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A HYBRID DIFFERENCE SCHEME FOR SINGULARLY PERTURBED SEMILINEAR REACTION-DIFFUSION PROBLEMS

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ABSTRACT. We consider a singularly perturbed semilinear reaction-diffusion problem. Its diffusion parameter ε is arbitrarily small, which induces boundary layers. To approximate the solution of this problem we propose a hybrid difference scheme on a generalized Shishkin mesh. We prove that the numerical approximations obtained from this method are almost fourth order uniformly convergent (in the maximum norm) with respect to the perturbation parameter. Numerical experiments are given that illustrate the theoretical order of convergence established for the numerical method.

Key Words: singular perturbation; semilinear reaction-diffusion; hybrid scheme; uniform convergence; generalized Shishkin mesh

AMS (MOS) Subject Classification: 65L10, 65L12

1. Introduction

Consider the following singularly perturbed semilinear reaction-diffusion problem

(1a) $$\overline{T}u := -\varepsilon u'' + f(x,u) = 0, \quad x \in \Omega = (0,1),$$

(1b) $$u(0) = 0, \quad u(1) = 0,$$

where ε is a small parameter, such that $0 < \varepsilon \ll 1$, and f is a sufficiently smooth function satisfying

(1.2) $$f_u(x,y) \geq \alpha \text{ for all } (x,y) \in \Omega \times \mathbb{R}, \quad \alpha > 0.$$

Under this assumption, problem (1) and the reduced problem $f(x, u_0(x)) = 0$, for all $x \in \Omega$, defined by setting $\varepsilon = 0$ in $(1a)$ have unique solutions u and u_0 respectively. The solution u generally has exponential boundary layers at $x = 0$ and $x = 1$ of width $O(\sqrt{\varepsilon}\ln(1/\sqrt{\varepsilon}))$. More precisely, u can be decomposed into two parts: $u = v + w$, where

(1.3) $$|v^{(s)}(x)| \leq C \text{ and } |w^{(s)}(x)| \leq C\varepsilon^{-s/2}(e^{(-x\sqrt{\alpha/\varepsilon})} + e^{(-(1-x)\sqrt{\alpha/\varepsilon})})$$

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for $x \in \overline{\Omega}$ and $s = 0, \ldots, 6$ [14]. The use of classical numerical methods on uniform meshes for solving such problems give rise to difficulties when perturbation parameter becomes sufficiently small. Then the mesh needs to be refined substantially to grasp the solution within the boundary layers. Properly layer-adapted meshes have been proven to overcome these difficulties and to yield methods that converge uniformly with respect to the perturbation parameter ε ([11, 3, 5, 6]). Among these meshes, Shishkin type meshes gained more popularity because of their simplicity and applicability to problems in higher dimensions.

Problem (1) were solved asymptotically in [2, 7] and numerically in [14, 9, 10, 4, 11, 17, 15, 12], and the references therein. Vulanović [14] considered the central difference scheme and achieved second order uniform convergence result on a special graded mesh of Bakhvalov type. Rao et al. [10] developed a cubic B-spline collocation method on a piecewise uniform Shishkin mesh and proved that the method is almost second order uniformly convergent. Rao and Kumar [9] derived an exponential spline difference scheme on the basis of spline in tension on a piecewise-uniform Shishkin mesh and proved an almost second order uniform convergence of the scheme. Higher order uniformly convergent schemes are always attractive, because they provide very accurate numerical approximations with a low computational cost. Herceg [4] considered a non-equidistant generalization of the fourth order three point finite-difference scheme, known as the Hermite or Numerov scheme, on a graded mesh of Bakhvalov type and achieved fourth order uniform convergence result (in the discrete maximum norm) under extra somewhat restrictive assumptions on the nonlinear term $f(x, u)$. It was these unpleasant assumptions on $f(x, u)$ that motivated the work in Vulanović and Herceg [17], Vulanović [15], and Sun and Stynes [12], which resulted in eliminating these assumptions on $f(x, u)$, usually at a certain price. Vulanović and Herceg [17] used the method which was essentially the same as in Herceg [4], but the numerical error was estimated differently, viz. using an L_1 discrete norm instead of the usual maximum norm. Vulanović [15] considered a more complicated Bakhvalov mesh. Sun and Stynes [12] considered the Hermite scheme on a piecewise-uniform Shishkin mesh and proved that the scheme is almost fourth order uniformly convergent (in the discrete maximum norm) when $\varepsilon \leq N^{-1}$.

In this article we propose a hybrid difference scheme, combining the Hermite scheme and the central difference scheme in a special way, on a generalized Shishkin mesh. The proposed numerical method preserves inverse monotonicity of the continuous problem. We prove that the numerical approximations obtained from this method are almost fourth order uniformly convergent (in the maximum norm). Numerical results are given to illustrate the efficiency of the proposed method.

Notation: Throughout the article we use C to denote a generic positive constant independent of ε and the discretization parameter. Let $\overline{\Omega}^N$ denote any mesh with points $0 = x_0 < x_1 < \cdots < x_N = 1$. For any function $g \in C(\overline{\Omega})$, define $g_j = g(x_j)$. We consider the maximum norm and denote it by $\|.\|_D$, where D is a closed and bounded subset of $\overline{\Omega}$. For a real valued function $v \in C(D)$, define $\|v\|_D = \max_{x\in D} |v(x)|$. Let z^N is a mesh function. Mesh functions will be identified with $\mathbb{R}^{N+1}$ column vectors $z^N = (z_0^N, \ldots, z_N^N)^T$. We use the maximum vector norm, $\|z^N\| = \max_{0\le i\le N} |z_i^N|$, and denote the corresponding subordinate matrix norm in the same way.

2. Discretization

2.1. **The Mesh.** We first construct a generalized Shishkin mesh using a suitable mesh generating function ξ as described in [16]. Let $\overline{\Omega}^N = \{x_j\}_{j=0}^N$ be the partitioning of $\overline{\Omega}$ with mesh spacing $h_j = x_j - x_{j-1}$, $j = 1, \ldots, N$. For simplicity, we assume that $N \ge 4$ is an even integer and that $x_{N-j} = 1 - x_j$, $j = 0, 1, \ldots, N$. It therefore suffices to describe the mesh on the interval $[0, 1/2]$. Let $L = L(N)$ satisfying $\ln(\ln N) \le L \le \ln N$ and

$$e^{-L} \le \frac{L}{N}. \tag{2.1}$$

Define

$$\sigma = \min\left\{ \frac{1}{4},\ \sigma_0\sqrt{\varepsilon}L \right\},$$

where σ_0 is a positive constant which we define later. The standard Shishkin mesh uses $L = \ln N$, and is constructed by forming a fine equidistant mesh with $N/4$ mesh steps on the interval $[0, \sigma]$, and a coarse equidistant mesh with $N/4$ steps on $[\sigma, 1/2]$. Next we define a generalized Shishkin mesh $S(L)$ that changes smoothly in the transition point $x_{N/4} = \sigma$ from the fine part to the coarse part. Let $S(L)$ be the mesh defined by $x_j = \xi(j/N)$, $j = 0, 1, \ldots, N/2$, where $\xi \in C^2[0, 1/2]$ is a mesh generating function

$$\xi(t) = \begin{cases} 4\sigma t & \text{for } t \in [0, 1/4]; \\ p(t-1/4)^3 + 4\sigma(t-1/4) + \sigma & \text{for } t \in [1/4, 1/2]. \end{cases} \tag{2.2}$$

The coefficient p is determined from $\xi(1/2) = 1/2$. Let $h_* = \max_j h_j$. For the generalized Shishkin mesh $S(L)$, the maximum mesh width h_* always correspond to the $N/2$ mesh width and $N/2+1$ mesh width, that is, $h_* = h_{N/2} = h_{N/2+1} \le CN^{-1}$.

2.2. **The Scheme.** On $S(L)$ we introduce a hybrid finite difference scheme in the form

$$\begin{aligned} &r_j^- u_{j-1}^N + r_j^c u_j^N + r_j^+ u_{j+1}^N + q_j^- f(x_{j-1}, u_{j-1}^N) + q_j^c f(x_j, u_j^N) \\ &\quad + q_j^+ f(x_{j+1}, u_{j+1}^N) = 0, \quad j = 1, \ldots, N-1,\ u_0^N = 0,\ u_N^N = 0. \end{aligned} \tag{2.3}$$

The coefficients $r_j^\star, j = 1, \ldots, N-1, \star = -, c, +$, are given by

$$r_j^- = \frac{-2\varepsilon}{h_j(h_j + h_{j+1})}, \quad r_j^c = \frac{2\varepsilon}{h_j h_{j+1}}, \quad r_j^+ = \frac{-2\varepsilon}{h_{j+1}(h_j + h_{j+1})} \tag{2.4}$$

The values of the coefficients $q_j^\star$, $j = 1, \ldots, N-1$, $\star = -, c, +$, depend on the location of the mesh points, and also on the relation between step sizes of the mesh and the perturbation parameter. For the mesh points located in $(0, \sigma) \cup (1-\sigma, 1)$, the coefficients $q_j^\star$, $\star = -, c, +$, are given by

$$q_j^- = \frac{h_j^2 - h_{j+1}^2 + h_j h_{j+1}}{6h_j(h_j + h_{j+1})}, \quad q_j^c = \frac{h_j^2 + h_{j+1}^2 + 3h_j h_{j+1}}{6h_j h_{j+1}}, \quad q_j^+ = \frac{h_{j+1}^2 - h_j^2 + h_j h_{j+1}}{6h_{j+1}(h_j + h_{j+1})}. \tag{2.5}$$

For the mesh points located in $[\sigma, 1-\sigma]$, depending on the relation between h_* and ε, the coefficients $q_j^\star$, $\star = -, c, +$, are defined in two different cases. Let $f_u(x, y) \leq \beta$, for all $(x, y) \in \overline{\Omega} \times \mathbb{R}$. If $\tau h_*^2 \beta \leq \varepsilon$, the coefficients $q_j^\star$, $j = N/4, \ldots, 3N/4$, $\star = -, c, +$, are defined again by (2.5). While for the case, when $\tau h_*^2 \beta > \varepsilon$, the coefficients $q_j^\star$, $j = N/4, \ldots, 3N/4$, $\star = -, c, +$, are given by

$$q_j^- = 0, \quad q_j^c = 1, \quad q_j^+ = 0. \tag{2.6}$$

The above definition of coefficients q_j's and r_j's show that the fourth order Hermite scheme [4] is considered within the boundary layer region $(0, \sigma) \cup (1-\sigma, 1)$. While in the regular region $[\sigma, 1-\sigma]$, the fourth order Hermite scheme is considered when $\tau h_*^2 \beta \leq \varepsilon$ and the central difference scheme is considered when $\tau h_*^2 \beta > \varepsilon$.

The Scheme (2.3) can be written in the form

$$Tu^N = 0, \tag{2.7}$$

where $T = A + B$, A is a $(N+1) \times (N+1)$ tridiagonal matrix defined by

$$A = \begin{pmatrix} 1 & 0 & 0 & & & & \\ r_1^- & r_1^c & r_1^+ & & & & \\ & \cdot & \cdot & \cdot & & & \\ & & \cdot & \cdot & \cdot & & \\ & & & & \cdot & \cdot & \cdot \\ & & & & r_{N-1}^- & r_{N-1}^c & r_{N-1}^+ \\ & & & & 0 & 0 & 1 \end{pmatrix},$$

and $B : \mathbb{R}^{N+1} \to \mathbb{R}^{N+1}$ is the mapping

$$(Bz^N)_j = \begin{cases} 0, & \text{for } j = 0; \\ q_j^- f(x_{j-1}, z_{j-1}^N) + q_j^c f(x_j, z_j^N) + q_j^+ f(x_{j+1}, z_{j+1}^N), & \text{for } j = 1, \ldots, N-1; \\ 0, & \text{for } j = N. \end{cases}$$

2.3. **Existence and Uniqueness.** The Frechet-derivative T' of T at any arbitrary vector z^N is a $(N+1)\times(N+1)$ tridiagonal matrix and is given by

$$T'(z^N) = \begin{pmatrix} 1 & 0 & 0 & & & & \\ \eta_1^- & \eta_1^c & \eta_1^+ & & & & \\ & \cdot & \cdot & \cdot & & & \\ & & \cdot & \cdot & \cdot & & \\ & & & \cdot & \cdot & \cdot & \\ & & & & \eta_{N-1}^- & \eta_{N-1}^c & \eta_{N-1}^+ \\ & & & & 0 & 0 & 1 \end{pmatrix},$$

where for $j = 1, 2, \ldots, N-1$,

$$\eta_j^- = r_j^- + q_j^- f_u(x_{j-1}, z_{j-1}^N), \tag{2.8}$$

$$\eta_j^c = r_j^c + q_j^c f_u(x_j, z_j^N), \tag{2.9}$$

$$\eta_j^+ = r_j^+ + q_j^+ f_u(x_{j+1}, z_{j+1}^N). \tag{2.10}$$

For what follows, we assume that $N \geq N_0$, where N_0 is sufficiently large such that

$$4\sigma_0^2\beta/3 < N_0^2/\ln^2 N_0.$$

Let $\tau = 1/6$. The choice of the coefficients r_j's and q_j's together with the definition mesh step sizes prove that T' is an M-matrix with

$$\eta_j^c - |\eta_j^-| - |\eta_j^+| \geq \alpha > 0 \quad \text{for } j = 1, \ldots, N-1.$$

By Theorem A of [13], we get

$$\|T'(z^N)^{-1}\| \leq \frac{1}{\min\{1, \alpha\}}. \tag{2.11}$$

Then (2.7) has a solution by Hadamard's theorem ([8]). An immediate consequence of (2.11) is the following uniform stability result.

Lemma 2.1. *Let y^N and z^N be any two mesh functions such that $y_0^N = z_0^N$ and $y_N^N = z_N^N$. Then*

$$\|y^N - z^N\| \leq \frac{1}{\min\{1, \alpha\}} \|Ty^N - Tz^N\|.$$

The above lemma implies that the solution of (2.7) is unique.

3. **Error Analysis**

We first estimate the truncation error of T on $S(L)$. For what follows, let us assume that $\sigma = \sigma_0\sqrt{\varepsilon}L$, as otherwise N^{-1} is exponentially small compared with $\sqrt{\varepsilon}$. We have

$$(Tu)_j = \Lambda u(x_j), \quad j = 1, \dots, N-1,$$

where

$$\Lambda u(x_j) = r_j^- u(x_{j-1}) + r_j^c u(x_j) + r_j^+ u(x_{j+1}) + \varepsilon(q_j^- u''(x_{j-1}) + q_j^c u''(x_j) + q_j^+ u''(x_{j+1})).$$

Consider $x_j \in (0,\sigma) \cup (1-\sigma, 1)$. Note that $h_j = h_{j+1} \le C\sqrt{\varepsilon}N^{-1}L$. By Taylor expansions we have

$$|\,\Lambda u(x_j)\,| \le C\varepsilon h_j^4 \|u^{(6)}\|_{[x_{j-1},x_{j+1}]}.$$

Using $\|u^{(6)}\|_{\overline{\Omega}} \le C\varepsilon^{-3}$, we get

$$|\Lambda u(x_j)| \le C(L/N)^4 \quad \text{for } x_j \in (0,\sigma) \cup (1-\sigma,1).$$

Next consider $x_j \in [\sigma, 1-\sigma]$. We now consider two distinct cases: $\tau h_*^2 \beta \le \varepsilon$ and $\tau h_*^2 \beta > \varepsilon$. In the first case the Hermite scheme is used. For any $y \in C^6(\overline{\Omega})$, Taylor expansions give

$$|\Lambda y(x_j)| \le C\varepsilon(P_j + R_j), \tag{3.1}$$

where

$$P_j = |h_{j+1} - h_j|(h_{j+1} + h_j)^2 |y^{(5)}(x_j)|, \quad R_j = (h_j^4 + h_{j+1}^4)\|y^{(6)}\|_{[x_{j-1},x_{j+1}]}.$$

For generalized Shishkin mesh $S(L)$, the mesh width h_j, for $j = N/4, \dots, 3N/4$, satisfies the following

(i) For some $\vartheta_j \in (j/N, j+1/N)$,

$$h_{j+1} = \xi((j+1)/N) - \xi(j/N) \le N^{-1} \max_{(j/N,(j+1)/N)} \xi'(\vartheta_j) \le CN^{-1}. \tag{3.2}$$

(ii) For some $\varphi_j \in ((j-1)/N, (j+1)/N)$,

$$\begin{aligned} |h_{j+1} - h_j| &= |\xi((j+1)/N) - 2\xi(j/N) + \xi((j-1)/N)| \\ &\le N^{-2} \max_{((j-1)/N,(j+1)/N)} \xi''(\varphi_j) \le CN^{-2}. \end{aligned} \tag{3.3}$$

These properties are important in estimating the term P_j. Using the decomposition $u = v + w$, we get

$$|\Lambda u(x_j)| \le |\Lambda v(x_j)| + |\Lambda w(x_j)|. \tag{3.4}$$

The first term is bounded using (3.1)–(3.3), and (1.3). We get $|\Lambda v(x_j)| \le CN^{-4}$. To bound the second term we use (3.1)–(3.3), (1.3) and $\tau h_*^2 \beta \le \varepsilon$ to get

$$|\Lambda w(x_j)| \le C\|e^{-x\sqrt{\alpha/\varepsilon}} + e^{-(1-x)\sqrt{\alpha/\varepsilon}}\|_{[x_{j-1},x_{j+1}]}.$$

For $x_j \in [\sigma, 1-\sigma]$,

$$\|e^{-x\sqrt{\alpha/\varepsilon}} + e^{-(1-x)\sqrt{\alpha/\varepsilon}}\|_{[x_{j-1},x_{j+1}]} \le e^{(-x_{N/4-1}\sqrt{\alpha/\varepsilon})} + e^{(-(1-x_{3N/4+1})\sqrt{\alpha/\varepsilon})}$$
$$= 2e^{(-x_{N/4-1}\sqrt{\alpha/\varepsilon})} = 2e^{(-\sigma\sqrt{\alpha/\varepsilon})}e^{(h_{N/4}\sqrt{\alpha/\varepsilon})} \le Ce^{-\sigma_0\sqrt{\alpha}L}.$$

Using that it holds $e^{-L} \le L/N$ and taking σ_0 such that $\sigma_0\sqrt{\alpha} \ge 4$, it follows that

$$\|e^{-x\sqrt{\alpha/\varepsilon}} + e^{-(1-x)\sqrt{\alpha/\varepsilon}}\|_{[x_{j-1},x_{j+1}]} \le C(L/N)^4.$$

Hence

$$|\Lambda u(x_j)| \le C(L/N)^4 \text{ for } x_j \in [\sigma, 1-\sigma], \tau h_*^2\beta \le \varepsilon.$$

Now consider the case $\tau h_*^2\beta > \varepsilon$. Using the decomposition $u = v + w$, we get

$$|\Lambda u(x_j)| \le |\Lambda v(x_j)| + |\Lambda w(x_j)|. \tag{3.5}$$

For the first term, Taylor expansions give

$$|\Lambda v(x_j)| \le C\varepsilon(P_j + R_j), \tag{3.6}$$

where

$$P_j = |h_{j+1} - h_j|\,|v^{(3)}(x_j)|, \quad R_j = h_{j+1}^2\|v^{(4)}\|_{[x_{j-1},x_{j+1}]}.$$

Using (3.2), (3.3) and (1.3) we get $|\Lambda v(x_j)| \le C\varepsilon N^{-2} \le CN^{-4}$. For the second term, by Taylor expansions and (1.3) we obtain

$$|\Lambda w(x_j)| \le C\varepsilon\|w''\|_{[x_{j-1},x_{j+1}]} \le C(L/N)^4.$$

Hence

$$|\Lambda u(x_j)| \le C(L/N)^4 \text{ for } x_j \in [\sigma, 1-\sigma], \tau h_*^2\beta > \varepsilon.$$

Thus we have

$$\|Tu\| \le C(L/N)^4. \tag{3.7}$$

Then using Lemma 2.1, we get the following main result of this paper.

Theorem 3.1. *Let u be the solution of the problem* (1) *and u^N that of the hybrid difference scheme* (2.7) *on $S(L)$. Then*

$$\|u - u^N\| \le C(L/N)^4.$$

We now extend the nodal parameter-uniform error estimate obtained in Theorem 3.1 to the global parameter-uniform error estimate. For the purpose we define a cubic C^0-spline $\mathcal{P}u^N$ that approximates u on the whole domain, by clustering three adjacent and equidistant mesh intervals and fitting a cubic function through the numerical approximation on the four associated mesh points.

To define a cubic C^0-spline $\mathcal{P}u^N$, we modify the mesh slightly. First we construct a generalized Shishkin mesh with $N/3$ mesh intervals as described in Section 2.1. Then we subdivide each mesh interval into three subintervals of equal length. This gives us a modified generalized Shishkin mesh $\widetilde{S}(L)$. This modification is necessary because the stability constant of the operator $\mathcal{P}$ depends on the local mesh size ratio. With this modification the stability constant of the operator $\mathcal{P}$ is independent of ε and N. Note that the nodal parameter-uniform error estimate of Theorem 3.1 is true on $\widetilde{S}(L)$ also.

Theorem 3.2. *Let u be the solution of problem* (1) *and u^N that of the hybrid difference scheme* (2.7) *on $\widetilde{S}(L)$. Then*

$$\|u - \mathcal{P}u^N\|_{\overline{\Omega}} \le C(L/N)^4.$$

Proof. By a triangle inequality we get

$$\|u - \mathcal{P}u^N\|_{\overline{\Omega}} \le \|u - \mathcal{P}u\|_{\overline{\Omega}} + \|\mathcal{P}(u - u^N)\|_{\overline{\Omega}}. \tag{3.8}$$

For the second term we use the uniform stability of the operator $\mathcal{P}$ to get

$$\|\mathcal{P}(u - u^N)\|_{\overline{\Omega}} \le C\|u - u^N\| \le C(L/N)^4. \tag{3.9}$$

Suppose I denote the cluster of three adjacent and equidistant mesh intervals of length h_I. We have the following standard interpolation error estimates

$$\|g - \mathcal{P}g\|_I \le Ch_I^4\|g^{(4)}\|_I \text{ and } \|g - \mathcal{P}g\|_I \le C\|g\|_I \quad \text{for any } g \in C^4(I). \tag{3.10}$$

First consider the case when I lies in the layer regions. Using $h_I \le C\sqrt{\varepsilon}N^{-1}L$, $\|u^{(4)}\|_{\overline{\Omega}} \le C\varepsilon^{-2}$, and the first bound of (3.10) it follows that

$$\|u - \mathcal{P}u\|_I \le C(L/N)^4.$$

Next consider the case when I lies in the regular region. Using the decomposition of u we get

$$\|u - \mathcal{P}u\|_I \le \|v - \mathcal{P}v\|_I + \|w - \mathcal{P}w\|_I$$

For the regular part v, we use the first estimate of (3.10), $h_I \le CN^{-1}$ and (1.3), while for the layer part w, we use the second estimate of (3.10) and (1.3). Thus we get

$$\begin{aligned}\|u - \mathcal{P}u\|_I &\le CN^{-4} + C\|e^{-x\sqrt{\alpha/\varepsilon}} + e^{-(1-x)\sqrt{\alpha/\varepsilon}}\|_I \\ &\le CN^{-4} + C(L/N)^4 \\ &\le C(L/N)^4.\end{aligned}$$

Collecting the various bounds for the interpolation error we have

$$\|u - \mathcal{P}u\|_{\overline{\Omega}} \le C(L/N)^4. \tag{3.11}$$

Combining (3.8), (3.9) and (3.11) we get the desired result. □

4. **Numerical Results**

To demonstrate the efficiency of the proposed method we consider the following test problem

$$-\varepsilon u'' + \frac{u-1}{2-u} + g(x) = 0, \quad u(0) = u(1) = 0,$$

where $g(x)$ is chosen so that the exact solution is

$$u(x) = 1 - \frac{e^{-x/\sqrt{\varepsilon}} + e^{-(1-x)/\sqrt{\varepsilon}}}{1 + e^{-1/\sqrt{\varepsilon}}}.$$

A version of this problem, when $g \equiv 0$, represents one of the models for the Michaelis-Menten process in biology, Bohl [1], and its solution behaves similarly to u above.

To solve the nonlinear system of equations, the Newton's method is used with the initial guess $u^{N,0} = (0, u_0(x_1), \ldots, u_0(x_{N-1}), 0)^T$, where u_0 is the solution of the reduced problem. In all our computations 5 Newton iterations were sufficient to get discrete solutions within the tolerance of 10^{-15}. For different values of N and ε, we compute the maximum nodal errors, $E_\varepsilon^N = \|u - u^N\|$. From these values we compute uniform errors by $E^N = \max_{\forall \varepsilon} E_\varepsilon^N$. Define L_N, the value of L with N elements, that is, $L_N = L(N)$. Assuming the convergence of order $(L_N/N)^r$, we compute the uniform convergence rates r^N using

$$r^N = \frac{\ln(E^N) - \ln(E^{2N})}{\ln(\frac{2L_N}{L_{2N}})}.$$

By using $L < \ln N$ instead of $\ln N$ we are trying to bring the point x_1 closer to $x = 0$ and this provide the higher density of mesh points in the layers. The motivation for this is the fact that the better performance of the mesh can be governed by the high density of mesh points in the layers. The smallest value of L is chosen to be $L^* = L^*(N)$ which satisfies

$$e^{-L^*} = L^*/N.$$

Table 1 represents the maximum nodal errors E_ε^N of the hybrid difference scheme on $S(L^*)$. The last two rows in the table represent the uniform nodal errors E^N and the uniform convergence rates r^N.

The hybrid difference scheme (2.7) is also implemented on generalized Shishkin mesh $S(L)$ with $L = \ln N$ and on standard Shishkin mesh $\widehat{S}(L)$ with $L = \ln N$. The comparison of the hybrid difference scheme on these meshes is given in Table 2. From the last two rows in the table we observe that the numerical results are identical for $S(\ln N)$ and $\widehat{S}(\ln N)$. The reason for this is that these meshes are identical in layer regions. From Table 2 we observe that the hybrid difference scheme is more accurate on $S(L^*)$ when compared with $S(\ln N)$ and $\widehat{S}(\ln N)$.

To compute the maximum global errors we additionally take N divisible by three and construct the modified generalized Shishkin mesh $\widetilde{S}(L)$ as described in Section 3.

We define $\mathcal{P}u^N$ on macro intervals $[x_{3j}, x_{3(j+1)}]$, $j = 0, \ldots, N/3-1$. Let $\breve{\Omega}^N := \{\breve{x}_j\}_0^{2N}$ be the mesh that contains the mesh points of the original mesh and their midpoints, that is,

$$\breve{x}_{2j} = x_j, \quad j = 0, \ldots, N, \quad \breve{x}_{2j+1} = (x_j + x_{j+1})/2, \quad j = 0, \ldots, N-1.$$

For different values of N and ε, we compute the maximum global errors, $\widetilde{E}_\varepsilon^N = \max_{\breve{x}_j \in \breve{\Omega}^N} |u(\breve{x}_j) - (\mathcal{P}u^N)(\breve{x}_j)|$. Then uniform global errors are calculated by $\widetilde{E}^N = \max_{\forall \varepsilon} \widetilde{E}_\varepsilon^N$. We compute the uniform convergence rates $\widetilde{r}^N$ using the formula

$$\widetilde{r}^N = \frac{\ln(\widetilde{E}^N) - \ln(\widetilde{E}^{2N})}{\ln(\frac{2L_{N/3}}{L_{2N/3}})}.$$

Table 3 represents the maximum global errors $\widetilde{E}_\varepsilon^N$ of the hybrid difference scheme on $\widetilde{S}(L^*)$. The last two rows in the table represent the uniform global errors $\widetilde{E}^N$ and the uniform convergence rates $\widetilde{r}^N$. Clearly numerical results given in Tables 1 and 3 are in good agreement with our theoretical results.

5. Conclusions

We proposed a hybrid difference scheme, combining the Hermite scheme and the central difference scheme in a special way, on generalized Shishkin mesh $S(L)$ for solving singularly perturbed semilinear reaction diffusion problems. The proposed numerical method preserves inverse monotonicity of the continuous problem. It is observed that the fine parts of standard Shishkin mesh $\widehat{S}(L)$ and generalized Shishkin mesh $S(L)$ are identical, but the coarse part of $S(L)$ is a smooth continuation of the fine mesh and is no longer equidistant. Using this fact we established almost fourth order pointwise uniform convergence of the present scheme on $S(L)$. Furthermore, on a slightly modified generalized Shishkin mesh, we proved that the present scheme is almost fourth order global uniformly convergent. Numerical results illustrate the efficiency of the present method.

Table 1. Nodal errors E_ε^N, E^N and uniform convergence rates r^N of the hybrid difference scheme (2.7) on $S(L)$ with $L = L^*$.

$\varepsilon = 10^{-k}$	$N = 2^7$	2^8	2^9	2^{10}	2^{11}	2^{12}
k=1	6.20E-10	3.87E-11	2.42E-12	1.50E-13	1.27E-14	1.02E-15
2	3.57E-08	2.23E-09	1.39E-10	8.72E-12	5.45E-13	3.85E-14
3	3.54E-06	2.22E-07	1.39E-08	8.69E-10	5.43E-11	3.40E-12
4	3.34E-04	2.19E-05	1.39E-06	8.68E-08	5.43E-09	3.39E-10
5	5.84E-04	6.70E-05	6.98E-06	7.01E-07	6.69E-08	6.17E-09
6	5.84E-04	6.70E-05	6.98E-06	7.01E-07	6.69E-08	6.17E-09
7	5.84E-04	6.70E-05	6.98E-06	7.01E-07	6.69E-08	6.17E-09
8	5.84E-04	6.70E-05	6.98E-06	7.01E-07	6.69E-08	6.17E-09
E^N	5.84E-04	6.70E-05	6.98E-06	7.01E-07	6.69E-08	6.17E-09
r^N	3.94	4.00	3.98	4.00	4.00	

Table 2. Comparison of the errors E_ε^N of the hybrid difference scheme (2.7) on Shishkin mesh $\widehat{S}(L)$ and on generalized Shishkin mesh $S(L)$ for $\varepsilon = 10^{-6}$.

	$N = 2^7$	2^8	2^9	2^{10}	2^{11}	2^{12}
$S(L^*)$	5.84E-04	6.70E-05	6.98E-06	7.01E-07	6.69E-08	6.17E-09
$S(\ln N)$	1.92E-03	2.06E-04	2.18E-05	2.091E-06	1.92E-07	1.70E-08
$\widehat{S}(\ln N)$	1.92E-03	2.06E-04	2.18E-05	2.091E-06	1.92E-07	1.70E-08

Table 3. Global errors $\widetilde{E}_\varepsilon^N$, $\widetilde{E}^N$ and uniform convergence rates $\widetilde{r}^N$ of the hybrid difference scheme (2.7) on $\widetilde{S}(L)$ with $L = L^*$.

$\varepsilon = 10^{-k}$	$N = 3 \times 2^6$	3×2^7	3×2^8	3×2^9	3×2^{10}	3×2^{11}
k=1	2.82E-09	1.78E-10	1.12E-11	8.13E-13	5.36E-14	4.27E-15
2	2.68E-07	1.74E-08	1.10E-09	6.96E-11	3.90E-12	3.01E-13
3	2.31E-05	1.61E-06	1.06E-07	6.83E-09	4.33E-10	2.72E-11
4	1.34E-03	1.28E-04	9.45E-06	6.44E-07	4.20E-08	2.68E-09
5	1.34E-03	2.09E-04	2.72E-05	3.13E-06	3.30E-07	3.25E-08
6	1.34E-03	2.09E-04	2.72E-05	3.13E-06	3.30E-07	3.25E-08
7	1.34E-03	2.09E-04	2.72E-05	3.13E-06	3.30E-07	3.25E-08
8	1.34E-03	2.09E-04	2.72E-05	3.13E-06	3.30E-07	3.25E-08
$\widetilde{E}^N$	1.34E-03	2.09E-04	2.72E-05	3.13E-06	3.30E-07	3.25E-08
$\widetilde{r}^N$	3.50	3.70	3.83	3.90	3.95	

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