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NUMERICAL APPROACH VIA GENERALIZED MONOTONE METHOD FOR SCALAR CAPUTO FRACTIONAL DIFFERENTIAL EQUATIONS

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ABSTRACT. The generalized monotone method for Caputo fractional differential equations using coupled lower and upper solutions is very useful, since it does not require any additional assumption. In this work we provide theoretical as well as computational methodology to compute coupled lower and upper solutions of type I to any desired interval. Further the convergence can be accelerated using Gauss-Seidel method. We have provided numerical results as well.

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1. INTRODUCTION

Nonlinear problems (nonlinear dynamic systems) occur naturally as mathematical models in many branches of science, engineering, finance, economics, etc. So far, in literature, most models are differential equations with integer derivative. However, the qualitative and quantitative study of fractional differential and integral equations has gained importance recently due to its applications. See [1, 3, 8, 6] for details. In solving nonlinear problems, monotone method combined with method of upper and lower solutions is a popular choice, because the existence of solution by monotone method is both theoretical and computational. In addition the interval of existence is guaranteed. Monotone method for various nonlinear problems has been developed in [4]. Monotone method (monotone iterative technique) combined with method of lower and upper solutions yields monotone sequences, which converges to minimal and maximal solutions of nonlinear differential equation.

In many nonlinear problems (nonlinear dynamic systems), the nonlinear term is the sum of an increasing and decreasing functions. Monotone method extended to such systems is called generalized monotone method. Generalized monotone method for first order nonlinear initial value problems and for fractional order nonlinear initial

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value problem has been developed in [10, 7] respectively. See [9] for generalized monotone method for fractional order of N systems. The generalised monotone method for nonlinear fractional differential equations with initial conditions has an added advantage over the usual monotone method, since the former method does not need the computation of Mittag-Leffler function. However the difficulty is in computing the coupled upper and lower solutions of type I (see [7] for details) to any desired interval. In this work we provide a methodology to compute coupled lower and upper solutions of type I for scalar Caputo fractional differential equation with initial conditions on any given interval. We also develop accelerated convergence results using generalized monotone method. Finally, we provide a numerical example as an application of all our theoretical results.

2. PRELIMINARY RESULTS

In this section, we recall known results, which are needed for our main results. Initially, we recall some definitions.

Definition 2.1. Caputo fractional derivative of order q is given by equation

$$^{c}D^{q}u(t) = \frac{1}{\Gamma(1-q)}\int_0^t (t-s)^{-q}u'(s)ds,$$

where $0 < q < 1$.

Also, consider nonlinear Caputo fractional differential equation with initial condition of the form

$$^{c}D^{q}u(t) = f(t,u(t)), \quad u(0) = u_0, \tag{2.1}$$

where $f \in C[J \times \mathbb{R}, \mathbb{R}]$ and $J = [0,T]$.

The integral representation of (2.1) is given by equation

$$u(t) = u_0 + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} f(s,u(s))ds, \tag{2.2}$$

where $\Gamma(q)$ is the Gamma function.

In order to compute the solution of linear fractional differential equation with constant coefficients we need Mittag Leffler function.

Definition 2.2. Mittag Leffler function is given by

$$E_{\alpha,\beta}(\lambda(t-t_0)^{\alpha}) = \sum_{k=0}^{\infty} \frac{(\lambda(t-t_0)^{\alpha})^k}{\Gamma(\alpha k+\beta)},$$

where $\alpha, \beta > 0$. Also, for $t_0 = 0$, $\alpha = q$ and $\beta = 1$, we get

$$E_{q,1}(\lambda t^{q}) = \sum_{k=0}^{\infty} \frac{(\lambda t^{q})^k}{\Gamma(qk+1)},$$

where $q > 0$.

Also, consider linear Caputo fractional differential equation,

$$^cD^q u(t) = \lambda u(t) + f(t), \quad u(0) = u_0, \text{ on } J \tag{2.3}$$

where $J = [0, T]$, λ is a constant and $f(t) \in C[J, \mathbb{R}]$.

The solution of (2.3) exists and is unique. The explicit solution of (2.3) is given by

$$u(t) = u_0 E_{q,1}(\lambda t^q) + \int_0^t (t-s)^{q-1} E_{q,q}(\lambda t^q) f(s) ds.$$

See [5] for details.

In particular, if $\lambda = 0$, the solution $u(t)$ is given by

$$u(t) = u_0 + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s) ds, \tag{2.4}$$

where $\Gamma(q)$ is the Gamma function.

Also we recall known results related to scalar Caputo nonlinear fractional differential equations of the following form:

$$^cD^q u(t) = f(t, u) + g(t, u), \quad u(0) = u_0 \text{ on } J = [0, T], \tag{2.5}$$

where $0 \leq q < 1$. Here $f, g \in C(J \times \mathbb{R}, \mathbb{R})$, $f(t, u)$ is non-decreasing in u on J and $g(t, u)$ is non-increasing in u on J.

We recall a basic lemma relative to the Reimann-Liouville fractional derivative.

Lemma 2.3. *Let $m(t) \in C_p[J, \mathbb{R}]$ (where $J = [0, T]$) be such that for some $t_1 \in (0, T]$, $m(t_1) = 0$ and $m(t) \leq 0$, on $[0, T]$, then $D^q m(t_1) \geq 0$.*

Proof. See [2, 5] for details. □

The above lemma is true for Caputo derivative also, using the relation $^cD^q x(t) = D^q(x(t) - x(0))$ between the Caputo derivative and the Reimann-Liouville derivative. This is the version we will be using to prove our comparison results.

We recall the following known definitions which are needed for our main results.

Definition 2.4. The functions $v_0, w_0 \in C^1([0, T], \mathbb{R})$ are called natural lower and upper solutions of (2.5) if:

$$^cD^q v_0(t) \leq f(t, v_0) + g(t, v_0), \quad v_0(0) \leq u_0,$$

and

$$^cD^q w_0(t) \geq f(t, w_0) + g(t, w_0), \quad w_0(0) \geq u_0.$$

Definition 2.5. The functions $v_0, w_0 \in C^1(J, \mathbb{R})$ are called coupled lower and upper solutions of (2.5) of type I if:

$$^cD^q v_0(t) \leq f(t, v_0) + g(t, w_0), \quad v_0(0) \leq u_0,$$

and

$$^cD^q w_0(t) \geq f(t, w_0) + g(t, v_0), \quad w_0(0) \geq u_0.$$

See [7] for other types of coupled lower and upper solutions relative to (2.5).

Denoting $F(t,u) = f(t,u) + g(t,u)$, we state the next comparison result.

Theorem 2.6. *Let v, w be natural lower and upper solutions of (2.5), respectively. Suppose that $F(t,w) - F(t,v) \leq L(w - v)$ where L is a constant such that $L > 0$, then $v(0) \leq w(0)$ implies that $v(t) \leq w(t)$, $t \in J$.*

Proof. See [5] for details. □

Next, we recall a corollary of Theorem 2.6, which we will use in our main result.

Corollary 2.7. *Let $p \in C^1[J, \mathbb{R}]$. $^cD^q p(t) \leq Lp(t)$, where $L \geq 0$ and $p(0) \leq 0$. Then $p(t) \leq 0$ on J.*

We define the following sector Ω for convenience. That is, $\Omega = [(t,u) : v(t) \leq u(t) \leq w(t), t \in J]$.

Theorem 2.8. *Suppose $v, w \in C^1[J, \mathbb{R}]$ are coupled lower and upper solutions of (2.5) such that $v(t) \leq w(t)$ on J and $F \in C(\Omega, \mathbb{R})$. Then there exists a solution $u(t)$ of (2.5) such that $v(t) \leq u(t) \leq w(t)$ on J, provided $v(0) \leq u_0 \leq w(0)$.*

Proof. See [5] for details. □

Theorem 2.9. *Assume that*

(i) *$v_0, w_0 \in C^1[J, \mathbb{R}]$. v_0, w_0 are coupled lower and upper solutions of (2.5) of type I, with $v_0(t) \leq w_0(t)$ on J.*
(ii) *$f(t,u)$, $g(t,u) \in C[J \times \mathbb{R}, \mathbb{R}]$, where $f(t,u)$ is increasing in u on J, and $g(t,u)$ is decreasing in u on J.*

Then there exist monotone sequences, $v_n(t)$ and $w_n(t)$, such that $v_n(t) \to v(t)$ and $w_n(t) \to w(t)$ uniformly and monotonically, where $v(t)$ and $w(t)$ are coupled minimal and maximal solutions of equation (2.5) on J. That is, for any solution $u(t)$, of (2.5), with $v_0 \leq u \leq w_0$ on J, we get natural sequences, $\{v_n\}$ and $\{w_n\}$, satisfying, $v_0(t) \leq v_1(t) \leq v_2(t) \leq \cdots \leq v_n(t) \leq u(t) \leq w_n(t) \leq \cdots \leq w_2(t) \leq w_1(t) \leq w_0(t)$, for each $n \geq 1$ on J, where $v(t)$ and $w(t)$ satisfy the coupled system,

$$\begin{aligned} {}^cD^q v(t) &= f(t, v(t)) + g(t, w(t)), \quad v(0) = u_0, \\ {}^cD^q w(t) &= f(t, w(t)) + g(t, v(t)), \quad w(0) = u_0. \end{aligned} \tag{2.6}$$

Here we use type (ii) iterative schemes,

$$
\begin{aligned}
{}^cD^q v_{n+1}(t) &= f(t, v_n(t)) + g(t, w_n(t)), \quad v_{n+1}(0) = u_0, \\
{}^cD^q w_{n+1}(t) &= f(t, w_n(t)) + g(t, v_n(t)), \quad w_{n+1}(0) = u_0.
\end{aligned}
$$

Also, $v(t) \le u(t) \le w(t)$ *on* J.

Theorem 2.10. *Let all the hypothesis of Theorem* 2.9 *be satisfied. Further, let*

$$
\begin{aligned}
f(t, u_1) - f(t, u_2) &\le L_1(u_1 - u_2), \\
g(t, u_1) - g(t, u_2) &\ge -M_1(u_1 - u_2),
\end{aligned}
$$

where L_1 *and* M_1 *are constants, whenever* $v_0 \le u_2 \le u_1 \le w_0$*, then* $v = w = u$ *is the unique solution of equation* (2.5)*.*

The next result is monotone method for (2.5) where we use natural lower and upper solutions.

Theorem 2.11. *Assume that*

(i) $v_0, w_0 \in C^1(J, \mathbb{R})$ *are natural lower and upper solutions* (2.5) *with* $v_0(t) \le w_0(t)$ *on* J.

(ii) $f, g \in C(J \times \mathbb{R}, \mathbb{R}), f(t, u)$ *is nondecreasing in* u *and* $g(t, u)$ *is nonincreasing in* u *on* J.

Then there exists monotone sequences $v_n(t)$ *and* $w_n(t)$ *on* J *such that* $v_n(t) \to v(t)$ *and* $w_n(t) \to w(t)$ *uniformly and monotonically and* (v, w) *are coupled minimal and maximal solutions, respectively to equation* (2.5)*. That is,* (v, w) *satisfy*

$$
\begin{aligned}
{}^cD^q v(t) &= f(t, v) + g(t, w), \quad v(0) = u_0, \quad \text{on} \quad J, \\
{}^cD^q w(t) &= f(t, w) + g(t, v), \quad w(0) = u_0, \quad \text{on} \quad J,
\end{aligned}
\tag{2.7}
$$

provided $v_0 \le v_1$ *and* $w_1 \le w_0$ *on* J.

See [5] for details of the proofs of Theorems 2.9, 2.10, 2.11. Also note that the iterative schemes used in Theorems 2.9 and 2.11 are one and the same. Theorem 2.11, uses v_0, w_0 as natural lower and upper solutions. Then v_1, w_1 will be coupled lower and upper solutions only on some interval $[0, t_1)$ and not necessarily on $[0, T]$. This is the motivation for our main result relative to equation (2.5).

3. MAIN RESULTS

The generalized monotone method is well known for scalar Caputo differential equations with initial conditions using coupled lower and upper solutions of type I as described in our preliminaries. It is easy to observe that coupled lower and upper solutions of type I implies that they are also natural lower and upper solution. However, the converse is not true. In theory, we know that the existence of natural lower and upper solutions, where the lower solution is less than or equal to the upper solution, we have a solution of (2.5) such that $v_0 \le u \le w_0$ on J, whenever $v_0(0) \le u_0 \le w_0(0)$. In the generalized monotone method, if we use natural lower and upper solution we need an extra assumption, that is $v_0(t) \le v_1(t)$ and $w_1(t) \le w_0(t)$ on J. Note that in this case, the sequences are developed as in Theorem 2.9.

Consider the example

$$^cD^{\frac{1}{2}}u(t) = u - u^2, \quad u(0) = \frac{1}{2}, \quad t \in [0, T], \quad T \ge 1.$$

Then $v_0(t) = 0$ and $w_0(t) = 1$ are natural lower and upper solutions respectively. Then using the iterations as in Theorem 2.11, we get

$$v_1(t) = 0.5 - 1.1284t^{\frac{1}{2}} \quad \text{and} \quad w_1(t) = 0.5 + 1.1284t^{\frac{1}{2}}.$$

It is easy to observe $v_1(t) \ge v_0(t)$ and $w_1(t) \le w_0(t)$ on $[0, 0.1963]$. However in order to apply Theorem 2.9, we need

$$v_1(t) \ge v_0(t) \quad \text{and} \quad w_1(t) \le w_0(t) \quad \text{on} \quad [0, T].$$

This is the motivation for our main result. Our aim is to develop a method to construct coupled lower and upper solutions on the interval $J = [0, T]$, so that we can apply Theorem 2.9 to compute the coupled minimal and maximal solutions for equation (2.5). If f and g satisfies one sided Lipschitz condition, we can also prove that the coupled minimal and maximal solutions of (2.5) will converge to the unique solution of (2.5). The next result provides a method to construct coupled lower and upper solutions to any desired interval using natural lower and upper solutions.

Theorem 3.1. *Assume that*

(i) $v_0, w_0 \in C[J, \mathbb{R}]$ *are natural lower and upper solutions of* (2.5) *such that* $v_0(t) \le w_0(t)$ *on* J.

(ii) $f, g \in C[J \times \mathbb{R}, \mathbb{R}], f(t, u)$ *is nondecreasing and* $g(t, u)$ *is nonincreasing in* u *on* J. *Then there exists monotone sequences* $\{v_n(t)\}$ *and* $\{w_n(t)\}$ *on* J *such that* $v_n(t) \to v(t)$ *and* $w_n(t) \to w(t)$ *uniformly and monotonically to* v *and* w *where* v *and* w *are coupled lower and upper solutions of* (2.5) *such that* $v \le w$ *on* J. *The iterative scheme is given by*

$$^cD^q v_{n+1}(t) = f(t, v_n) + g(t, w_n), \quad on \quad [0, t_n], \quad v_{n+1}(0) = u_0$$

$$^cD^q w_{n+1}(t) = f(t, w_n) + g(t, v_n), \quad on \quad [0, \overline{t_n}\ ], \quad w_{n+1}(0) = u_0,$$

where $v_n(t) \ge v_0(t)$ *on* $[0, t_n)$ *and* $w_n(t) \le w_0(t)$ *on* $[0, \overline{t_n})$.

Also define $v_{n+1}(t), w_{n+1}(t)$ *on* $[t_n, T]$ *and* $[\overline{t_n}, T]$ *respectively as the solution of*

$$^cD^q v_{n+1}(t) = f(t, v_0) + g(t, w_0), \quad v_{n+1}(t_n) = \lim_{h \to 0} v_{n+1}(t_n - h),$$

$$^cD^q w_{n+1}(t) = f(t, w_0) + g(t, v_0) \quad w_{n+1}(\overline{t_n}) = \lim_{h \to 0} w_{n+1}(\bar{t}_n - h).$$

Proof. From Theorem 2.11 we have $v_0(t) \leq v_1(t)$ on $[0, t_1]$ and $w_1(t) \leq w_0(t)$ on $[0, \overline{t_1}\,]$. If $t_1 \geq T$, and $\overline{t_1} \geq T$ there is nothing to prove since one can use Theorem 2.9 to compute coupled minimal and maximal solutions. If not, certainly $t_1 < T$ and $\overline{t_1} < T$. Also $v_1(t_1) = v_0(t_1)$ and $w_1(\overline{t_1}) = w_0(\overline{t_1})$. We will now redefine $v_1(t)$ and $w_1(t)$ on $[0, T]$ as follows

$$^cD^q v_1(t) = f(t, v_0) + g(t, w_0), \;\; v_1(0) = u_0 \;\; on \;\; [0, t_1],$$

$$^cD^q w_1(t) = f(t, w_0) + g(t, v_0), \;\; w_1(0) = u_0 \;\; on \;\; [0, \overline{t_1}\,],$$

and

$$v_1(t) = v_0(t) \;\; \text{on} \;\; [t_1, T],$$

$$w_1(t) = w_0(t) \;\; \text{on} \;\; [\,\overline{t_1}, T].$$

Proceeding in this manner, we will have $v_n(t_n) = v_0(t_n)$, and $w_n(\overline{t_n}) = w_0(\overline{t_n})$. Now we can redefine v_n, w_n as follows.

$$^cD^q v_n(t) = f(t, v_{n-1}) + g(t, w_{n-1}), \quad v_n(0) = u_0 \;\; \text{on} \;\; [0, t_n)$$

$$v_n(t) = v_0(t) \quad \text{on} \quad [t_n, T],$$

Similarly,

$$^cD^q w_n(t) = f(t, w_{n-1}) + g(t, v_{n-1}), \quad w_n(0) = u_0 \;\; \text{on} \;\; [0, \bar{t}_n)$$

$$w_n(t) = w_0(t) \quad \text{on} \quad [\,\bar{t}_n, T]$$

v_n, w_n intersect v_0 and w_0 at $t_n, \;\; \overline{t_n}$ respectively. If $t_n \geq T$, and $\overline{t_n} \geq T$ we can stop the process. Certainly $v_n \leq w_n$ and v_n and w_n are coupled minimum and maximum solutions of (2.5) respectively.

Now we can show that the sequences $\{v_n(t)\}$ *and* $\{w_n(t)\}$ constructed above are equicontinuous and uniformly bounded on J. Hence by Ascoli Arzela's theorem, a subsequence converges uniformly and monotonically. Since the sequences are monotone, the entire sequence converges uniformly and monotonically to v and w respectively.

It is easy to observe that

$$^cD^q v_n(t) = f(t, v_{n-1}) + g(t, w_{n-1}), \;\; v_n(0) = u_0 \;\; \text{on} \quad [0, t_n),$$

$$v_n(t) = v_0(t) \quad \text{on} \quad [t_{n-1}, T], \;\; \text{such that} \;\; v_n(t_{n-1}) = v_0(t_n),$$

and

$$^cD^q w_n(t) = f(t, w_{n-1}) + g(t, v_{n-1}), \quad w_n(0) = u_0 \;\; \text{on} \;\; [0, \bar{t}_n),$$

$$w_n(t) = w_0(t) \quad \text{on} \quad [\,\bar{t}_{n-1}, T], \;\; \text{such that} \;\; w_n(\,\bar{t}_n) = w_0(\,\bar{t}_{n-1}),$$

for all $n \geq 1$.

As $n \to \infty, \quad t_n, \overline{t_n} \to T, v_n(t) \to v(t)$, and $w_n(t) \to w(t)$, uniformly and monotonically.

Further,

$$^cD^q v(t) = f(t,v) + g(t,w), \quad v(0) = u_0 \quad \text{on} \quad J,$$

and

$$^cD^q w(t) = f(t,w) + g(t,v), \quad w(0) = u_0 \quad \text{on} \quad J.$$

Hence v, w are coupled lower and upper solutions of (2.5) on J. This concludes the proof. □

Remark 3.2. Note that Theorem 3.1 provides coupled lower and upper solutions of (2.5) on J. Now we can develop sequences $\{v_n\}$ and $\{w_n\}$ using Theorem 2.9. These sequences converge uniformly and monotonically to coupled minimal and maximal solutions. Further if uniqueness condition is satisfied, the sequences converge to the unique solution of (2.5). However, in generalized monotone method even for scalar equations like (2.5), we can apply Gauss-Seidel method such that sequences converge faster. This is precisely the next result.

Theorem 3.3. *Let all the hypothesis of Theorem* 2.9 *hold. Then there exist monotone sequences* v_n *and* w_n*, where the iterative scheme is given by*

$$\begin{aligned} ^cD^q v^*_{n+1} &= f(t, v^*_n) + g(t, w^*_n), \quad v^*_{n+1}(0) = u_0, \\ ^cD^q w^*_{n+1} &= f(t, w^*_n) + g(t, v^*_{n+1}), \quad w^*_{n+1}(0) = u_0, \end{aligned} \tag{3.1}$$

where $v^*_0 = v_1$ *and* w^*_0 *is the solution of* $^cD^q w^*_0 = f(t, w_0) + g(t, v_1), w^*_0(0) = u_0$*, or*

$$\begin{aligned} ^cD^q v^*_{n+1} &= f(t, v^*_n) + g(t, w^*_{n+1}), \quad v^*_{n+1}(0) = u_0, \\ ^cD^q w^*_{n+1} &= f(t, w^*_n) + g(t, v^*_n), \quad w^*_{n+1}(0) = u_0, \end{aligned} \tag{3.2}$$

where $w^*_0 = w_1$ *and* v^*_0 *is the solution of* $^cD^q v^*_0 = f(t, v_0) + g(t, w_1), v^*_0(0) = u_0$.

Proof. We provide a brief proof. One can easily see that $v_0(t) \le v_1(t)$ on J. Now it is enough if we prove that $w^*_0 \le w_1$. Let

$$\begin{aligned} p(t) &= w^*_0 - w_1, \quad p(0) = 0 \\ ^cD^q p(t) &= {}^cD^q {w_0}^* - {}^cD^q w_1 \\ &= f(t, w_0) + g(t, v_1) - (f(t, w_0) + g(t, v_0)) \\ &= g(t, v_1) - g(t, v_0) \le 0 \end{aligned}$$

since $v_1(t) \ge v_0(t)$ on J.

This implies $p(t) \le 0$ on J, using Corollary 2.7. That is $w^*_0 \le w_1$ on J. Continuing the process, we can show that that the sequences $\{v^*_n\}$ and $\{w^*_n\}$ converges faster than the sequences $\{v_n\}$ and $\{w_n\}$ which are computed using Theorem 2.9. □

4. NUMERICAL RESULTS

In this section, we provide numerical examples justifying our results of section 3. Initially we take a simple logistic equation and apply Theorem 2.9. In order to apply Theorem 2.9, we assume that v_1 and w_1 should satisfy $v_0 \leq v_1, \quad w_1 \leq w_0$ on $[0, T]$.

Consider the example

$$ {}^cD^{\frac{1}{2}}u(t) = u - u^2, \quad u(0) = \frac{1}{2}, \quad t \in [0, T], \quad T \geq 1. \tag{4.1} $$

It is easy to observe that $v_0(t) = 0$ and $w_0(t) = 1$ are natural lower and upper solutions respectively of (4.1) such that $v_0 \leq w_0$ on $[0, T]$.

Using our main result namely Theorem 3.1 we can compute coupled lower and upper solutions on $[0, T]$. In the graph below we apply Theorem 3.1 to the above example.

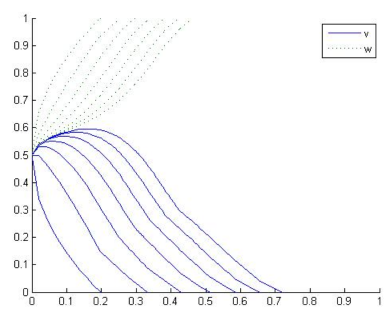

FIGURE 1. Coupled Lower and Upper Solutions of (4.1) using Theorem 3.1

In the above graph we have computed v and w such that $v \leq w$ on the interval $[0, 0.4618]$.

In the next graph we use the coupled upper and lower solutions of Figure 1 and apply Theorem 2.9 to obtain the coupled minimal and maximal solutions.

We have plotted the above graph on the interval $[0, 0.45]$ showing six iterations.

In the next graph we use the coupled upper and lower solutions of Figure 1 and apply Theorem 3.3 to obtain the coupled minimal and maximal solutions.

We can observe that Figure 3 took only five iterations compared to six iterations in Figure 2, as we have used accelerated convergence.

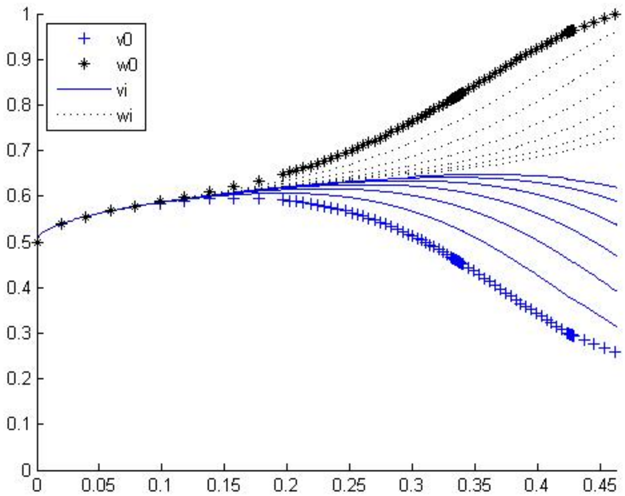

FIGURE 2. Coupled Lower and Upper Solutions of (4.1) using Theorem 2.9

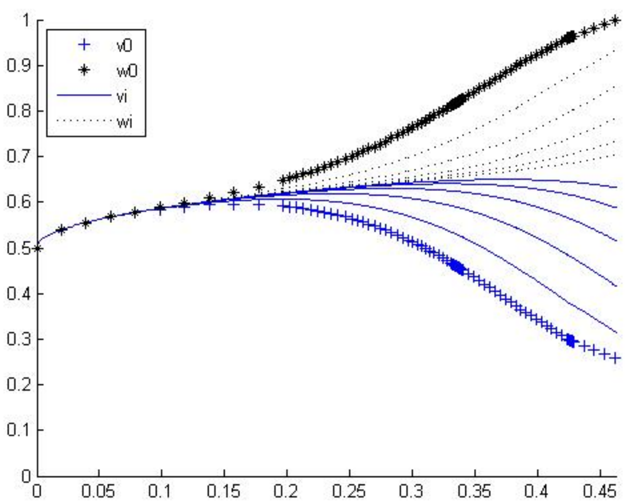

FIGURE 3. Coupled Lower and Upper Solutions of (4.1) using Theorem 3.3

5. CONCLUSION

In general in order to compute the coupled lower and upper solutions of linear fractional differential equations, using the usual monotone method we need the Mittag Leffler function. In this work, we have computed the coupled lower and upper solutions of scalar Caputo fractional differential equations, on a desired interval using the generalised monotone method. The advantage of the generalised monotone method over the usual monotone method is that it does not require the computation of Mittag Leffler function. As we are using generalised monotone method, even for scalar equations we were able to accelerate the convergence by using Gauss-Seidel method. In our future work we would like to extend this method to system of fractional differential equations like the Volterra-Lotka model.

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