Neural, Parallel, and Scientific Computations 21 (2013) 31-42

UPPER AND LOWER SOLUTIONS FOR GENERAL TWO-POINT BOUNDARY VALUE PROBLEMS ON TIME SCALES

K. R. PRASAD[1], P. MURALI[2], AND N. V. V. S. SURYANARAYANA[3]

[1,2]Department of Applied Mathematics, Andhra University
Visakhapatnam, 530003, India
rajendra92@rediffmail.com, penugurthi.murali@gmail.com
[3]Department of Mathematics, VITAM College of Engineering
Visakhapatnam, 531173, A.P, India
suryanarayana_nvvs@yahoo.com

ABSTRACT. We consider a general two point boundary value problem for dynamical equations on time scales and establish a criterion for the existence of solution by using a fixed point theorem on cones. We also establish the existence of solution to two-point boundary value problem by the method of upper and lower solutions.

Key words: Time scale, dynamic equation, lower solution, upper solution.

AMS Subject Classification: 34B99, 39A99.

1. Introduction

The existence of a solution via lower and upper solutions, coupled with a monotone iterative technique, provides an effective and flexible mechanism that offers theoretical as well as constructive results for nonlinear boundary value problems on a closed set. The lower and upper solutions for two point boundary value problems on time scales, an improvement by a monotone iterative process, serve as bounds for solution. The idea imbedded in this technique has proved to be of immense value and has played an important role in unifying a variety of nonlinear problems.

The history of the existence of solutions via upper and lower solutions for boundary value problems (BVPs) also enjoys a good history, first for BVPs associated with differential equations, then finite difference equations, and recently, unifying the theory on time scales. The development of the theory has gained attention by many researchers; to mention a few, we list some papers Habets and Zanolin [9], and Lee [13] for ordinary differential equations, Kelley and Peterson [12], and Atici and Cabada [5] for finite difference equations, and the results for time scales by Akin [2]. Later the results extended to infinite interval by Agarwal, Bohner and O' Regan [1]. We extend these results to second order nonlinear differential equation on time scales

Received October 17, 2012

satisfying two-point nonhomogeneous Sturm Liouville boundary conditions when f is taken as general nonlinear function.

This paper considers the existence of solution and the existence of a solution via upper and lower solutions to the second order dynamic equation,

$$y^{\Delta^2}(x) + f(x, y(x), y^{\Delta}(x)) = 0,\ t \in [a, b] \tag{1.1}$$

subject to the Sturm Liouville boundary conditions,

$$\alpha y(a) + \beta y^{\Delta}(a) = A, \tag{1.2}$$

$$\gamma y(\sigma^2(b)) + \delta y^{\Delta}(\sigma(b)) = B, \tag{1.3}$$

where $\alpha, \beta, \gamma, \delta, A, B \in \mathbb{R}$ such that $\alpha > 0, \beta \leq 0, \gamma > 0, \delta \geq 0$, and $|\alpha^{-1}\delta + \beta\gamma^{-1}| < \sigma^2(b) - a$.

We make the following assumptions throughout:

(A1) $f : [a, \sigma^2(b)] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous, and
(A2) $f : [a, \sigma^2(b)] \times \mathbb{R}^2 \to \mathbb{R}$ is increasing in its last argument.

The rest of the paper is organized as follows. In Section 2, we briefly describe some salient features of time scales and we state some basic concepts which are needed for later discussion. In Section 3, we establish the existence of solution for the BVP (1.1)–(1.3) by using fixed point theorem on cones. In Section 4, we establish the existence of solution for the BVP (1.1)–(1.3) via upper and lower solutions. Finally, as an application, we give examples to demonstrate our result.

2. **Preliminaries**

For the information on time scale calculus and notation for delta differentiation, as well as concepts for dynamic equations on time scales, we refer to the introductory book on time scales by Bohner and Peterson [7]. By a time scale we mean a nonempty closed subset of $\mathbb{R}$. We denote the time scale by the symbol $\mathbb{T}$. By an interval we mean the intersection of the real interval with a given time scale. The jump operators introduced on a time scale $\mathbb{T}$ may be connected or disconnected. To overcome this topological difficulty the concept of jump operators is introduced in the following way. The operators σ and ρ from $\mathbb{T} \to \mathbb{T}$, defined as $\sigma(x) = \inf\{\xi \in \mathbb{T} : \xi > x\}$ and $\rho(x) = \sup\{\xi \in \mathbb{T} : \xi < x\}$ are called jump operators. If σ is bounded above and ρ is bounded below then we define $\sigma(\max\mathbb{T})=\max\mathbb{T}$ and $\rho(\min\mathbb{T})=\min\mathbb{T}$. These operators allow us to classify the points of time scale $\mathbb{T}$. A point $x \in \mathbb{T}$ is said to be right-dense if $\sigma(x) = x$, left-dense if $\rho(x) = x$, right-scattered if $\sigma(x) > x$, left-scattered if $\rho(x) < x$, isolated if $\rho(x) < x < \sigma(x)$ and dense if $\rho(x) = x = \sigma(x)$. The set $\mathbb{T}^{\kappa}$ which is derived from the time scale $\mathbb{T}$ as follows

$$\mathbb{T}^\kappa = \begin{cases} \mathbb{T}\backslash(\rho(\sup\mathbb{T}), \sup\mathbb{T}], & \text{if } \sup\mathbb{T} < \infty \\ \mathbb{T}, & \text{if } \sup\mathbb{T} = \infty \end{cases}.$$

Finally, if $f : \mathbb{T} \to \mathbb{R}$ is a function, then we define the function $f^\sigma : \mathbb{T} \to \mathbb{R}$ by $f^\sigma(x) = f(\sigma(x))$ for all $x \in \mathbb{T}$.

We define the set D=$\{y : y^{\Delta^2}$ is continuous on $[a, b]$ $\}$. For any $u, v \in D$, we define the sector $[u, v]$ by $[u, v] = \{w \in D : u(x) \le w(x) \le v(x),\ x \in [a, \sigma^2(b)]\}$.

Definition 2.1. A real valued function $u(x) \in D$ on $[a, \sigma^2(b)]$ is a lower solution for the BVP (1.1)–(1.3), if

$$-u^{\Delta^2}(x) \le f(x, u(x), u^\Delta(x)),\ x \in [a, b]$$

$$\alpha u(a) + \beta u^\Delta(a) \le A$$

and

$$\gamma u(\sigma^2(b)) + \delta u^\Delta(\sigma(b)) \le B.$$

Definition 2.2. A real valued function $v(x) \in D$ on $[a, \sigma^2(b)]$ is an upper solution for the BVP (1.1)–(1.3), if

$$-v^{\Delta^2}(x) \ge f(x, v(x), v^\Delta(x)),\ x \in [a, b]$$

$$\alpha v(a) + \beta v^\Delta(a) \ge A$$

and

$$\gamma v(\sigma^2(b)) + \delta v^\Delta(\sigma(b)) \ge B.$$

3. **An Existence Theorem**

In this section, we establish the general solution for the BVP (1.1)–(1.3), and then we establish the existence of solutions for the BVP (1.1)–(1.3) by using the Schauder-Tychonov fixed point theorem [10].

Define C^1 to be the Banach space of all continuously differentiable functions on $[a, \sigma^2(b)]$ equipped with the norm $\|\cdot\|$ defined by

$$\max\left\{\max_{x\in[a,\sigma^2(b)]}|y(x)|, \max_{x\in[a,\sigma(b)]}|y^\Delta(x)|\right\}.$$

Theorem 3.1. *The solution of the BVP* (1.1)–(1.3) *is*

$$y(x) = l(x) + \int_a^{\sigma(b)} G(x,\xi) f(\xi, y(\xi), y^\Delta(\xi))\Delta\xi,$$

where $G(x, \xi)$ is the Green's function for the associated homogeneous BVP of (1.1)–(1.3) *and*

$$l(x) = \frac{(B\alpha - A\gamma)x + (\gamma\sigma^2(b) + \delta)A - (a\alpha + \beta)B}{\alpha\gamma(\sigma^2(b) - a) + \alpha\delta - \beta\gamma},$$

for all $x \in [a, \sigma^2(b)]$.

Proof. We define $l(x) = \frac{(B\alpha - A\gamma)x + (\gamma\sigma^2(b)+\delta)A - (a\alpha+\beta)B}{\alpha\gamma(\sigma^2(b)-a)+\alpha\delta-\beta\gamma}$ on $[a, \sigma^2(b)]$, where $l(x)$ satisfies the boundary conditions

$$\alpha l(a) + \beta l^{\Delta}(a) = A,$$

$$\gamma l(\sigma^2(b)) + \delta l^{\Delta}(\sigma(b)) = B.$$

Hence, if $y(x)$ is a solution of the BVP (1.1)–(1.3) then $w(x) = y(x) - l(x)$ is a solution of the following second order dynamical equation,

$$w^{\Delta^2}(x) + F(x, w(x), w^{\Delta}(x)) = 0 \tag{3.1}$$

satisfying the two-point boundary conditions,

$$\alpha w(a) + \beta w^{\Delta}(a) = 0, \tag{3.2}$$

$$\gamma w(\sigma^2(b)) + \delta w^{\Delta}(\sigma(b)) = 0, \tag{3.3}$$

where $F(x, w(x), w^{\Delta}(x)) = f(x, w(x)+l(x), w^{\Delta}(x)+l^{\Delta}(x))$. The Green's function for the BVP $-w^{\Delta^2}(x) = 0$ satisfying the boundary conditions (3.2)–(3.3) can be obtained easily by elementary methods and is given by

$$G(x,\xi) = \begin{cases} \frac{[\gamma(\sigma^2(b)-\sigma(\xi))+\delta][\alpha(x-a)-\beta]}{\alpha\gamma(\sigma^2(b)-a)+\alpha\delta-\beta\gamma}, & a \le x \le \xi < \sigma^2(b), \\ \frac{[\gamma(\sigma^2(b)-x)+\delta][\alpha(\sigma(\xi)-a)-\beta]}{\alpha\gamma(\sigma^2(b)-a)+\alpha\delta-\beta\gamma}, & a < \sigma(\xi) \le x \le \sigma^2(b), \end{cases}$$

for all $(x, \xi) \in [a, \sigma^2(b)] \times [a, b]$. The solution for the BVP (3.1)–(3.3) is

$$w(x) = \int_a^{\sigma(b)} G(x,\xi) F(\xi, w(\xi), w^{\Delta}(\xi)) \Delta\xi, \tag{3.4}$$

for all $x \in [a, \sigma^2(b)]$. Hence, $y(x)$ is the solution of the BVP (1.1)–(1.3) and is given by

$$y(x) = l(x) + \int_a^{\sigma(b)} G(x,\xi) f(\xi, y(\xi), y^{\Delta}(\xi)) \Delta\xi,$$

for all $x \in [a, \sigma^2(b)]$. □

Theorem 3.2. *Let* $K = \{y \in C^1 : |y(x)| \le 2M, |y^{\Delta}(x)| \le 2N\}$, *then* K *is closed, bounded and convex set.*

Proof. Let $\{y_n\}_{n=1}^{\infty} \subseteq K$ and let $y_0 \in C^1$ be such that $\|y_n - y_0\| \to 0$ as $n \to \infty$, which implies $\{y_n\}$ uniformly converges to y_0 on $[a, \sigma^2(b)]$ and $\{y_n^{\Delta}\}$ uniformly converges to y_0^{Δ} on $[a, \sigma(b)]$, thus $|y_0(x)| \le 2M, |y_0^{\Delta}(x)| \le 2N$ on $[a, \sigma(b)]$, implies that $y_0 \in K$ and hence K is closed. Clearly K is bounded. Let $y, z \in K$ and consider $\lambda y(x) + (1-\lambda)z(x)$, $0 \le \lambda \le 1$,

$$\begin{aligned} |\lambda y(x) + (1-\lambda)z(x)| &\le \lambda |y(x)| + (1-\lambda)|z(x)| \\ &\le \lambda 2M + (1-\lambda)2M \\ &= 2M, \end{aligned}$$

for all $x \in [a, \sigma^2(b)]$. Similarly $|\lambda y^\Delta(x) + (1-\lambda)z^\Delta(x)| \le 2N$, for all $x \in [a, \sigma(b)]$. Thus $\lambda y(x) + (1-\lambda)z(x) \in K$, and hence K is convex. □

Define $T : K \to C^1$ by

$$Ty(x) = l(x) + \int_a^{\sigma(b)} G(x,\xi) f(\xi, y(\xi), y^\Delta(\xi))\Delta\xi, \tag{3.5}$$

for all $x \in [a, \sigma^2(b)]$. Let us take

$$K_1 = \max_{x\in[a,\sigma^2(b)]} \int_a^{\sigma(b)} G(x,\xi)\Delta\xi \text{ and } K_2 = \max_{x\in[a,\sigma^2(b)]} \int_a^{\sigma(b)} G^\Delta(x,\xi)\Delta\xi.$$

Theorem 3.3. *The operator T, as defined in equation* (3.5)*, is continuous on K.*

Proof. The set K is compact subset of $[a, \sigma^2(b)] \times \mathbb{R}^2$, where K is defined as in Theorem 3.2. Hence, given $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that for each (x_1, y_1, y_2), $(x_2, z_1, z_2) \in K$ with $|x_1 - x_2| < \delta, |y_1 - z_1| < \delta, |y_2 - z_2| < \delta$, it follows that $|f(x_1, y_1, y_2) - f(x_2, z_1, z_2)| < \epsilon$. Let $y, z \in K$ with $\| y - z \| < \delta$, then $|y - z| < \delta$ and $|y^\Delta - z^\Delta| < \delta$, it follows that

$$|Ty(x) - Tz(x)| < \epsilon K_1$$

and

$$|Ty^\Delta(x) - Tz^\Delta(x)| < \epsilon K_2.$$

Therefore,

$$\| Ty(x) - Tz(x) \| < \epsilon[K_1 + K_2],$$

whenever $\| y - z \| < \delta$. Hence T is continuous on K. □

Theorem 3.4. *Assume that the condition* (A1) *holds. Let $Q > 0$ satisfies $Q \ge \max\left\{|f(x, y, y^\Delta)| : x \in [a, \sigma(b)], |y(x)| \le 2M, |y^\Delta(x)| \le 2N\right\}$, and if $M = \max\left\{\frac{A}{\alpha}, \frac{B}{\gamma}\right\}$, $K_1 < \frac{M}{Q}$, $\frac{|B\alpha - A\gamma|}{p} \le N$ and $K_2 < \frac{N}{Q}$, then the BVP* (1.1)–(1.3) *has a solution.*

Proof. From Theorem 3.2, K is a closed, bounded and convex subset of C^1. The operator T, as defined in equation (3.5), is continuous. Let $y \in K$ and consider

$$\begin{aligned}
|Ty(x)| &= |l(x) + \int_a^{\sigma(b)} G(x,\xi) f(\xi, y(\xi), y^\Delta(\xi))\Delta\xi| \\
&\le |l(x)| + \int_a^{\sigma(b)} |G(x,\xi)||f(\xi, y(\xi), y^\Delta(\xi))|\Delta\xi \\
&\le M + Q\int_a^{\sigma(b)} |G(x,\xi)|\Delta\xi \\
&\le M + QK_1 \\
&\le 2M
\end{aligned}$$

and

$$\begin{aligned}
|Ty^\Delta(x)| &= |l^\Delta(x) + \int_a^{\sigma(b)} G^\Delta(x,\xi) f(\xi, y(\xi), y^\Delta(\xi))\Delta\xi| \\
&\le |l^\Delta(x)| + \int_a^{\sigma(b)} |G^\Delta(x,\xi)||f(\xi, y(\xi), y^\Delta(\xi))|\Delta\xi \\
&\le N + Q\int_a^{\sigma(b)} |G^\Delta(x,\xi)|\Delta\xi \\
&\le N + QK_2 \\
&\le 2N,
\end{aligned}$$

for all $x \in [a, \sigma^2(b)]$, which implies, $|Ty(x)| \le 2M$, and $|Ty^\Delta(x)| \le 2N$. Hence $T : K \to K$. Using the Arzela-Ascoli theorem [8] it can be shown that $T : K \to K$ is compact operator. Hence T has a fixed point y in K by the Schauder-Tychonov fixed point theorem [10]. □

4. **Existence Via Upper and Lower Solutions**

In this section, we establish the existence of solutions for the BVP (1.1)–(1.3) via upper and lower solutions .

Theorem 4.1. *Assume that the conditions* $(A1)$ *and* $(A2)$ *are satisfied. Let* $u(x)$ *and* $v(x)$ *be a lower and an upper solutions for the* BVP (1.1)*–*(1.3) *respectively, such that* $u(x) \le v(x)$ *for all* $x \in [a, \sigma^2(b)]$*. Then the* BVP (1.1)*–*(1.3) *has a solution* $y(x)$ *with* $u(x) \le y(x) \le v(x)$ *for all* $x \in [a, \sigma^2(b)]$*.*

Proof. Define

$$F(x, y(x), y^\Delta(x)) = \begin{cases} f(x, v(x), y^\Delta(x)) - \frac{y(x)-v(x)}{1+|y(x)-v(x)|}, & y(x) \ge v(x), \\ f(x, y(x), y^\Delta(x)), & u(x) \le y(x) \le v(x), \\ f(x, u(x), y^\Delta(x)) + \frac{y(x)-u(x)}{1+|y(x)-u(x)|}, & y(x) \le u(x). \end{cases}$$

Clearly, the function F is continuous and bounded on $[a, \sigma^2(b)] \times \mathbb{R}^2$. Thus, by Theorem 3.4, there exists a solution $y(x)$ of the BVP

$$y^{\Delta^2}(x) + F(x, y(x), y^{\Delta}(x)) = 0$$

subject to the boundary conditions (1.2)–(1.3). We claim that $u(x) \leq y(x) \leq v(x)$ on $[a, \sigma^2(b)]$. We establish the second inequality, $y(x) \leq v(x)$ on $[a, \sigma^2(b)]$. To the contrary assume that $y(x) > v(x)$ at some point in $[a, \sigma^2(b)]$. From the boundary conditions we know that $y(x) - v(x)$ has a positive maximum at some point, say $c \in (a, \sigma^2(b))$, $y(c) - v(c) > 0$ and $y(x) - v(x) < y(c) - v(c)$ on $[a, \sigma^2(b)]$. Consequently, we have $(y^{\Delta} - v^{\Delta})(c) \leq 0$ and $(y^{\Delta^2} - v^{\Delta^2})(c) \leq 0$. But

$$\begin{aligned} -y^{\Delta^2}(c) &= F(c, y(c), y^{\Delta}(c)) \\ &= f(c, v(c), y^{\Delta}(c)) - \frac{y(c) - v(c)}{1 + |y(c) - v(c)|}. \end{aligned}$$

From $(A2)$, $f(c, v(c), y^{\Delta}(c)) \leq f(c, v(c), v^{\Delta}(c))$ and

$$\begin{aligned} f(c, v(c), y^{\Delta}(c)) - \frac{y(c) - v(c)}{1 + |y(c) - v(c)|} &\leq f(c, v(c), v^{\Delta}(c)) - \frac{y(c) - v(c)}{1 + |y(c) - v(c)|} \\ &< f(c, v(c), v^{\Delta}(c)) \\ &< -v^{\Delta^2}(c). \end{aligned}$$

Hence, we have

$$(y^{\Delta^2} - v^{\Delta^2})(c) > 0,$$

which is a contradiction. It follows that $y(x) \leq v(x)$ on $[a, \sigma^2(b)]$. Similarly, we can prove, $u(x) \leq y(x)$ on $[a, \sigma^2(b)]$. Thus $y(x)$ is a solution of the BVP (1.1)–(1.3) and lies between $u(x)$ and $v(x)$. □

Example 1. Now, we give an example to illustrate the result. Consider the BVP,

$$y^{\Delta^2} + \cos y = 0, \ \ x \in [0, \pi],$$

$$y(0) = 0 = y(\sigma^2(\pi)).$$

We observe that, $u \equiv 0$ is the lower solution on $[0, \sigma^2(\pi)]$, since

$$u^{\Delta^2}(x) = 0 > -\cos 0 = -1$$

and

$$u(0) = 0, \ \ u(\sigma^2(\pi)).$$

Next, let $v(x) = \int_0^x (c - s)\Delta s$, where $c = (1/\sigma^2(\pi)) \int_0^{\sigma^2(\pi)} s\Delta s$. Then

$$v^{\Delta^2}(x) = -1 \leq -\cos v$$

and

$$v(0) = 0, \ \ v(\sigma^2(\pi)).$$

So v is an upper solution on $[0, \sigma^2(\pi)]$. Since v is a solution of the BVP

$$v^{\Delta^2}(x) + 1 = 0$$

and

$$v(0) = 0,\ v(\sigma^2(\pi)).$$

It follows that $v(x) \geq 0$ on $[0, \sigma^2(\pi)]$. Therefore, by Theorem 4.1, we conclude that there is a solution y with

$$0 \leq y(x) \leq \int_0^x (c - s)\Delta s,$$

for all $x \in [0, \sigma^2(\pi)]$.

Now, we will give graphical representation of the solution (taking $\pi = 3.141$) :

Case (i): For $\mathbb{T} = \mathbb{R}$,
Lower solution $u(x) = 0$.
Upper solution $v(x) = \frac{x}{2}(\pi - x)$.
Numerical solution of above problem is in **Figure 1**.

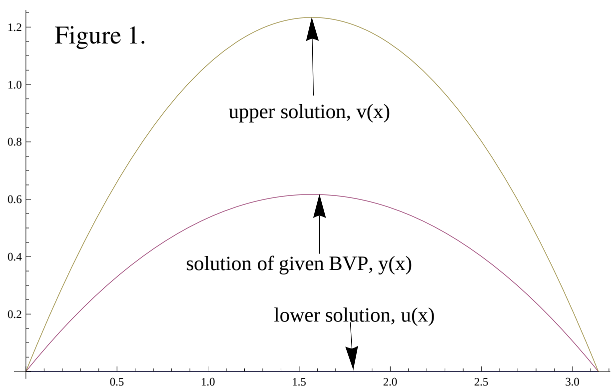

Fig. 1. There exists a solution $y(x)$ to the above BVP such that $u(x) \leq y(x) \leq v(x)$ for $x \in [0, \pi]$.

Case (ii): For $\mathbb{T} = \mathbb{Z} = \{0, 1, 2, 3, \pi, \pi + 1, \pi + 2\}$,
Lower solution $u(x) = 0$.
Upper solution $v(x) = \frac{3(\pi+3)}{\pi+2}x - \sum_{i=0}^{x} i$.

x-values	0	1	2	3	π	$\pi + 1$	$\pi + 2$
Lower solution $u(x)$	0	0	0	0	0	0	0
Upper solution $v(x)$	0	$\frac{2\pi+7}{\pi+2}$	$\frac{3(\pi+4)}{\pi+2}$	$\frac{3(\pi+5)}{\pi+2}$	$\frac{2\pi^2+\pi-12}{\pi+2}$	$\frac{\pi^2+\pi-3}{\pi+2}$	0

Numerical solution of above problem is in **Figure 2**.

Case (iii): For $\mathbb{T} = \{0\} \cup \{\frac{\pi+2}{2^{n-1}} : n \in \mathbb{N}\}$
Lower solution $u(x) = 0$.

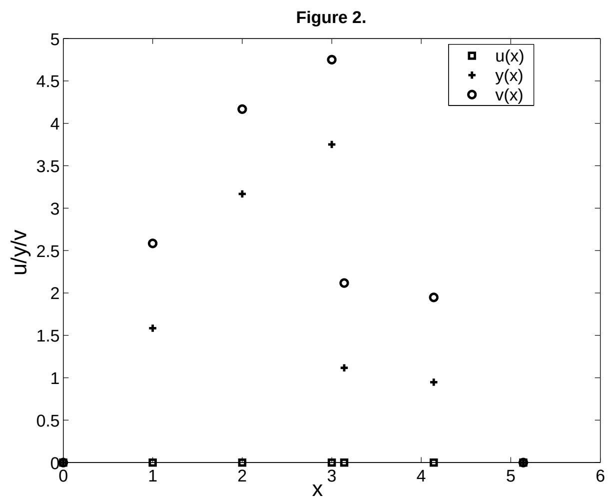

Fig. 2. There exists a solution $y(x)$ to the above BVP such that $u(x) \leq y(x) \leq v(x)$ for $x \in [0, \pi + 2]$.

Upper solution $v(x) = 2x - \sum_{i=0}^{x} i$.

x-values	0	$\pi + 2$	$\frac{\pi+2}{2}$	$\frac{\pi+2}{4}$	$\frac{\pi+2}{8}$	$\frac{\pi+2}{16}$	$\frac{\pi+2}{32}$	$\frac{\pi+2}{64}$	$\frac{\pi+2}{128}$
Lower solution $u(x)$	0	0	0	0	0	0	0	0	0
Upper solution $v(x)$	0	0	$\pi + 1$	$\frac{\pi+1}{2}$	$\frac{\pi+1}{4}$	$\frac{\pi+1}{8}$	$\frac{\pi+1}{16}$	$\frac{\pi+1}{32}$	$\frac{\pi+1}{64}$

Numerical solution of above problem is in **Figure 3**.

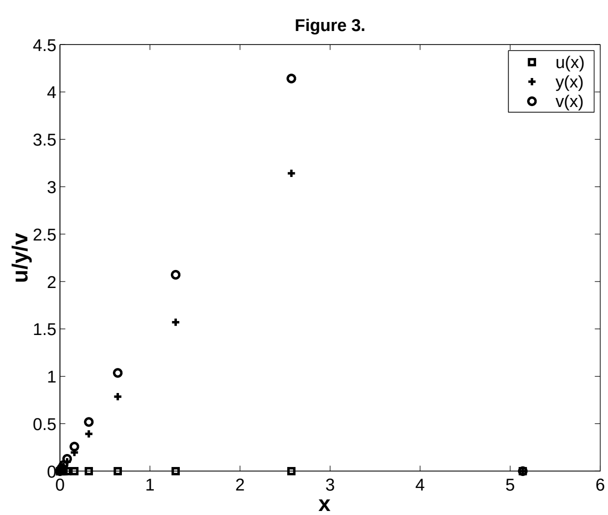

Fig. 3. There exists a solution $y(x)$ to the above BVP such that $u(x) \leq y(x) \leq v(x)$ for $x \in [0, \pi + 2]$.

Case (iv): For $\mathbb{T} = \{0\} \cup \{\frac{\pi}{2^n} : n \in \mathbb{N}\} \cup [\frac{\pi}{2}, \pi]$,
Lower solution $u(x) = 0$.
Upper solution $v(x) = \frac{(3\pi+8)}{8}x - \int_0^x s\Delta s$.

$x:$	0	π	$\frac{\pi}{2}$	$\frac{\pi}{4}$	$\frac{\pi}{8}$	$\frac{\pi}{16}$	$\frac{\pi}{32}$	$\frac{\pi}{64}$	$\frac{\pi}{128}$
$u(x):$	0	0	0	0	0	0	0	0	0
$v(x):$	0	0	$\frac{\pi(3\pi-8)}{16}$	$\frac{\pi(3\pi-8)}{32}$	$\frac{\pi(3\pi-8)}{64}$	$\frac{\pi(3\pi-8)}{128}$	$\frac{\pi(3\pi-8)}{256}$	$\frac{\pi(3\pi-8)}{512}$	$\frac{\pi(3\pi-8)}{1024}$

Numerical solution of above problem is in **Figure 4**.

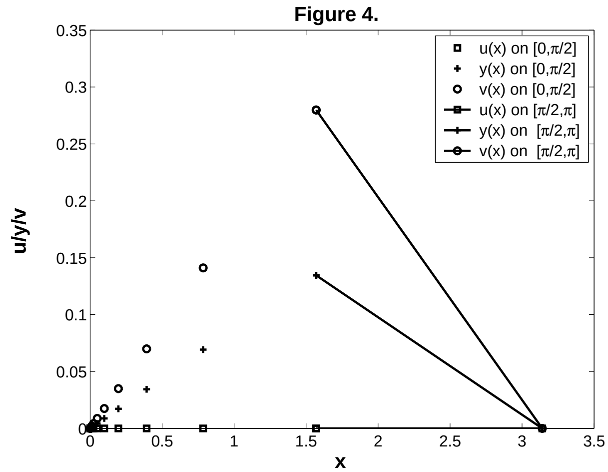

Fig. 4. There exists a solution $y(x)$ to the above BVP such that $u(x) \leq y(x) \leq v(x)$ for $x \in [0, \pi]$.

Example 2. Now, we give another example to illustrate the result. Consider the following dynamical equation,

$$y^{\Delta^2} + \cos(y(1 + y^{\Delta})) = 0, \; x \in [0, b], \; b > 1$$

satisfying the two-point boundary conditions,

$$y(0) = 0, \; y(\sigma^2(b)) + y^{\Delta}(\sigma(b)) = (3/2)b^2.$$

$u \equiv 0$ is the lower solution on $[0, \sigma^2(b)]$, since

$$-u^{\Delta^2}(x) = 0 < 1$$

and

$$u(0) = 0, \; u(\sigma^2(b)) + u^{\Delta}(\sigma(b)) = 0 < (3/2)b^2.$$

Next, let $v(x) = \int_0^x (c - s)\Delta s$, where $c = (4/\sigma^2(b)) \int_0^{\sigma^2(b)} s\Delta s$. Then

$$-v^{\Delta^2}(x) = +1 \geq \cos(v(1 + v^{\Delta}))$$

and

$$v(0) = 0, \; v(\sigma^2(b)) + v^{\Delta}(\sigma(b)) = (3/2)\sigma(b)\sigma^2(b) + \sigma(b) \geq (3/2)b^2.$$

So v is an upper solution on $[0, \sigma^2(b)]$. Since v is a solution of the BVP

$$v^{\Delta^2}(x) + 1 = 0$$

and

$$v(0) = 0, \; v(\sigma^2(b)) + v^{\Delta}(\sigma(b)) = (3/2)\sigma(b)\sigma^2(b) + \sigma(b).$$

It follows that $v(x) \geq 0$ on $[0, \sigma^2(b)]$. Therefore, by Theorem 4.1, we conclude that there is a solution y with

$$0 \leq y(x) \leq \int_0^x (c-s)\Delta s,$$

for all $x \in [0, \sigma^2(b)]$.

Acknowledgement. The authors thank editor for his encouragement and valuable suggestions. One of the authors (Dr. P. Murali) is thankful to C.S.I.R. of India for awarding him an RA.

REFERENCES

[1] Agarwal, R. P., Bohner, M. and O'Regan, D., Time scale boundary value problems on infinite intervals. Dynamic equations on time scales. *J. Comput. Appl. Math.* **141** (2000), no. 1-2, 27–34.

[2] Akin, E., Boundary value problems for a differential equation on a measure chain, *Panamer. Math. J.* **10** (2000), no. 3, 17–30.

[3] Akin-Bohner, E., Atici, F. M. and Kaymakcalan, B., Lower and upper solutions of boundary value problems. *Advances in dynamic equations on time scales*, 165-188, Birkhuser Boston, Boston, MA, 2003.

[4] Anderson, D. R., Solutions to second-order three-point problems on time scales, *J. Difference Equ. Appl.* **8** (2002), no. 8, 673–688.

[5] Atici, F. M. and Cabada, A., Existence and uniqueness results for dis crete second order periodic boundary value problems, *J. Computers Math. Applic.* **45** (2003), 1417–1427.

[6] Atici, F. M. and Guseinov, G. Sh., On Green's functions and positive solutions for boundary value problems on time scales, *J. Comput. Appl. Math.* **141** (2002), no. 1-2, 75–99.

[7] Bohner, M. and Peterson, A. C., *Dynamic equations on time scales, An introduction with applications*, Birkhauser, Boston, MA, (2001).

[8] Folland, G., *Real Analysis: Modern techinics and their applications*, John Wiley and Sons, Inc., New York, Second Edition, 1999.

[9] Habets, P. and Zanolin, F., Upper and lower solutions for a generalized Emden-Fowler equation, *J. Math. Anal. Appl.* **181** (1994), no. 3, 684–700.

[10] Hartman, P., *Ordinary differential equations*, John Wiley and Sons, Inc., New york, 1964.

[11] Henderson, J., Peterson, A. C. and Tisdell, C. C., On the existence and uniqueness of solutions to boundary value problems on time scales, *Adv. Difference Equ.*, (2004), no. 2, 93–109.

[12] Kelley, W. G. and Peterson, A. C., *Difference equations: An introduction with applications*, Acadamic press, San Diego, Second Edition, 2001.

[13] Lee, Y. H., A multiplicity result of positive solutions for the generalized Gelfand type singular boundary value problems, Proceedings of the Second World Congress of Nonlinear Analysts, Part 6 (Athens, 1996); *Nonlinear Anal.* **30** (1997), no. 6, 3829–3835.