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NUMERICAL APPROACH VIA GENERALIZED MONOTONE METHOD FOR SYSTEM OF CAPUTO FRACTIONAL DIFFERENTIAL EQUATIONS

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ABSTRACT. The generalized monotone method to compute solutions for two system of Caputo fractional differential equations using coupled lower and upper solutions is very useful, since it does not require any additional assumption. In this work we provide theoretical as well as computational methodology to compute coupled lower and upper solutions of type I to any desired interval. Further the computation of coupled lower and upper solutions can be accelerated by Gauss-Seidel method. We have applied our theoretical results to population models and have obtained corresponding numerical results.

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1. INTRODUCTION

Nonlinear problems (nonlinear dynamic systems) occur naturally as mathematical models in many branches of science, engineering, finance, economics, etc. So far, in literature, most models are differential equations with integer derivative. However, the qualitative and quantitative study of fractional differential and integral equations has gained importance recently due to its applications. See[1, 3, 8, 6] for details. In solving nonlinear problems, monotone method combined with method of upper and lower solutions is a popular choice, because the existence of solution by monotone method is both theoretical and computational. In addition the interval of existence is guaranteed. Monotone method for various nonlinear problems has been developed in [4]. Monotone method(monotone iterative technique) combined with method of lower and upper solutions yields monotone sequences, which converges to minimal and maximal solutions of nonlinear differential equation.

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In many nonlinear problems (nonlinear dynamic systems), the nonlinear term is the sum of an increasing and decreasing functions. Monotone method extended to such systems is called generalized monotone method. Generalized monotone method for first order nonlinear initial value problems and for fractional order nonlinear initial value problem has been developed in [11, 7] respectively. See [10] for generalized monotone method for fractional order of N systems. The generalised monotone method for nonlinear fractional differential equations with initial conditions has an added advantage over the usual monotone method, since the former method does not need the computation of Mittag-Leffler function. However the difficulty is in computing the coupled upper and lower solutions of type I (see [7] for details) to any desired interval. In this work we provide a methodology to compute coupled lower and upper solutions of type I for two system of Caputo fractional differential equation with initial conditions on any given interval. We also develop accelerated convergence results using generalized monotone method. Finally, we provide a numerical example as an application of all our theoretical results. In our numerical results we have considered two nonlinear fractional differential systems which represent cooperative, competitive and prey-predator models.

2. PRELIMINARY RESULTS

In this section, we recall known results, which are needed for our main results. Initially, we recall some definitions.

Definition 2.1. Caputo fractional derivative of order q is given by equation

$$ {}^cD^q u(t) = \frac{1}{\Gamma(1-q)} \int_0^t (t-s)^{-q} u'(s)ds, $$

where $0 < q < 1$.

Also, consider nonlinear Caputo fractional differential equation with initial condition of the form:

$$ {}^cD^q u(t) = f(t, u(t)), \quad u(0) = u_0, \tag{2.1} $$

where $f \in C[J \times \mathbb{R}, \mathbb{R}]$ and $J = [0, T]$.

The integral representation of (2.1) is given by equation

$$ u(t) = u_0 + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, u(s))ds, \tag{2.2} $$

where $\Gamma(q)$ is the Gamma function.

The equivalence of (2.1) and (2.2) is established in [3]. This equivalence means if we prove the existence of the solution of (2.2) on J, then we have proved the existence of solution of (2.1) and vice versa.

In order to compute the solution of linear fractional differential equation with constant coefficients we need Mittag Leffler function.

Definition 2.2. Mittag Leffler function is given by equation

$$E_{\alpha,\beta}(\lambda(t-t_0)^\alpha) = \sum_{k=0}^{\infty} \frac{(\lambda(t-t_0)^\alpha)^k}{\Gamma(\alpha k+\beta)},$$

where $\alpha, \beta > 0$. Also, for $t_0 = 0$, $\alpha = q$ and $\beta = 1$, we get equation

$$E_{q,1}(\lambda t^q) = \sum_{k=0}^{\infty} \frac{(\lambda t^q)^k}{\Gamma(qk+1)},$$

where $q > 0$.

Also, consider linear Caputo fractional differential equation,

$$^cD^q u(t) = \lambda u(t) + f(t), \quad u(0) = u_0, \text{ on } J, \tag{2.3}$$

where $J = [0, T]$, λ is a constant and $f(t) \in C[J, \mathbb{R}]$.

The solution of (2.3) exists and is unique. The explicit solution of (2.3) is given by the equation

$$u(t) = u_0 E_{q,1}(\lambda t^q) + \int_0^t (t-s)^{q-1} E_{q,q}(\lambda t^q) f(s) ds.$$

See [5] for details.

In particular, if $\lambda = 0$, the solution $u(t)$ is given by equation

$$u(t) = u_0 + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s) ds, \tag{2.4}$$

where $\Gamma(q)$ is the Gamma function.

Note that, in generalized monotone method, we need to compute the solution of the type (2.3) for $\lambda = 0$ only, whether it is for scalar or vector nonlinear fractional differential equations. The next result is related to the Reimann-Liouville fractional derivative. For that purpose we define C_p continuous function.

Definition 2.3. Let $p = 1 - q$. A function $\phi(t) \in C[(0, T], \mathbb{R}]$ is a C_p function if $t^p\phi(t) \in C([0, T], \mathbb{R})$. The set of C_p functions is denoted $C_p[J, \mathbb{R}]$. Further, given a function $\phi(t) \in C_p(J, \mathbb{R})$ we call the function $t^p\phi(t)$ the continuous extension of $\phi(t)$.

Lemma 2.4. *Let $m(t) \in C_p[J, \mathbb{R}]$ (where $J = [0, T]$) be such that for some $t_1 \in (0, T]$, $m(t_1) = 0$ and $m(t) \leq 0$, on J, then $D^q m(t_1) \geq 0$.*

Proof. See [2, 5] for details. □

However note that we have not assumed $m(t)$ to be Holder continuous as in [5]. The above lemma is true for Caputo derivative also, using the relation ${}^cD^qx(t) = D^q(x(t) - x(0))$ between the Caputo derivative and the Reimann-Liouville derivative. This is the version we will be using to prove our comparison results. The next lemma states the Caputo derivative version.

Lemma 2.5. *Let $m(t) \in C^1[J, \mathbb{R}]$ (where $J = [0, T]$) be such that $m(t) \leq 0$ on J and for $t_1 > 0$, if $m(t_1) = 0$, then ${}^cD^qm(t_1) \geq 0$.*

Also we recall known results related to two system of first order fractional differential equations of the following form:

$$ {}^cD^qu_i = f_i(t, u) + g_i(t, u), \quad u_i(0) = u_{0i} \text{ on } J \text{ for } i = 1, 2, \tag{2.5} $$

where $f_i, g_i \in C(J \times \mathbb{R}^2, \mathbb{R}^2)$.

Here and throughout this paper we assume $f_i(t, u_1, u_2)$ is non-decreasing in u_1 and u_2, and $g_i(t, u_1, u_2)$ is non-increasing in u_1 and u_2 for $i = 1, 2$ and for $t \in [0, T] = J$.

We recall the following known definitions which are needed for our main results.

Definition 2.6. Let v_i, w_i for $i = 1, 2$ be $C[J, \mathbb{R}]$. Then v_i and w_i are called natural lower and upper solutions of (2.5), if they satisfy the following inequalities:

$$ {}^cD^qv_i \leq f_i(t, v_1, v_2) + g_i(t, v_1, v_2), \quad v_i(0) \leq u_{0i}, \tag{2.6} $$

$$ {}^cD^qw_i \geq f_i(t, w_1, w_2) + g_i(t, w_1, w_2), \quad w_i(0) \geq u_{0i}. \tag{2.7} $$

Definition 2.7. Let v_i, w_i for $i = 1, 2 \in C[J, \mathbb{R}]$. Then v_i and w_i are called coupled lower and upper solutions of (2.5), if they satisfy the following inequalities:

$$ {}^cD^qv_i \leq f_i(t, v_1, v_2) + g_i(t, w_1, w_2), \quad v_i(0) \leq u_{0i}, \tag{2.8} $$

$$ {}^cD^qw_i \geq f_i(t, w_1, w_2) + g_i(t, v_1, v_2), \quad w_i(0) \geq u_{0i}. \tag{2.9} $$

The next result is the existence theorem for the solutions of the system (2.5) by generalized monotone method. The first result proves the existence of a solution of (2.5), when we have coupled lower and upper solutions as in definition 2.7.

Theorem 2.8. *Let $v_i(t), w_i(t) \in C^1[J, \mathbb{R}^2]$ and $f_i, g_i \in C[\Omega_i, \mathbb{R}^2]$, for $i = 1, 2$, where $\Omega_i = \{(t, u_i), t \in J, v_i \leq u_i \leq w_i\}$ such that $v_i(t) \leq w_i(t)$ on J and $v_i(t)$ and $w_i(t)$ are coupled lower and upper solutions as in definition 2.7 for (2.5). Then there exists a solution $u_i \in C^1[J, \mathbb{R}^2]$ of (2.5) such that $v_i(t) \leq u_i(t) \leq w_i(t)$ on J, provided $v_{i,0} \leq u_{i,0} \leq w_{i,0}$.*

Proof. See [9] for details. □

Theorem 2.9. *Let $f_i, g_i \in C[J \times \mathbb{R}^2, \mathbb{R}^2]$ such that $f_i(t,u)$ is nondecreasing in u_i components and $g_i(t,u)$ is nonincreasing in u_i components for $t \in J$, and for each $i = 1,2$. Let $v_{0,i}, w_{0,i} \in C^1[J, \mathbb{R}^2]$ be coupled lower and upper solutions of (2.5), such that $v_{0,i}(t) \le w_{0,i}(t)$ for $i = 1,2$, on J. Then, there exists monotone sequences $\{v_{n,i}\}$ and $\{w_{n,i}\}$ which converges uniformly and monotonically to coupled minimal and maximal solutions of (2.5) such that $v_{n,i} \to v_i$ and $w_{n,i} \to w_i$ as $n \to \infty$, provided $v_{0,i}(0) \le u_i(0) \le w_{0,i}(0)$, for $i = 1,2$. Further, if u_i for $i = 1,2$, is any solution of (2.5) such that $v_{0,i} \le u_i \le w_{0,i}$, then $v_i \le u_i \le w_i$ on J.*

The following result is a comparison theorem related to coupled lower and upper solutions.

Theorem 2.10. *Let $(v_{0,1}, v_{0,2})$ and $(w_{0,1}, w_{0,2})$ be coupled lower and upper solutions of (2.5). Further let*

(i) *$f_i(t,u)$ is nondecreasing in u_i components and $g_i(t,u)$ is nonincreasing in u_i components for $i = 1,2$;*
(ii) *$f_i(t,u)$ and $g_i(t,u)$ satisfy the one sided Lipschitz condition of the form,*
$f_i(t,u) - f_i(t,\overline{u}) \le L_i \sum_{j=1}^{2}(u_j - \overline{u_j})$, $L_i > 0$, $i = 1,2$ and
$g_i(t,u) - g_i(t,\overline{u}) \ge -M_i \sum_{j=1}^{2}(u_j - \overline{u_j})$, $M_i > 0$, $i = 1,2$,
whenever $u_i \ge \overline{u_i}$ for $i = 1,2$.

Then $v_i(t) = w_i(t) = u_i(t)$ for $i = 1,2$, where $u_i(t)$ is the unique solution of (2.5).

The following Corollary is useful in the generalized monotone method.

Corollary 2.11. *Let*

$$^{c}D^{q}p_i(t) \le \sum_{j=1}^{2}(L_{ij} + M_{ij})p_j, \quad \text{for} \quad i = 1,2.$$

Then we have $p_i(t) \le 0$ for $i = 1,2$ on $J = [0,T]$, whenever $p_i(0) \le 0$ for $i = 1,2$.

The next result is monotone method for (2.5) where we use natural lower and upper solutions.

Theorem 2.12. *Assume that*

(i) *$v_{0,i}$ and $w_{0,i}$ are natural lower and upper solutions of (2.5) with $v_{0,i}(t) \le w_{0,i}(t)$ for $i = 1,2$ on J .*
(ii) *$f_i, g_i \in C[J \times \mathbb{R}^2, \mathbb{R}^2]$, $f_i(t,u_1,u_2)$ is nondecreasing in u_i components and $g_i(t,u_1,u_2)$ is nonincreasing in u_i components for $i = 1,2$ on J.*

Then there exists monotone sequences $\{v_{n,i}\}$ and $\{w_{n,i}\}$ on J such that $v_{n,i}(t) \to v_i(t)$ and $w_{n,i}(t) \to w_i(t)$ uniformly and monotonically and (v_i, w_i) are coupled minimal *and maximal solutions, respectively to equation* (2.5)*. That is,* (v_i, w_i) *satisfy*

$$^cD^q v_i \leq f_i(t, v_1, v_2) + g_i(t, w_1, w_2), \quad v_i(0) \leq u_{0,i},$$
$$^cD^q w_i \geq f_i(t, w_1, w_2) + g_i(t, v_1, v_2), \quad w_i(0) \geq u_{0,i}$$

provided $v_{0,i} \leq v_{1,i}$ *and* $w_{1,i} \leq w_{0,i}$ *on* J.

See [10] for details of the proofs of Theorems 2.9, 2.10, 2.12. Also note that the iterative schemes used in Theorems 2.9 and 2.12 are one and the same. Theorem 2.12, uses $v_{0,i}, w_{0,i}$ as natural lower and upper solutions. The natural lower and upper solutions are easy to compute. For example the equilibrium solutions provide natural lower and upper solutions. Then $v_{1,i}, w_{1,i}$ will be coupled lower and upper solutions only on some interval $[0, t_{1,i}]$ and not necessarily on $[0, T]$ in general. However if we have coupled lower and upper solutions on $[0, T]$ then generalized monotone method can be used to compute solutions on $[0, T]$. This is the motivation for our main result relative to equation (2.5).

3. MAIN RESULTS

Theorem 3.1. *Assume that*

(i) $v_{0,i}, w_{0,i} \in C[J, \mathbb{R}^2]$ *for* $i = 1, 2$ *are natural lower and upper solutions of system* (2.5) *such that* $v_{0,i}(t) \leq w_{0,i}(t)$ *on* J.

(i) $f_i, g_i \in C[J \times \mathbb{R}^2, \mathbb{R}^2]$ *such that* $f_i(t, u)$ *is nondecreasing in* u_i *components and* $g_i(t, u)$ *is nonincreasing in* u_i *components for* $t \in J$, *and for each* $i = 1, 2$. *Then there exists monotone sequences* $\{v_{n,i}(t)\}$ *and* $\{w_{n,i}(t)\}$ *on* J *such that* $v_{n,i}(t) \to v_i(t)$ *and* $w_{n,i}(t) \to w_i(t)$ *uniformly and monotonically and* (v_i, w_i) *are coupled lower and upper solutions of* (2.5) *such that* $v_i \leq w_i$ *on* J. *The iterative scheme for the two system is given by*

$$^cD^q v_{n+1,1} = f_1(t_1, v_{n,1}, v_{n,2}) + g_1(t_1, w_{n,1}, w_{n,2}), \quad v_{n,1}(0) = u_{0,1} \text{ on } [0, t_{n,1}],$$
$$^cD^q v_{n+1,2} = f_2(t_1, v_{n,1}, v_{n,2}) + g_2(t_1, w_{n,1}, w_{n,2}), \quad v_{n,2}(0) = u_{0,2} \text{ on } [0, t_{n,2}],$$
$$^cD^q w_{n+1,1} = f_1(\overline{t_1}, w_{n,1}, w_{n,2}) + g_1(\overline{t_1}, v_{n,1}, v_{n,2}), \quad w_{n,1}(0) = u_{0,1} \text{ on } [0, \bar{t}_{n,1}],$$
$$^cD^q w_{n+1,2} = f_2(\overline{t_1}, w_{n,1}, w_{n,2}) + g_2(\overline{t_1}, v_{n,1}, v_{n,2}), \quad w_{n,2}(0) = u_{0,2} \text{ on } [0, \bar{t}_{n,2}],$$

and

$$v_{n,1}(t) = v_{0,1}(t) \text{ on } [t_{n,1}, T], \quad v_{n,2}(t) = v_{0,2}(t) \text{ on } [t_{n,2}, T],$$
$$w_{n,1}(t) = w_{0,1}(t) \text{ on } [\bar{t}_{n,1}, T], \quad w_{n,2}(t) = w_{0,2}(t) \text{ on } [\bar{t}_{n,2}, T].$$

Proof. Note that $v_{n,i}$, $w_{n,i}$ are the *nth* elements of the sequences $\{v_{n,i}\}$, $\{w_{n,i}\}$ such that $v_{0,i} \leq v_{n,i}$ and $w_{n,i} \leq w_{0,i}$ on $[0, t_{n,i}]$ and $[0, \bar{t}_{n,i}]$ for $i = 1, 2$ respectively. To continue the proof we relabel $v_{n,i}$ and $w_{n,i}$ such that they are sequences on the interval $[0, T]$ using the $v_{n,i}$, $w_{n,i}$ computed above. That is $v_{n,i} = v_{n,i}$ on $[0, t_{n,i}]$ and $v_{n,i} = v_{0,i}$ on $[t_{n,i}, T]$. Similarly $w_{n,i} = w_{n,i}$ on $[0, \overline{t_{n,i}}]$ and $w_{n,i} = w_{0,i}$ on $[\overline{t_{n,i}}, T]$. Thus the sequences $\{v_{n,i}\}$, $\{w_{n,i}\}$ are defined on $[0, T]$. The proof follows similar to the generalised monotone method except that the sequences $\{v_{n,i}\}$, $\{w_{n,i}\}$ are piecewise fractional differentiable functions on J. We can prove the sequences are equicontinuous and uniformly bounded on J. Hence by Ascoli Arzela's theorem, a subsequence converges uniformly and monotonically. Since the sequences are monotone, the entire sequence converges uniformly and monotonically to v_i and w_i respectively. □

Note that for some $n > N$,

$$
\begin{aligned}
{}^cD^q v_{n,i} &= f_i(t, v_{N,1}, v_{N,2}) + g_i(t, w_{N,1}, w_{N,2}) \text{ on } J,\\
{}^cD^q w_{n,i} &= f_i(t, w_{N,1}, w_{N,2}) + g_i(t, v_{N,1}, v_{N,2}) \text{ on } J,
\end{aligned}
$$

for $i = 1, 2$. Further it follows,

$$
\begin{aligned}
{}^cD^q v_{N,i} &\le f_i(t, v_{N,1}, v_{N,2}) + g_i(t, w_{N,1}, w_{N,2}),\\
{}^cD^q w_{N,i} &\ge f_i(t, w_{N,1}, w_{N,2}) + g_i(t, w_{N,1}, w_{N,2}),
\end{aligned}
$$

for $i = 1, 2$. Hence v_i, w_i, for $i = 1, 2$ are coupled lower and upper solutions of (2.5) on J.

Remark 3.2. Note that Theorem 3.1 provides a method to compute coupled lower and upper solutions of (2.5) on the desired interval [0,T]. We can develop an accelerated convergence result for the system (2.5) similar to Theorem 3.1. This is precisely our next result.

Theorem 3.3. *Let all the hypothesis of Theorem 2.9 hold. Then there exists sequences $\{v^*_{n,i}\}, \{w^*_{n,i}\}$ for $i = 1, 2$, on $[0, T]$, such that it converges uniformly and monotonically to coupled minimal and maximal solutions of (2.5). These sequences converge at a much faster pace than the sequences of Theorem 2.9. The sequences $\{v^*_{n,i}\}$, and $\{w^*_{n,i}\}$, are developed as follows: where the iterative scheme is given by*

$$
\begin{aligned}
{}^cD^q {v^*}_{n+1,1} &= f_1(t, v^*_{n,1}, v^*_{n,2}) + g_1(t, w^*_{n,1}, w^*_{n,2}), \quad v_{n,1}(0) = u_{0,1},\\
{}^cD^q {v^*}_{n+1,2} &= f_2(t, v^*_{n+1,1}, v^*_{n,2}) + g_2(t, w^*_{n,1}, w^*_{n,2}), \quad v_{n,2}(0) = u_{0,2},\\
{}^cD^q {w^*}_{n+1,1} &= f_1(t, w^*_{n,1}, w^*_{n,2}) + g_1(t, v^*_{n+1,1}, v^*_{n+1,2}), \quad w_{n,1}(0) = u_{0,1},\\
{}^cD^q {w^*}_{n+1,2} &= f_2(t, w^*_{n+1,1}, w^*_{n,2}) + g_2(t, v^*_{n+1,1}, v^*_{n+1,2}), \quad w_{n,2}(0) = u_{0,2}.
\end{aligned}
$$

Proof. Let $v_{1,1} = v^*_{0,1}$,then

$$\begin{aligned} {}^cD^q {v^*}_{0,2} &= f_2(t, v^*_{0,1}, v_{0,2}) + g_2(t, w_{0,1}, w_{0,2}), \quad v^*_{0,2}(0) = u_{0,2}, \\ {}^cD^q {w^*}_{0,1} &= f_1(t, w_{0,1}, w_{0,2}) + g_1(t, v^*_{0,1}, v^*_{0,2}), \quad w^*_{0,1}(0) = u_{0,1}, \\ {}^cD^q {w^*}_{0,2} &= f_2(t, w^*_{0,1}, w_{0,2}) + g_2(t, v^*_{0,1}, v^*_{0,2}), \quad w^*_{0,2}(0) = u_{0,2}. \end{aligned}$$

We will prove that $v^*_{0,2} \geq v_{1,2}$ on J. For that purpose, set $p(t) = v^*_{0,2} - v_{1,2}$, $p(0) = 0$. Then ${}^cD^q p(t) = {}^cD^q {v^*}_{0,2} - {}^cD^q v_{1,2} = f_2(t, v^*_{0,1}, v_{0,2}) + g_2(t, w_{0,1}, w_{0,2}) - (f_2(t, v_{0,1}, v_{0,2}) + g_2(t, w_{0,1}, w_{0,2})) = f_2(t, v_{1,1}, v_{0,2}) - f_2(t, v_{0,1}, v_{0,2}) \geq 0$, using the fact that $v_{1,1} \geq v_{0,1}$. This proves $v^*_{0,2} \geq v_{1,2}$. Similarly, we can prove $w^*_{0,1} \leq w_{1,1}$ using the information $v_{0,1} \leq v_{1,1} = v^*_{0,1}$ and $v_{0,2} \leq v_{1,2} \leq v^*_{0,2}$. Continuing this process we can show the sequences $\{v^*_{n,i}\}$ and $\{w^*_{n,i}\}$ converges faster than the sequence $\{v_{n,i}\}$ and $\{w_{n,i}\}$ computed using Theorem 2.9. □

4. NUMERICAL RESULTS

In this section, we provide numerical examples justifying our main results. We consider three Volterra-Lotka models namely, the prey-predator model, competitive model and cooperative model and apply Theorem 2.9. In order to apply Theorem 2.9, we assume that $v_{1,i}$ and $w_{1,i}$ should satisfy $v_{0,i} \leq v_{1,i}$, $w_{1,i} \leq w_{0,i}$ for $i = 1, 2$ on $[0, T]$. In all these examples first we will apply Theorem 3.1 to obtain the coupled lower and upper solutions $v_{1,i}$ and $w_{1,i}$ for $i = 1, 2$ on a desired interval. Then using these coupled lower and upper solutions we apply Theorem 2.9 to obtain the coupled minimal and maximal solutions. Also we use Theorem 3.3 to accelerate the rate of convergence.

First, we consider the prey-predator model

$$\text{(4.1)} \qquad \begin{cases} {}^cD^{\frac{1}{2}} u_1(t) = 5u_1 - 2u_1^2 - 3u_1u_2, \\ {}^cD^{\frac{1}{2}} u_2(t) = -2u_2 + u_1u_1 + u_2^2. \end{cases}$$

It is easy to observe that $(v_{0,1}, v_{0,2}) = (0, 0)$, and $(w_{0,1}, w_{0,2}) = (1, 1)$ are natural lower and upper solutions respectively of (4.1) such that $v_{0,1} \leq w_{0,1}$ and $v_{0,2} \leq w_{0,2}$ on $[0, T]$. Using our main result namely Theorem 3.1 we can compute coupled lower and upper solutions on $[0, 1]$. In figure 1 we apply Theorem 3.1 to example (4.1).

In figure 1, using Theorem 3.1 we have computed $v_{1,i}$ and $w_{1,i}$ such that $v_{1,i} \leq w_{1,i}$ for $i = 1, 2$ on the interval $[0, 0.02]$. Further $v_{1,i}, w_{1,i}$, for $i = 1, 2$ are coupled lower and upper solutions of (4.1) on $[0, 0.02]$. In figure 2, we use the coupled upper and lower solutions obtained in figure 1 and apply Theorem 2.9 to compute the coupled minimal and maximal solutions.

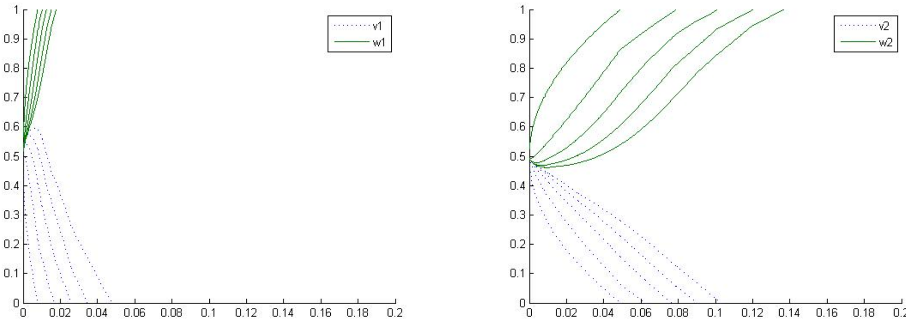

FIGURE 1. Coupled lower and upper solutions of (4.1) using Theorem 3.1

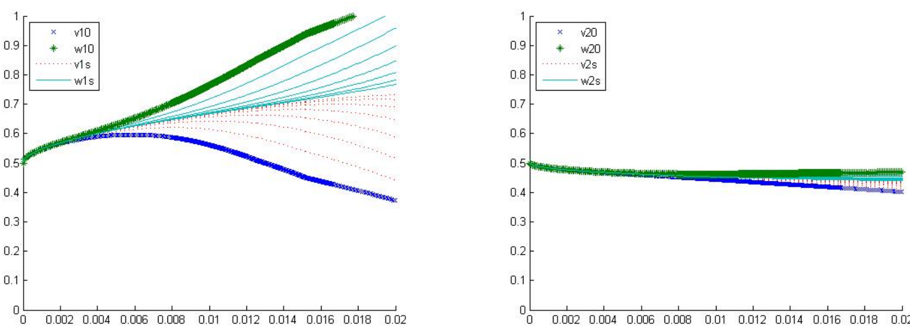

FIGURE 2. Coupled min and max solutions of (4.1) using Theorem 2.9

We have plotted figure 2 on the interval $[0, 0.02]$ showing seven iterations. In figure 3, again using the coupled lower and upper solutions obtained in figure 1, we compute the coupled minimal and maximal solutions of example (4.1) with less number of iterations, using the accelerated convergence of Theorem 3.3.

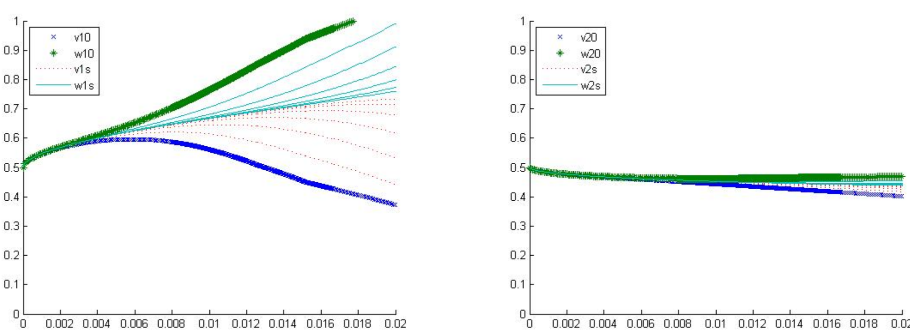

FIGURE 3. Coupled min and max solutions of (4.1) using Theorem 3.3

We can observe that figure 3 took only six iterations compared to seven iterations in figure 2, as we have used accelerated convergence.

Next, we consider the Cooperative model

$$(4.2)\qquad \begin{cases} {}^cD^{\frac{1}{2}}u_1(t) = 2u_1 + 2u_1u_2 - 3u_1^2, \\ {}^cD^{\frac{1}{2}}u_2(t) = u_2 + 4u_1u_2 - 6u_2^2. \end{cases}$$

It is easy to observe that $(v_{0,1}, v_{0,2}) = (0,0)$, and $(w_{0,1}, w_{0,2}) = (\frac{7}{5}, \frac{33}{30})$ are natural lower and upper solutions respectively of (4.2) such that $v_{0,1} \leq w_{0,1}$ and $v_{0,2} \leq w_{0,2}$ on $[0,T]$.

Using our main result namely Theorem 3.1 we can compute coupled lower and upper solutions on $[0,1]$. In figure 4 we apply Theorem 3.1 to example (4.2).

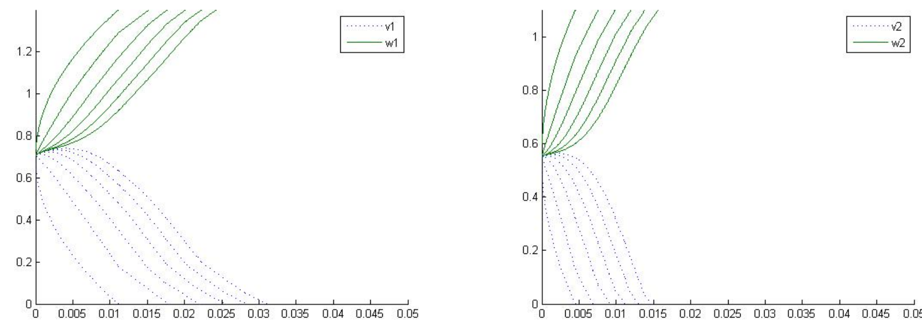

FIGURE 4. Coupled lower and upper solutions of (4.2) using Theorem 3.1

In figure 4 we have computed $v_{1,i}$ and $w_{1,i}$ such that $v_{1,i} \leq w_{1,i}$ for $i = 1,2$ on the interval $[0, 0.0120]$. In figure 5 we use the coupled upper and lower solutions obtained in figure 4 and apply Theorem 2.9 to obtain the coupled minimal and maximal solutions of example (4.2).

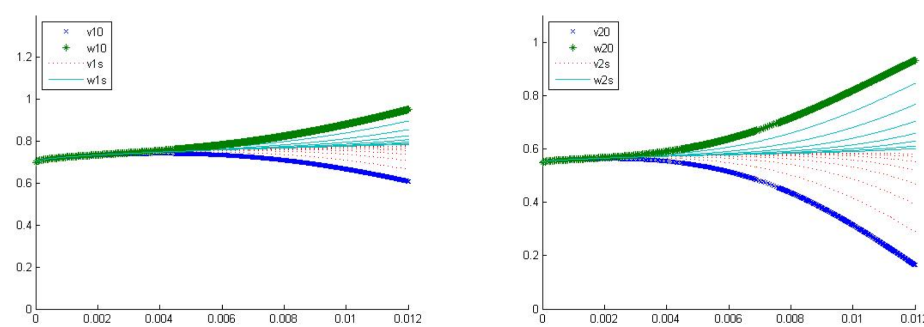

FIGURE 5. Coupled min and max solutions of (4.2) using Theorem 2.9

We have plotted figure 5 on the interval $[0, 0.0120]$ showing seven iterations. In figure 6, again using the coupled lower and upper solutions obtained in figure 4, we compute the coupled minimal and maximal solutions of example (4.2) with less number of iterations, using the accelerated convergence of Theorem 3.3.

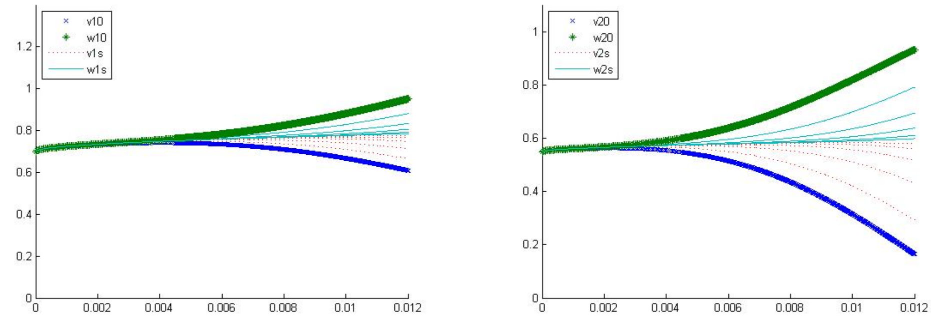

Figure 6. Coupled min and max solutions of (4.2) using Theorem 3.3

We can observe that figure 6 took only five iterations compared to seven iterations in figure 5, as we have used accelerated convergence.

Finally, we consider the Competitive model

$$\begin{cases} {}^cD^{\frac{1}{2}}u_1(t) = 6u_1 - 3u_1^2 - 2u_1u_2, \\ {}^cD^{\frac{1}{2}}u_2(t) = 6u_2 - 2u_1u_2 - u_2^2. \end{cases} \tag{4.3}$$

It is easy to observe that $(v_{0,1}, v_{0,2} = (0,0)$ and $(w_{0,1}, w_{0,2} = (6,-6)$ are natural lower and upper solutions respectively of (4.3) such that $v_{0,1} \leq w_{0,1}$ and $v_{0,2} \leq w_{0,2}$ on $[0,T]$. Using our main result namely Theorem 3.1 we can compute coupled lower and upper solutions on $[0,1]$.

In figure 7 we apply Theorem 3.1 to example (4.3).

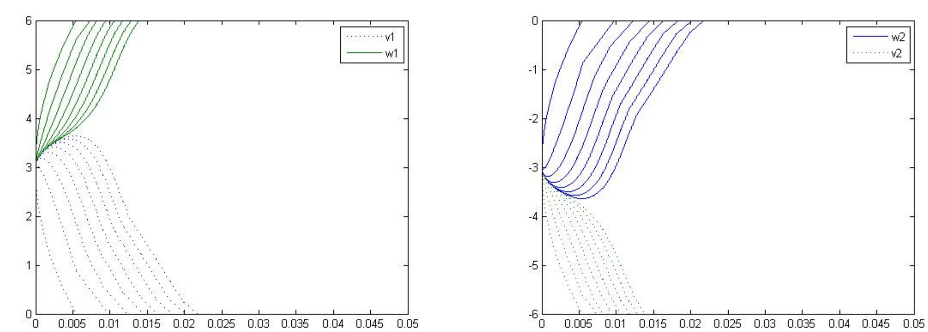

Figure 7. Coupled lower and upper solutions of (4.3) using Theorem 3.1

In figure 7 we have computed $v_{1,i}$ and $w_{1,i}$ such that $v_{1,i} \leq w_{1,i}$ for $i = 1,2$ on the interval $[0, 0.0128]$. In figure 8 we use the coupled upper and lower solutions of figure 7 and apply Theorem 2.9 to obtain the coupled minimal and maximal solutions of example (4.3).

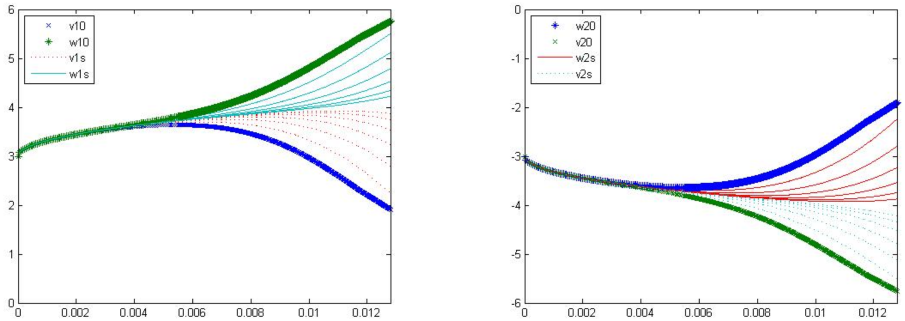

FIGURE 8. Coupled min and max solutions of (4.3) using Theorem 2.9

We have plotted figure 8 on the interval $[0, 0.0128]$ showing six iterations. In figure 9, again using the coupled lower and upper solutions obtained in figure 7, we compute the coupled minimal and maximal solutions of example (4.3) showing less number of iterations, using the accelerated convergence of Theorem 3.3.

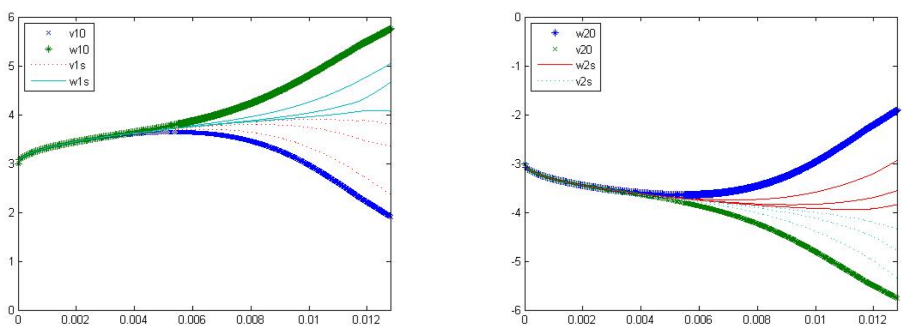

FIGURE 9. Coupled min and max solutions of (4.3) using Theorem 3.3

We can observe that figure 9 took only three iterations compared to six iterations in figure 8, as we have used accelerated convergence.

In all the examples we have considered, if the coupled minimal and maximal solutions v_i, w_i, for $i = 1, 2$ satisfy the Lipschitz condition then those solutions converge to a unique solution u_i, for $i = 1, 2$.

5. **CONCLUSION**

In general in order to compute the coupled lower and upper solutions of fractional differential equations, using the usual monotone method we need the Mittag Leffler function. In this work, we have computed the coupled lower and upper solutions of two system of Caputo fractional differential equations, on a desired interval using the generalised monotone method. The advantage of the generalised monotone method over the usual monotone method is that it does not require the computation of Mittag Leffler function. As we are using generalised monotone method, even we were able to accelerate the convergence by using Gauss-Seidel method. The disadvantage of the generalized monotone method is that its rate of convergence is linear. The use of Mittag Leffler function in computing the solution may give us a faster convergence. In our future work we want to use the generalized quasilinearization method to compute the solutions, which uses the Mittag Leffler function. We expect to get a faster convergence using the generalized quasilinearization method.

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