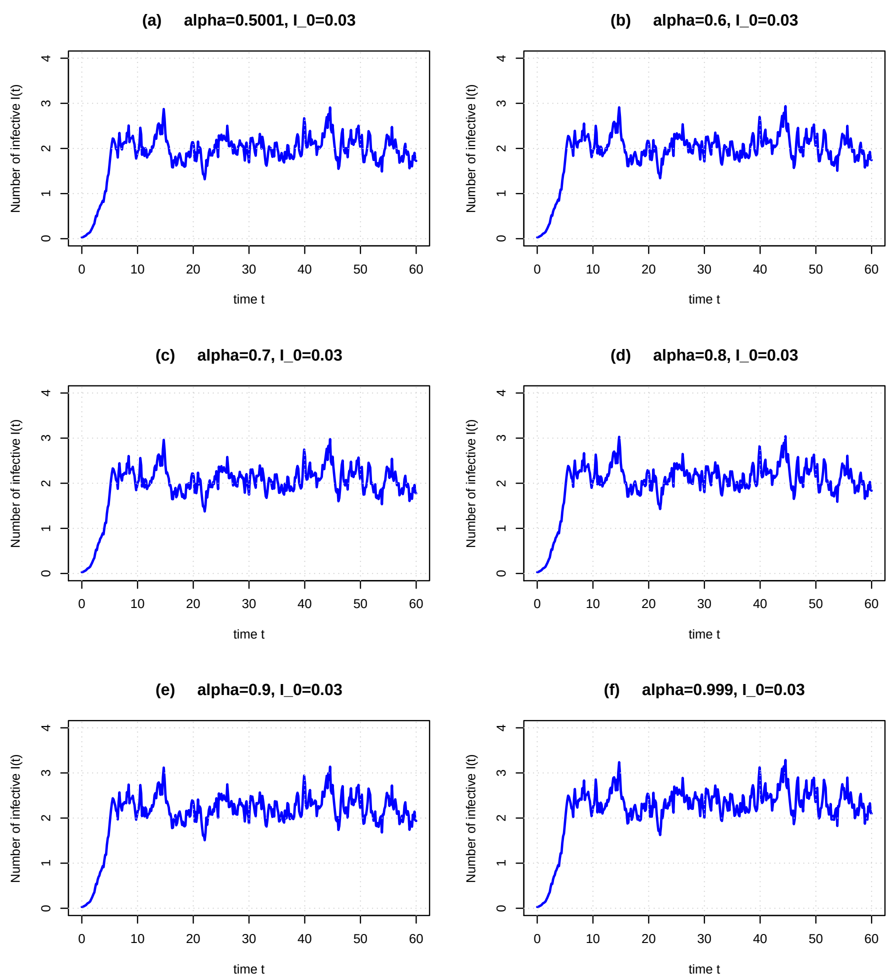

FIGURE 6. Some sample paths of the solution $I(t)$ with $\theta = 0.5$, $\widetilde{N} = 2$, $\sigma_1 = 0.2$, $\sigma_2 = 0.2$, the initial condition is $I_0 = 0.03$. $\alpha = 0.5001$ (a), $\alpha = 0.6$ (b), $\alpha = 0.7$ (c), $\alpha = 0.8$ (d), $\alpha = 0.9$ (e), and $\alpha = 0.999$ (f).

Observations: We note the the sample paths of $I(t)$ averages about the curve of the two-time scale deterministic fractional differential $dI = \theta(\widetilde{N} - I)Idt + \sigma_2(t, I)(dt)^\alpha$ discussed in subsection 4.3 above. This means that the average impact of the epidemic on the community is as explained in the subsection 4.3.

5. **Conclusion**

In this work, we presented a numerical scheme for three time-scale fractional stochastic differential equations whose fundamental properties and analysis have carried out by Pedjeu and Ladde [20]. This numerical scheme is a combination of the Euler-Maruyama numerical scheme for Itô-Doob stochastic differential equations [15] and predictor corrector method for deterministic fractional differential [4, 5]. We showed that the presented scheme strongly converges globally. This result can also be used for a local convergence. Finally, we applied the numerical scheme to the epidemiological processes developed in [20]. Currently, we are working on the improvement of this scheme while also considering the case where α not restricted to the interval $\left(\frac{1}{2}, 1\right)$; additional applications are also underway.

Acknowledgments. This research was supported by the Mathematical Science Division, US Army Research Office, Grant No. W911NF12-1-0090.

REFERENCES

[1] L. Arnold, *Stochastic Differential Equations: Theory and Appl.* John Wiley. 1974.

[2] R. L. Burden, J. D. Faires *Numerical Analysis 8th ed.* Thomson Brooks/Cole. 2005.

[3] K. Diethelm, N. J. Ford, A. D. Freed, *A Predictor-Corrector Approach for the Numerical Solution of Fractional Diff. Equations.* Nonlinear Dyn. 29 (2002) 3–22.

[4] K. Diethelm, N. J. Ford, A. D. Freed, *Detailed error analysis for a fractional Adams method.* Numerical Algorithms 36 (2004) 31–52.

[5] K. Diethelm, N. J. Ford, A. D. Freed, Y. Luchko, *Algorithms for the fractional calculus: A selection of numerical methods.* Comput. Methods Appl. Mech. Engrg 194 (2005) 743–773.

[6] K. Diethelm, *The Analysis of Fractional Differential Equations: An Application-Oriented Exposition Using Differential Operators of Caputo Type.* Springer. 2010.

[7] F. Eperson, *An Introduction to Numerical Methods and Applications.* Wiley. 2007.

[8] R. W. Hamming, *Numerical Methods for Scientists and Engineers.* McGraw Hill. 1973

[9] D. J. Higham, X. Mao and A.M . Stuart, *Strong convergence of Euler-type methods for nonlinear stochastic differential equations.* SIAM J. Num. Analysis 40 (2002) 1041–1063.

[10] G. Jumarie, *Fractional Brownian motions via random walk in the complex plane and via fractional derivative. Comparison and further results on their Fokker-Planck equations*, Chaos, Solitons and Fractals, 22 pages 907–925, 2004.

[11] G. Jumarie, *New stochastic fractional models for Malthusian growth, the Poissonian birth process and optimal management of populations.* Mathematical and Computer Modelling, 44 (2006) 231–254.

[12] G. Jumarie, *Modified Riemann-Liouville Derivative and Fractional Taylor Series of Nondifferentiable Functions Further Results*. Computers and Mathematics with Applications, 51 (2006) 1367–1376.

[13] D. Kershaw, *Some Results for Abel-Volterra Integral Equations of the Second Kind, Treatment of Tntegral Equations by Numerical Methods (Durham, 1982).* Academic Press, London, 1982, pp. 273–282.

[14] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier. 2006.

[15] P. E. Kloeden, E. Platen, *Numerical Solution of Stochastic Differential Equations*, Springer. 1992.

[16] A. G. Ladde, G. S. Ladde, *An Introduction to Differential Equations II: Stochastic Modeling, Methods and Analysis*, World Scientific. 2012.

[17] V. Lakshmikantham, S. Leela, J. V. Devi, *Theory of Fractional Dynamic Systems*, Cambridge Scientific Publishers Ltd., Cottenham. 2009.

[18] G. N. Milstein and M. V. Tretjakov, *Numerical integration of stochastic differential equations with nonglobally Lipschitz coefficients.* SIAM J. Numer. Anal. 43 (2005), 1139–1154.

[19] B. Øksendal, *Stochastic Differential Equations. An Introduction with Applications*, sixth edition, Springer. 2003.

[20] Jean-C. Pedjeu & G. S. Ladde, *Stochastic fractional differential equations: Modeling, Method and Analysis*, Chaos, Solitons and Fractals 45 (2012) 279–293.

[21] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego. 1999.

[22] S. G. Samko, A. A. Kilbas, O. I. Marichev, *Fractional Integrals and Derivatives: Theory and Applications*, Gordon and Breach. 1993.