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DISTRIBUTIVE LAWS AND THE ALGEBRAIC STRUCTURE OF FUZZY NUMBERS

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ABSTRACT. This paper provides the proof for lemma 4.13 in [7] which states that for any crisp number $s > 0$, and a fuzzy numbers a and c, where the support of a is contained in $\mathbb{R}^+$, then $c(a - s) \equiv ca - cs$. With this lemma, we proved that the distribution laws hold, up to equivalence defined, for fuzzy number multiplication.

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1. Introduction

We continue to look at the multiplication of the fuzzy numbers, in particular, the distributive laws. It is very interesting because the law in general fails to be true in a narrow and strick sense. In [7], we first gave another proof that the distributive law holds from a crisp number distributed to fuzzy numbers. That is, given fuzzy numbers a and b, and a crisp (singleton fuzzy) number r,

$$r(a + b) = ra + rb.$$

In particular, $-(a + b) = -a - b$. We then proved an equivalent condition for the law to be true from a fuzzy number distributed to crisp numbers, which states that for any crisp numbers r and s, and a non-crisp fuzzy number c, we have

$$c(r + s) = cr + cs \quad \text{if and only if} \quad r \cdot s \geq 0.$$

This result was also proved earlier in [1]. Moreover, we also proved in [7] that for any arbitrary crisp numbers r and s,

$$c(r + s) \equiv cr + cs,$$

where the actual equality indeed fails to be true in general. We then extended the result to fuzzy-to-fuzzy distribution in the case their supports are both positive or both negative. That is, given (normal) fuzzy numbers a, b, and c, with the supports of a and b either both in $\mathbb{R}^+$ or both in $\mathbb{R}^-$, then

$$c(a + b) = ca + cb.$$

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This result was also mentioned in [1], proved by Dubois and Prade in 1988, that the distributive law

$$c(a+b) = ca + cb$$

holds if and only if at least one of the following is true,

- $c \in \mathbb{R}$ is crisp
- the supports of a and b are both contained in $\mathbb{R}^+$, or both contained in $\mathbb{R}^-$
- a and b are symmetric

This results leads to the final result in our paper [7], which states that, modulo by the equivalence classes defined by symmetric fuzzy numbers, the fuzzy number system forms a commutative ring with unity. The final result was based on the following lemma,

Lemma 4.13 [7]: *Given crisp number $s > 0$, and a fuzzy numbers a and c, where the support of a is contained in $\mathbb{R}^+$, then*

$$c(a-s) \equiv ca - cs.$$

The proof of the lemma was left out in [7] due to limitation on the number of pages. We shall provide the proof of the lemma in this paper.

2. The Proof of the Lemma

Proof of lemma 4.13 [7]. First, if the support of a is contained in (s, ∞), then the support of $a-s$ will be contained in $\mathbb{R}^+$. Notice $s > 0$, from the results mentioned above,

$$c[(a-s)+s] = c(a-s) + cs$$

Subtracting cs from both sides, we have

$$c(a-s) \equiv c(a-s) + cs - cs = c[(a-s)+s] - cs = ca - cs.$$

The equivalence holds. Thus, we may assume $s \in$ support of a.

First, let's consider the case of $s = 1$. It suffices to show $c(a-1) - ca + c$ is symmetric. We know that

$$\begin{aligned}
\mu_{c(a-1)-ca}(t) &= \max_{xu-yv=t} \min\{\mu_c(x), \mu_{a-1}(u), \mu_c(y), \mu_a(v)\} \\
&= \max_{xu-yv=t} \min\{\mu_c(x), \mu_a(u+1), \mu_c(y), \mu_a(v)\} \\
&= \max_{x(u-1)-yv=t} \min\{\mu_c(x), \mu_a(u), \mu_c(y), \mu_a(v)\}
\end{aligned}$$

Therefore,

$$\begin{aligned}\mu_{c(a-1)-ca+c}(t) &= \max_{x(u-1)-yv+z=t} \min\{\mu_c(x), \mu_a(u), \mu_c(y), \mu_a(v), \mu_c(z)\} \\ &= \max_{xu-yv-x+z=t} \min\{\mu_c(x), \mu_a(u), \mu_c(y), \mu_a(v), \mu_c(z)\} \\ &= \max_{yv-xu+x-z=-t} \min\{\mu_c(x), \mu_a(u), \mu_c(y), \mu_a(v), \mu_c(z)\} \\ &= \mu_{c(a-1)-ca+c}(-t).\end{aligned}$$

Therefore, $c(a-1)-ca+c$ is symmetric. Hence $c(a-1)-ca+c \equiv 0$. Thus

$$c(a-1) \equiv ca - c.$$

For any $s \in \mathbb{R} > 0$, since we do have the distribution from crisp numbers to fuzzy numbers, we have

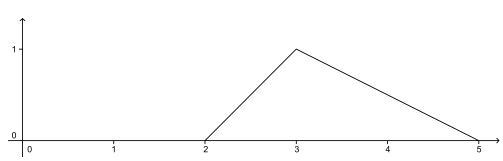
$$c(a-s) = cs\left[\frac{1}{s}(a-s)\right] = cs\left(\frac{a}{s}-1\right) \equiv cs\frac{a}{s} - cs = ca - cs. \qquad \square$$

3. An Example

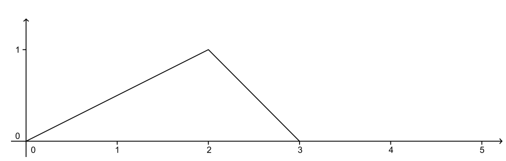
We conclude the paper by the following example to show the computations and that the narraw distributive law indeed fails to be true and it is only true up to equivalence as we have proved in the paper.

Example 3.1. We define fuzzy numbers a and c as following,

$$\mu_c(x) = \begin{cases} x-2, & 2 \le x < 3 \\ -\frac{1}{2}(x-5), & 3 \le x < 5 \\ 0, & \text{Otherwise} \end{cases} \qquad \mu_a(x) = \begin{cases} \frac{1}{2}x, & 0 \le x < 2 \\ -(x-3), & 2 \le x < 3 \\ 0, & \text{Otherwise} \end{cases}$$



Fuzzy number c



Fuzzy number a

To compute ca, it is clear we have $\mu_{ca}(t) = 0$ whenever $t \le 0$ or $t \ge 15$. For t otherwise, we have

$$\begin{aligned}\mu_{ca}(t) &= \max_{xy=t} \min\{\mu_c(x), \mu_a(y)\} = \max_{y} \min\{\mu_c(x), \mu_a(t/x)\} \\ &= \max\{\max_{2\le x\le 3} \min\{x-2, \mu_a(t/x)\}, \max_{3\le x\le 5} \min\{-(x-5)/2, \mu_a(t/x),\}\}.\end{aligned}$$

Case 1: $0 \le t \le 4$.

When $2 \le x \le 3$, we know that $0 \le \frac{t}{x} \le \frac{t}{2} \le 2$. Thus, $\mu_a(t/x) = \frac{t}{2x}$. From the equation $\frac{t}{2x} = x - 2$ we have $2x^2 - 4x - t = 0$, and

$$x = \frac{2 \pm \sqrt{4+2t}}{2}.$$

Since $x \ge 2$, only + fits. Thus,

$$\max_{2\le x\le 3} \min\{x-2, \mu_a(t/y)\} = \frac{2+\sqrt{4+2t}}{2} - 2 = \frac{\sqrt{4+2t}}{2} - 1.$$

When $3 \le x \le 5$, from the equation $\frac{t}{2x} = \frac{5-x}{2}$ we have $2x^2 - 10x + 2t = 0$, and

$$x = \frac{10 \pm \sqrt{100-16t}}{4}.$$

Since $x \ge 2$, only + fits. Thus,

$$\max_{3\le x\le 5} \min\{(5-x)/2, \mu_a(t/x)\} = \frac{5 - \frac{10+\sqrt{100-16t}}{4}}{2} = \frac{10-\sqrt{100-16t}}{8}.$$

It can be shown that, for $0 \le t \le 6$, we have $\frac{10-\sqrt{100-16t}}{8} \le \frac{\sqrt{4+2t}}{2} - 1$. Thus,

$$\mu_{ca}(t) = \frac{\sqrt{4+2t}}{2} - 1, \qquad 0 \le t \le 4.$$

Case 2: $4 \le t \le 6$.

We know that $2 \le \frac{t}{2} \le 3$. When $2 \le x \le \frac{t}{2}$ which implies $y = \frac{t}{x} \ge 2$, we have $\mu_a(y) = \mu_a(t/x) = 3 - \frac{t}{x}$. It can be shown that $3 - \frac{t}{x} \ge x - 2$ whenever $t \le 6$. Thus,

$$\max_{2\le x\le t/2} \min\{x-2, \mu_a(t/x)\} = \max_{2\le x\le t/2}\{x-2\} = \frac{t}{2} - 2.$$

When $\frac{t}{2} \le x \le 3$ which implies $y = \frac{t}{x} \le 2$, we have $\mu_a(y) = \mu_a(t/x) = \frac{t}{2x}$. From previous discussion, we have

$$\max_{2/t\le x\le 3} \min\{x-2, \mu_a(t/x)\} = \frac{2+\sqrt{4+2t}}{2} - 2 = \frac{\sqrt{4+2t}}{2} - 1.$$

When $3 \le x \le 5$, we have $y = \frac{t}{x} \le 2$. From previous discussion

$$\max_{3\le x\le 5} \min\{(5-x)/2, \mu_a(t/x)\} = \frac{5 - \frac{10+\sqrt{100-16t}}{4}}{2} = \frac{10-\sqrt{100-16t}}{8}.$$

We already know that, for $0 \le t \le 6$, we have $\frac{10-\sqrt{100-16t}}{8} \le \frac{\sqrt{4+2t}}{2} - 1$. Moreover, $\frac{t}{2} - 2 \le \frac{10-\sqrt{100-16t}}{8}$ when $0 \le t \le 6$. Thus,

$$\mu_{ca}(t) = \frac{\sqrt{4+2t}}{2} - 1, \qquad 4 \le t \le 6.$$

Case 3: $6 \le t \le 10$.

When $2 \le x \le 3$, we know $y = \frac{t}{x} \ge 2$. So we have $\mu_a(y) = \mu_a(t/x) = 3 - \frac{t}{x}$. It can be shown that $3 - \frac{t}{x} \le x - 2$ whenever $t \ge 6$. Thus,

$$\max_{2\le x\le 3} \min\{x-2, \mu_a(t/x)\} = \max_{2\le x\le 3}\left\{3 - \frac{t}{x}\right\} = 3 - \frac{t}{3}.$$

When $3 \le x \le \frac{t}{2}$ which implies $y = \frac{t}{x} \ge 2$, we have $\mu_a(y) = \mu_a(t/x) = 3 - \frac{t}{x}$. From the equation $3 - \frac{t}{x} = \frac{5-x}{2}$ we have $x^2 + x - 2t = 0$, and

$$x = \frac{-1 \pm \sqrt{1+8t}}{2}.$$

Since $x \ge 3$, only + fits. Thus,

$$\max_{3\le x\le t/2} \min\left\{\frac{5-x}{2}, \mu_a(t/x)\right\} = \frac{1}{2}\left(5 - \frac{-1+\sqrt{1+8t}}{2}\right) = \frac{11-\sqrt{1+8t}}{4}.$$

When $\frac{t}{2} \le x \le 5$ which implies $y = \frac{t}{x} \le 2$, we have $\mu_a(y) = \mu_a(t/x) = \frac{t}{2x}$. It can be shown that $\frac{t}{2x} \ge \frac{5-x}{2}$ whenever $t \ge 6$. Thus,

$$\max_{\frac{t}{2}\le x\le 5} \min\left\{\frac{5-x}{2}, \mu_a(t/x)\right\} = \max_{\frac{t}{2}\le x\le 5}\left\{\frac{5-x}{2}\right\} = \frac{10-t}{4}.$$

It can be shown that

$$3 - \frac{t}{3} \le \frac{10-t}{4} \le \frac{11-\sqrt{1+8t}}{4}, \qquad t \ge 6.$$

Thus,

$$\mu_{ca}(t) = \frac{11-\sqrt{1+8t}}{4}, \qquad 6 \le t \le 10.$$

Case 4: $10 \le t \le 16$.

Since $x \le 5$, we know that $\frac{t}{x} \ge \frac{t}{5} \ge \frac{10}{5} = 2$. Thus, $\mu_a(t/x) = 3 - \frac{t}{x}$ or 0 if $t/x \ge 3$.

When $2 \le x \le 3$, we know that $\frac{t}{x} \ge \frac{t}{3} \ge \frac{10}{3} > 3$. Thus, $\mu_a(t/x) = 0$, and

$$\max_{2\le x\le 3} \min\left\{x-2, \mu_a(t/x)\right\} = 0.$$

When $3 \le x \le 5$, from previous discussion we know

$$\max_{3\le x\le 5} \min\left\{\frac{5-x}{2}, \mu_a(t/x)\right\} = \frac{11-\sqrt{1+8t}}{4}.$$

Thus,

$$\mu_{ca}(t) = \frac{11-\sqrt{1+8t}}{4}, \qquad 10 \le t \le 15.$$

Putting together, as shown in figure 1, we have

$$\mu_{ca}(t) = \begin{cases} \dfrac{\sqrt{4+2t}}{2} - 1, & 0 \le t \le 6; \\ \dfrac{11-\sqrt{1+8t}}{4}, & 6 \le t \le 15; \\ 0, & \text{otherwise.} \end{cases}$$

To compute $ca - c$, we know that $\mu_{ca-c}(t) = \max_{x-y=t} \min\{\mu_{ca}(x), \mu_c(y)\}$. Therefore, $\mu_{ca-c}(t) = 0$ when $t \geq 13$ or $t \leq -5$.

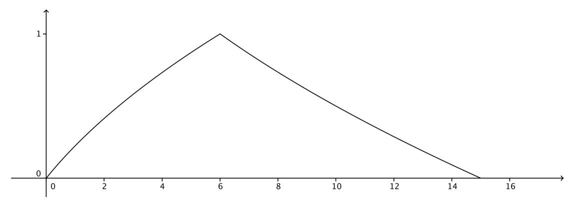


Figure 1: ca

Case 1: $-5 \leq t \leq 1$.

If $0 \leq x \leq 2+t$, we know $y = x-t \geq 2$ which implies $\mu_c(y) = 0$.

If $2+t \leq x \leq 3+t \leq 4$, we know $2 \leq y = x-t \leq 3$ which implies $\mu_c(y) = \mu_c(x-t) = (x-t)-2$. Also, $x \geq 0$ implies $t \geq -2$. Thus,

$$\max_{2+t \leq x \leq 3+t} \min\{\mu_{ca}(x), \mu_c(y)\} = \max_{\substack{2+t \leq x \leq 3+t \\ t \geq -2}} \min\{\mu_{ca}(x), \mu_c y\}$$

$$= \max_{\substack{2+t \leq x \leq 3+t \\ t \geq -2}} \min\left\{\frac{\sqrt{4+2x}}{2} - 1, x-t-2\right\}.$$

We know $x - t \geq 2$, it can be shown that

$$\frac{\sqrt{4+2x}}{2} - 1 \leq x - t - 2$$

Thus,

$$\max_{2+t \leq x \leq 3+t} \min\{\mu_{ca}(x), \mu_c(y)\} = \max_{\substack{2+t \leq x \leq 3+t \\ t \geq -2}} \left\{\frac{\sqrt{4+2x}}{2} - 1\right\}$$

$$= \frac{\sqrt{4+2(3+t)}}{2} - 1 = \frac{\sqrt{10+2t}}{2} - 1, \quad t \geq -2.$$

If $3+t \leq x \leq 5+t \leq 6$, we have $3 \leq x-t \leq 5$ which implies $\mu_c(y) = \mu_c(x-t) = \frac{1}{2}(5-(x-t))$. Thus,

$$\max_{3+t \leq x \leq 5+t} \min\{\mu_{ca}(x), \mu_c(y)\} = \max_{3+t \leq x \leq 5+t} \min\left\{\frac{\sqrt{4+2x}}{2} - 1, \frac{5-(x-t)}{2}\right\}.$$

From the equation $\frac{\sqrt{4+2x}}{2} - 1 = \frac{5-(x-t)}{2}$, we have

$$x = (8+t) \pm \sqrt{19+2t}.$$

Since $x \leq 2$, only $-$ fits. Thus,

$$\max_{3+t \leq x \leq 5+t} \min\{\mu_{ca}(x), \mu_c(y)\} = \frac{5+t-((8+t)-\sqrt{19+2t})}{2} = \frac{\sqrt{19+2t}-3}{2}.$$

If $x \geq 5+t$, we know $y = x-t \geq 5$ which implies $\mu_c(y) = 0$. Also, it can shown that, when $-5 \leq t \leq 1$,

$$\frac{\sqrt{10+2t}}{2} - 1 \leq \frac{\sqrt{19+2t}-3}{2}.$$

Thus,

$$\mu_{ca-c}(t) = \frac{\sqrt{19+2t}-3}{2}, \qquad -5 \leq t \leq 1.$$

Case 2: $1 \le t \le 3$.

If $0 \le x \le 2+t$, we know $y = x - t \ge 2$ which implies $\mu_c(y) = 0$.

If $2+t \le x \le 3+t \le 6$, we know $2 \le y = x - t \le 3$ which implies $\mu_c(y) = \mu_c(x-t) = (x-t) - 2$. From previous discussion,

$$\max_{2+t \le x \le 3+t} \min\{\mu_{ca}(x), \mu_c(y)\} = \frac{\sqrt{10+2t}}{2} - 1.$$

If $3+t \le x \le 6$, we have $3 \le x-t \le 5$ which implies $\mu_c(y) = \mu_c(x-t) = \frac{1}{2}(5-(x-t))$. From previous discussion,

$$\max_{3+t \le x \le 6} \min\{\mu_{ca}(x), \mu_c(y)\} = \frac{\sqrt{19+2t}-3}{2}.$$

If $6 \le x \le 5+t$, we have $3 \le x-t \le 5$ which implies $\mu_{ca}(x) = \frac{11-\sqrt{1+8x}}{4}$ and $\mu_c(y) = \mu_c(x-t) = \frac{1}{2}(5-(x-t))$. Thus,

$$\max_{6 \le x \le 5+t} \min\{\mu_{ca}(x), \mu_c(y)\} = \max_{6 \le x \le 5+t} \min\left\{\frac{11-\sqrt{1+8x}}{4}, \frac{5-(x-t)}{2}\right\}.$$

It can shown that, when $1 \le t \le 3$,

$$\frac{11-\sqrt{1+8x}}{4} \ge \frac{1}{2}(5-(x-t)).$$

Thus,

$$\max_{6 \le x \le 5+t} \min\{\mu_{ca}(x), \mu_c(y)\} = \max_{6 \le x \le 5+t}\left\{\frac{1}{2}(5-(x-t))\right\} = \frac{t-1}{2}.$$

If $x \ge 5+t$, we know $y = x-t \ge 5$ which implies $\mu_c(y) = 0$. Also, it can shown that, when $1 \le t \le 3$,

$$\frac{t-1}{2} \le \frac{\sqrt{10+2t}}{2} - 1 \le \frac{\sqrt{19+2t}-3}{2}.$$

Thus,

$$\mu_{ca-c}(t) = \frac{\sqrt{19+2t}-3}{2}, \qquad 1 \le t \le 3.$$

Case 3: $3 \le t \le 10$.

If $0 \le x \le 2+t$, we know $y = x - t \ge 2$ which implies $\mu_c(y) = 0$.

If $2+t \le x \le 6 \le 3+t$, we know $t \le 4$ and $2 \le y = x-t \le 3$ which implies $\mu_c(y) = \mu_c(x-t) = (x-t)-2$. We have

$$\max_{2+t \le x \le 6} \min\{\mu_{ca}(x), \mu_c(y)\} = \max_{\substack{2+t \le x \le 3+t \\ t \le 4}} \min\left\{\frac{\sqrt{4+2x}}{2} - 1, x-t-2\right\}.$$

We know $x - t \ge 2$, it can be shown that

$$\frac{\sqrt{4+2x}}{2} - 1 \ge x-t-2$$

Thus,

$$\max_{2+t \le x \le 6} \min\{\mu_{ca}(x), \mu_c(y)\} = \max_{\substack{2+t \le x \le 3+t \\ t \le 4}} \{x-t-2\} = 4-t, \quad t \le 4.$$

If $6 \leq x \leq 3+t$, we have $6-t \leq x-t \leq 3$ which implies $\mu_{ca}(x) = \frac{11-\sqrt{1+8x}}{4}$ and $\mu_c(y) = \mu_c(x-t) = (x-t)-2$ when $6-t \geq 2$ which implies $t \leq 4$. From the equation $\frac{11-\sqrt{1+8x}}{4} = (x-t)-2$, we have

$$x = \frac{10+2t \pm \sqrt{10+2t}}{2}.$$

Since $6 \leq x \leq 3+t$, only $-$ fits. Thus,

$$\max_{6\leq x\leq 3+t} \min\{\mu_{ca}(x), \mu_c(y)\} = \max_{\substack{6\leq x\leq 3+t \\ 3\leq t\leq 4}} \min\{\mu_{ca}(x), \mu_c(y)\} =$$

$$\frac{10+2t-\sqrt{10+2t}}{2} - t - 2 = 3 - \frac{\sqrt{10+2t}}{2}.$$

If $3+t \leq x \leq 5+t$, we have $3 \leq x-t \leq 5$ which implies $\mu_{ca}(x) = \frac{11-\sqrt{1+8x}}{4}$ and $\mu_c(y) = \mu_c(x-t) = \frac{1}{2}(5-(x-t))$. Thus,

$$\max_{3+t\leq x\leq 5+t} \min\{\mu_{ca}(x), \mu_c(y)\} = \max_{3+t\leq x\leq 5+t} \min\left\{\frac{11-\sqrt{1+8x}}{4}, \frac{5-(x-t)}{2}\right\}.$$

From the equation $\frac{11-\sqrt{1+8x}}{4} = \frac{5-(x-t)}{2}$, we have

$$x = \frac{1+2t \pm \sqrt{1+8t}}{2}.$$

Since $6 \leq 3+t \leq x$, only $+$ fits. Thus,

$$\max_{3+t\leq x\leq 5+t} \min\{\mu_{ca}(x), \mu_c(y)\} = \frac{1}{2}\left(5+t-\frac{1+2t+\sqrt{1+8t}}{2}\right) = \frac{9-\sqrt{1+8t}}{4}.$$

If $x \geq 5+t$, we know $y = x-t \geq 5$ which implies $\mu_c(y) = 0$. Also, it can shown that, when $3 \leq t \leq 10$,

$$4 - t \leq \frac{9-\sqrt{1+8t}}{4} \leq 3 - \frac{\sqrt{10+2t}}{2}.$$

Thus,

$$\mu_{ca-c}(t) = 3 - \frac{\sqrt{10+2t}}{2}, \qquad 3 \leq t \leq 10.$$

Case 4: $10 \leq t \leq 13$.

If $x \leq 2+t$, we know $y = x-t \leq 2$ which implies $\mu_c(y) = 0$.

If $6 \leq 2+t \leq x \leq 3+t \leq 15$ or $x \leq 15 \leq 3+t$, we know $2 \leq y = x-t \leq 3$ which implies $\mu_c(y) = \mu_c(x-t) = (x-t)-2$. Thus,

$$\max_{2+t\leq x\leq 3+t} \min\{\mu_{ca}(x), \mu_c(y)\} = \max_{2+t\leq x\leq 3+t} \min\left\{\frac{11-\sqrt{1+8x}}{4}, x-t-2\right\}.$$

From previous discussion, we have,

$$\max_{2+t\leq x\leq 3+t} \min\{\mu_{ca}(x), \mu_c(y)\} = 3 - \frac{\sqrt{10+2t}}{2}.$$

If $3+t \leq x \leq 15$, we have $t \leq 12$ and $3 \leq x-t \leq 5$ which implies $\mu_c(y) = \mu_c(x-t) = (x-t)-2$. Thus,

$$\max_{3+t\leq x\leq 15} \min\{\mu_{ca}(x), \mu_c(y)\} = \max_{\substack{3+t\leq x\leq 5+t \\ t\leq 12}} \min\left\{\frac{11-\sqrt{1+8x}}{4}, \frac{5-(x-t)}{2}\right\}.$$

From previous discussion, we have,

$$\max_{3+t\leq x\leq 15} \min\{\mu_{ca}(x), \mu_c(y)\} = \frac{9-\sqrt{1+8t}}{4}, \quad t \leq 12.$$

We have known that, when $10 \leq t \leq 13$,

$$4-t \leq \frac{9-\sqrt{1+8t}}{4} \leq 3 - \frac{\sqrt{10+2t}}{2}.$$

Thus,

$$\mu_{ca-c}(t) = 3 - \frac{\sqrt{10+2t}}{2}, \qquad 10 \leq t \leq 13.$$

Putting together, as shown in the figure 3 below we have

$$\mu_{ca-c}(t) = \begin{cases} \dfrac{\sqrt{19+2t}-3}{2}, & -5 \leq t \leq 3; \\ 3 - \dfrac{\sqrt{10+2t}}{2}, & 3 \leq t \leq 13; \\ 0, & \text{otherwise.} \end{cases}$$

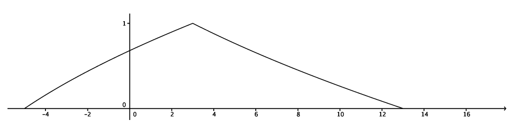
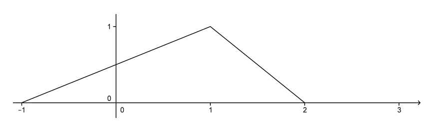


Figure 3. $ca-c$



Fuzzy number $a-1$

To compute $c(a-1)$, we first have $a-1$ as shown in the figure above, and

$$\mu_{a-1}(x) = \begin{cases} \frac{1}{2}(x+1), & -1 \leq x < 1 \\ -(x-2), & 1 \leq x < 2 \\ 0, & \text{Otherwise} \end{cases}$$

It is clear we have $\mu_{ca}(t) = 0$ whenever $t \leq -5$ or $t \geq 10$. For t otherwise, we have

$$\mu_{c(a-1)}(t) = \max_{xy=t} \min\{\mu_c(x), \mu_{a-1}(y)\} = \max_{x\neq 0} \min\{\mu_c(x), \mu_{a-1}(t/x)\}$$

$$= \max\{\max_{2\leq x\leq 3} \min\{x-2, \mu_{a-1}(t/x)\}, \max_{3\leq x\leq 5} \min\{-(x-5)/2, \mu_{a-1}(t/x),\}\}.$$

Case 1: $-5 \leq t \leq 2$.

When $2 \leq x \leq 3$, we know that $\frac{t}{x} \leq \frac{t}{2} \leq 1$. Thus, $\mu_a(t/x) = \frac{1}{2}(\frac{t}{x}+1)$. When $t \leq -2$, we know $\frac{t}{x} \leq -1$ and $\mu_{a-1}(t/x) = 0$. If $t \geq -2$, from the equation $\frac{1}{2}\left(\frac{t}{x}+1\right) = x-2$, we have $2x^2-5x-t=0$, and

$$x = \frac{5 \pm \sqrt{25+8t}}{4}.$$

Since $x \geq 2$, only $+$ fits. Thus,

$$\max_{2\leq x\leq 3} \min\{x-2, \mu_{a-1}(t/y)\} = \frac{5+\sqrt{25+8t}}{4} - 2 = \frac{\sqrt{25+8t}-3}{4}.$$

When $3 \leq x \leq 5$, we know $t \geq -3$, and from the equation $\frac{1}{2}\left(\frac{t}{x}+1\right) = \frac{5-x}{2}$ we have $x^2 - 4x + t = 0$, and

$$x = \frac{4 \pm \sqrt{16-4t}}{2} = 2 \pm \sqrt{4-t}.$$

Since $x \geq 3$, only $+$ fits. Thus,

$$\max_{3\leq x\leq 5} \min\{(5-x)/2, \mu_{a-1}(t/x)\} = \frac{5-(2+\sqrt{4-t})}{2} = \frac{3-\sqrt{4-t}}{2}.$$

It can be shown that, for $-3 \leq t \leq 0$, we have $\frac{\sqrt{25+8t}-3}{4} \leq \frac{3-\sqrt{4-t}}{2}$, and otherwise when $0 \leq t \leq 2$. Thus,

$$\mu_{c(a-1)}(t) = \begin{cases} \dfrac{3-\sqrt{4-t}}{2}, & -3 \leq t \leq 0; \\ \dfrac{\sqrt{25+8t}-3}{4}, & 0 \leq t \leq 2. \end{cases}$$

Case 2: $2 \leq t \leq 3$.

When $2 \leq x \leq t \leq 3$ which implies $y = \frac{t}{x} \geq 1$, we have $\mu_a(y) = \mu_{a-1}(t/x) = 2 - \frac{t}{x}$. It can be shown that $2 - \frac{t}{x} \geq x - 2$ whenever $t \geq 2$. Thus,

$$\max_{2\leq x\leq t} \min\{x-2, \mu_{a-1}(t/x)\} = \max_{2\leq x\leq t} \{x-2\} = t-2.$$

When $t \leq x \leq 3$ which implies $y = \frac{t}{x} \leq 1$, we have $\mu_{a-1}(y) = \mu_{a-1}(t/x) = \frac{1}{2}\left(\frac{t}{x}+1\right)$. From previous discussion, we have

$$\max_{t\leq x\leq 3} \min\{x-2, \mu_{a-1}(t/x)\} = \frac{\sqrt{25+8t}-3}{4}.$$

When $t \leq 3 \leq x \leq 5$ which implies $y = \frac{t}{x} \leq 1$, we have $\mu_{a-1}(y) = \mu_{a-1}(t/x) = \frac{1}{2}\left(\frac{t}{x}+1\right)$. From previous discussion, we have,

$$\max_{3\leq x\leq 5} \min\left\{\frac{5-x}{2}, \mu_{a-1}(t/x)\right\} = \frac{3-\sqrt{4-t}}{2},$$

and $\frac{\sqrt{25+8t}-3}{4} \geq \frac{3-\sqrt{4-t}}{2} \geq t-2$. Thus,

$$\mu_{c(a-1)}(t) = \frac{\sqrt{25+8t}-3}{4}, \qquad 2 \leq t \leq 3.$$

Case 3: $3 \leq t \leq 5$.

When $2 \leq x \leq 3 \leq t$, we have $y = \frac{t}{x} \geq 1$. From the equation $2 - \frac{t}{x} = x - 2$ we have $x^2 - 4x + t = 0$, and

$$x = \frac{4 \pm \sqrt{16-4t}}{2} = 2 \pm \sqrt{4-t}.$$

Since $x \geq 2$, only $+$ fits. Thus,

$$\max_{2\leq x\leq 3} \min\{x-2, \mu_a(t/x)\} = 2+\sqrt{4-t}-2 = \sqrt{4-t}.$$

When $3 \leq x \leq t \leq 5$, we have $y = \frac{t}{x} \geq 1$. From the equation $2 - \frac{t}{x} = \frac{1}{2}(5-x)$ we have $x^2 - x - 2t = 0$, and

$$x = \frac{1 \pm \sqrt{1+8t}}{2}.$$

Since $x \geq 3$, only $+$ fits. Thus,

$$\max_{3\leq x\leq t} \min\left\{\frac{5-x}{2}, \mu_{a-1}(t/x)\right\} = \frac{5-\frac{1+\sqrt{1+8t}}{2}}{2} = \frac{9-\sqrt{1+8t}}{4}.$$

When $t \leq x \leq 5$, we have $y = \frac{t}{x} \leq 1$, we know $\frac{5-x}{2} \leq (\frac{t}{x}+1)$, thus

$$\max_{t\leq x\leq 5} \min\left\{\frac{5-x}{2}, \mu_{a-1}(t/x)\right\} = \max_{t\leq x\leq 5}\left\{\frac{5-x}{2}\right\} = \frac{5-t}{2},$$

and $\frac{9-\sqrt{1+8t}}{8} \geq \frac{5-t}{2} \geq \sqrt{4-t}$. Thus,

$$\mu_{c(a-1)}(t) = \frac{9-\sqrt{1+8t}}{8} \qquad 3 \leq t \leq 5.$$

Case 4: $5 \leq t \leq 10$.

When $2 \leq x \leq 3 \leq t$, we have $y = \frac{t}{x} \geq 1$. From previous discussion,

$$\max_{2\leq x\leq 3} \min\{x-2, \mu_a(t/x)\} = 2+\sqrt{4-t}-2 = \sqrt{4-t}.$$

When $3 \leq x \leq 5 \leq t$, we have $y = \frac{t}{x} \geq 1$. From previous discussion,

$$\max_{3\leq x\leq 5} \min\left\{\frac{5-x}{2}, \mu_{a-1}(t/x)\right\} = \frac{5-\frac{1+\sqrt{1+8t}}{2}}{2} = \frac{9-\sqrt{1+8t}}{4}.$$

and $\frac{9-\sqrt{1+8t}}{4} \geq \sqrt{4-t}$. Thus,

$$\mu_{c(a-1)}(t) = \frac{9-\sqrt{1+8t}}{4} \qquad 5 \leq t \leq 10.$$

Putting together, as shown in the figure 2 below we have

$$\mu_{c(a-1)}(t) = \begin{cases} \dfrac{3-\sqrt{4-t}}{2}, & -5 \leq t \leq 0; \\ \dfrac{\sqrt{25+8t}-3}{4}, & 0 \leq t \leq 3. \\ \dfrac{9-\sqrt{1+8t}}{4}, & 3 \leq t \leq 10; \\ 0, & \text{otherwise.} \end{cases}$$

Compare Figures 3 and 2 we can see clearly that $ca - c \neq c(a-1)$. They even have different supports, although we have proved earlier in the paper they are equivalent up to a symmetric fuzzy number.

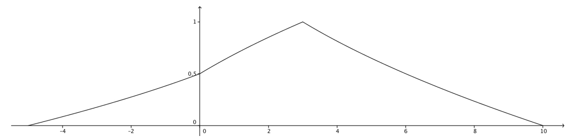


Figure 2: $c(a-1)$

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