Neural, Parallel, and Scientific Computations 21 (2013) 375-392

HEATLET APPROACH TO DIFFUSION EQUATION ON A SEMI-INFINITE REGION

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ABSTRACT. We develop Heatlets, the fundamental solutions of heat equation using wavelets, for numerically solving initial-boundary value problems of one dimensional diffusion equation on a quarter plane. This approach does not invovle the notion of artificial boundary conditions. We present the two scale properties of the heatlets and demonstrate the applicability of the heatlet approach by using them in the construction of numerical solution of diffusion equation on a semi-infinite region. The numerical results obtained are compared with those obtained using finite difference and finite element methods that use artificial boundary conditions.

AMS (MOS) Subject Classification. 35K05, 65M99

1. Introduction

In a recent paper [4] we studied a numerical method, based on heatlets, for finding the approximate solution of the initial value problems associated with the heat equation on $(-\infty, \infty)$:

$$u_t = u_{xx}, \qquad u(x,0) = f(x), \quad -\infty < x < \infty, t > 0, \tag{1.1}$$

The method was inspired by the work of Strang [8], where heatlets are obtained as fundamental solutions of the problem (1.1). Han and Huang [6] and Wu and Sun [10] obtain the numerical solution of a similar problem using the finite difference and finite element methods. However, employing the finite difference or finite element methods requires the use of artificial boundary conditions (ABC's) imposed on the finite computational domain to simulate the effects of diffusion into an infinite medium. There is an extensive literature on the issues concerning the treatment of ABC's and here we only mention the monograph [3]. It is well known that the adhoc discretization of the analytic ABC's induces numerical reflections at the artificial boundary and the stability properties of the underlying method could also be affected.

Received July 15, 2013

On the other hand, integral equation methods have also been employed for finding the numerical solution of these problems, albeit with a high computational cost. Recently Greengard and Lin [5] developed an efficient new algorithm based on the spectral approximation of the free space heat kernel and the non uniform fast Fourier transform.

Establishing a theoretical alliance between wavelets and heat equation, Shen and Strang [8] studied heatlets for their translation and scale invariance properties. The method proposed in [4] uses wavelets as building blocks to generate the heatlets, where the initial function $f(x)$ is either a scaling function or a wavelet. The main advantage of the heatlet method is that once the library of heatlets are built for a heat equation upto a desired level of accuracy, the computation of solution for any initial function or forcing term, requires only the knowledge of wavelet coefficients of these functions and the numerical solution of the problem can be easily obtained.

Here in this paper, we adapt the heatlet method developed in [4] to certain initial boundary value problems associated with diffusion equation on $[0,\infty)$. Several modifications need be made to the approach originally used in [4]. For example, the presence of the boundary point $x = 0$, makes it necessary to consider the 'edge wavelets' in the wavelet expansion and use the corresponding edge heatlets.

The organization of the paper is as follows. In Section 2, we provide some mathematical preliminaries on wavelets. Section 3 deals with the construction of heatlets, their properties and a description of the proposed numerical approach. Finally, numerical examples are considered in Section 4 and the results obtained are compared with those obtained using the finite difference and finite element methods.

2. **Preliminaries**

A multiresolution analysis (MRA) of $L^2(\mathbb{R})$, the real space of all square integrable functions on $\mathbb{R}$, equipped with the standard innerproduct $(\cdot,\cdot)$, is a chain of closed subspaces indexed by all integers:

$$\dots V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \dots, \tag{2.1}$$

such that

(i) $\overline{\lim_{n\to+\infty} V_n} = L^2(\mathbb{R})$,

(ii) $\overline{\lim_{n\to-\infty} V_n} = \{0\}$.

(iii) $f(\cdot) \in V_n \Leftrightarrow f(2\cdot) \in V_{n+1}$.

(iv) Let ϕ be a scaling function such that $\{\phi(\cdot - k) : k \in \mathbb{Z}\}$ constitute a complete orthonormal basis of V_0.

To obtain a multiresolution analysis, it suffices to construct the scaling function $\phi(x)$. The entire subspace chain can then be reconstructed from $\phi(x)$ according to

(iii) and (iv). Since $V_0 \subset V_1$ and $\{\phi(2\cdot -k) : k \in \mathbb{Z}\}$ form a complete orthonormal set for V_1 it follows from (iii) that the following two scale relation holds:

$$\phi(\cdot) = 2\sum_{k\in\mathbb{Z}} h_k\phi(2\cdot -k), \tag{2.2}$$

for a suitable set of coefficients $(\ldots, h_{-1}, h_0, h_1, \ldots)$. If the scaling function ϕ is compactly supported, we have

$$\phi(\cdot) = 2\sum_{k=0}^{L} h_k\phi(2\cdot -k),$$

with $h_0h_L \neq 0$. It is usually assumed in wavelet analysis that $\int \phi = 1$. This implies,

$$h_0 + h_1 + h_2 + \cdots + h_L = 1. \tag{2.3}$$

Let W_0 denote the orthogonal complement of V_0 in V_1. A function ψ whose integer translates $\{\psi(\cdot - k) : k \in \mathbb{Z}\}$ constitute an orthonormal basis of W_0 is called a wavelet. This wavelet function ψ satisfies the two scale relation

$$\psi(\cdot) = 2\sum_{k\in\mathbb{Z}} g_k\phi(2\cdot -k), \tag{2.4}$$

for a suitable set of coefficients $(\ldots, g_{-1}, g_0, g_1, \ldots)$. From (i)–(iv) it is clear that

$$\{\psi_{jk}(\cdot) = 2^{j/2}\psi(2^j\cdot -k); j, k \in \mathbb{Z}\},$$

is an orthonormal basis of $L^2(\mathbb{R})$.

For any function $f \in L^2(\mathbb{R})$, define $P_J : L^2(\mathbb{R}) \to V_J$ to be the projection of f onto the J-th resolution space V_J. For $f \in L^2(\mathbb{R})$, $f_J = P_Jf$, can be decomposed as follows,

$$f_J = \sum_{k\in\mathbb{Z}} c_{0,k}(f)\phi_{0,k} + \sum_{0\leq j\leq J-1, k\in\mathbb{Z}} d_{j,k}(f)\psi_{j,k}. \tag{2.5}$$

For a detailed introduction to wavelet theory, we refer to Daubechies [2], and Hernández and Weiss[9].

For the well known Haar wavelet, where the scaling function $\phi(x)$ and the wavelet $\psi(x)$ are defined by:

$$\phi(x) = \begin{cases} 1, & 0 \leq x < 1, \\ 0, & otherwise, \end{cases} \quad \text{and } \psi(x) = \begin{cases} 1, & 0 \leq x < 1/2, \\ -1, & 1/2 \leq x < 1, \\ 0, & otherwise, \end{cases}$$

the scaling and wavelet coefficients are given by $h_0 = h_1 = 1/2$, and $g_0 = h_0$, $g_1 = -h_1$.

Similarly the scaling and wavelet coefficients of the smooth and compactly supported Daubechies wavelet of order two, are given by

$$h_0 = \frac{1+\sqrt(3)}{4}, \quad h_1 = \frac{3+\sqrt(3)}{4}, \quad h_2 = \frac{3-\sqrt(3)}{4}, \quad h_3 = \frac{1-\sqrt(3)}{4}$$

and $g_k = (-1)^k h_{N-k}$, where $0 \leq k \leq 3$, $N = 3$. The wavelets thus constructed are defined on the entire real line. If we wish to use them on $[0, \infty)$ or on a bounded interval by taking the restrictions of each of the wavelet basis function on the required part, it is well known that numerical instabilities occur. In general, let us assume that the interval of our interest is $[0, 1]$. Since each of the Haar basis functions are supported either in $[0, 1]$ or in $\mathbb{R}\backslash]0, 1[$, restriction of these basis functions to $[0, 1]$ leads to an orthonormal Haar basis for $L^2([0, 1])$. The construction of orthonormal wavelet basis on an interval for smoother wavelets needs to be done more carefully to avoid numerical instabilities.

We adapt here the construction procedure proposed in Cohen et al., [1] and give a brief summary below. In addition to the usual scaling function ϕ and wavelet ψ, [1] introduces the edge scaling function ϕ^0 and wavelet ψ^0. In this construction, one obtains orthonormal systems of functions

$$X_j = \{\phi^0_{j,k} \mid 0 \leq k \leq N-1\} \cup \{\phi_{j,k} \mid N \leq k \leq 2^{|j|-N-1}\} \cup \{\phi^1_{j,k} \mid 0 \leq k \leq N-1\},$$

$$Y_j = \{\psi^0_{j,k} \mid 0 \leq k \leq N-1\} \cup \{\psi_{j,k} \mid N \leq k \leq 2^{|j|-N-1}\} \cup \{\psi^1_{j,k} \mid 0 \leq k \leq N-1\},$$

where $j \leq 0$ so that $2^{|j|} \geq 2N$. The spaces $V_j^{new} :=$ span X_j and $W_j^{new} :=$ span Y_j give a multiresolution analysis of $L^2([0, 1])$.

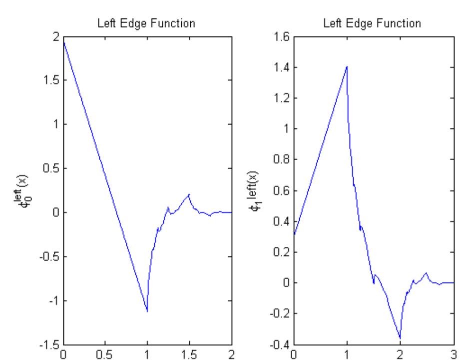
These spaces V_j^{new}, W_j^{new} have the following properties,

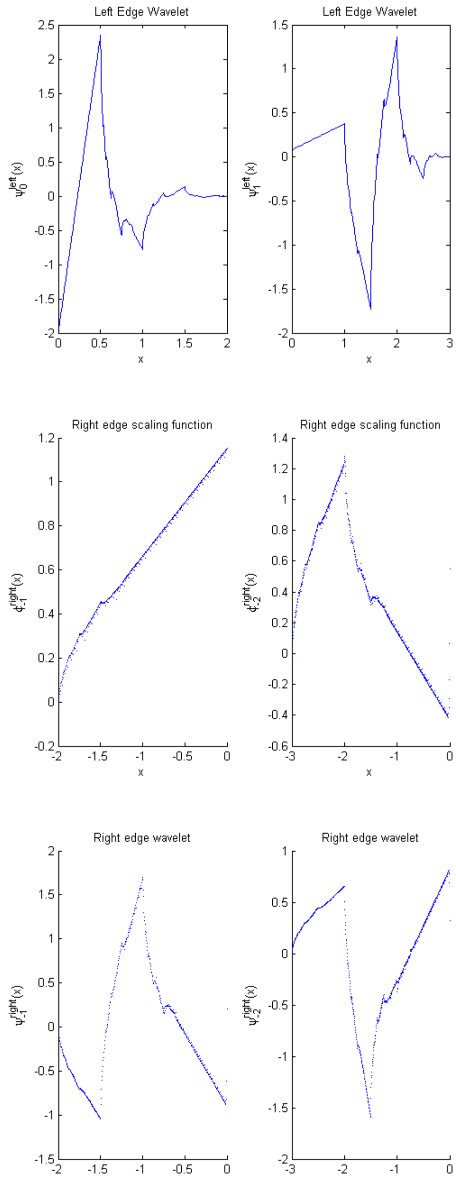
(i) $dim\ V_j^{new} = dim\ W_j^{new} = 2^{|j|}$,

(ii) $V_j^{new} \perp W_j^{new}$,

(iii) $V_{j-1}^{new} = V_j^{new} \oplus W_j^{new}$.

The functions ϕ_{jk}, ψ_{jk} are interior scaling functions and wavelets. $\phi^0_{j,k}, \psi^0_{j,k}$ are left edge functions and $\phi^1_{j,k}, \psi^1_{j,k}$ are right edge functions. The graphs of these edge functions constructed using the second order Daubechies wavelet, are given in figures.





The $\phi^i_{j,k}$, $\psi^i_{j,k}$, $i = 0, 1$, satisfy the following two scale relations:

$$\phi^i_{j,k} = \sum_{l=0}^{N-1} h^i_{k,l}\phi^i_{j-1,l} + \sum_{l=N}^{N+2k} h^i_{k,l}(\phi_N)_{j-1,l} \ ,$$

$$\psi^i_{j,k} = \sum_{l=0}^{N-1} g^i_{k,l}\phi^i_{j-1,l} + \sum_{l=N}^{N+2k} g^i_{k,l}(\phi_N)_{j-1,l} \ .$$

The coefficients $h^0_{k,l}, g^0_{k,l}$ and $h^1_{k,l}, g^1_{k,l}$ can be computed and we can find the values in [1].

Consider the Daubechies wavelet of order N, where $N \geq 2$, and let supp $\phi = [-N+1, N]$. Then, for $j = 0$, $\phi_{j,k}$, $k \geq N-1$, are interior scaling functions.

Define

$$\phi^0(x) = \sum_{k=-N+1}^{N-2} \phi(x-k). \tag{2.6}$$

Clearly, ϕ^0 has compact support and is orthogonal to all the interior $\phi_{0,k}$. Also, $\mathrm{span}\{\phi^0, \phi_{0,k}, k \geq N-1\} \subset \mathrm{span}\{\phi^0(2\cdot), \phi_{1,k}, k \geq N-1\}$.

Given N, for $k = 0, 1, \ldots N-1$, define

$$\tilde{\phi}^k(x) = \sum_{n=k}^{2N-2} \begin{pmatrix} n \\ k \end{pmatrix} \phi(x+n-N+1). \tag{2.7}$$

$\tilde{\phi}^k$'s are compactly supported and thier supports are $[0, 2N-1-k]$

Proposition 2.1 (see Proposition 4.1 [1]). *The N functions $\tilde{\phi}^k$, $k = 0, \ldots, N-1$, are independent, and orthogonal to the $\phi_{0,m}$, $m \geq N$. Together with the $\phi_{0,m}$, $m \geq N$, they generate all the polynomials up to degree $N-1$ on $[0, \infty]$. Finally, there exist constants $a_{k,l}, b_{k,m}$ such that*

$$\tilde{\phi}^k = \sum_{l=0}^{k} a_{k,l} \tilde{\phi}^l(2x) + \sum_{m=N}^{3N-2-2k} b_{k,m} \phi(2x-m).$$

Define

$$V_{-j}^{left} = \overline{\mathrm{Span}[\{\tilde{\phi}^k(2^j \cdot); k = 0, 1, \ldots, N-1\} \cup \{\phi_{-j,m}; m \geq N\}]}$$

The above Proposition establishes that the V_{-j}^{left} constitute a multiresolution hierarchy

$$\cdots \subset V_2^{left} \subset V_1^{left} \subset V_0^{left} \subset V_{-1}^{left} \subset V_{-2}^{left} \subset \cdots.$$

To obtain an orthonormal basis for V_0^{left}, we use *Gram-Schmidt* process starting with $\tilde{\phi}^{N-1}$ and working down to lower values of k. The constructed $\phi_{0,k}^{left}$, $k = 0, 1, \ldots N-1$ together with the orthonormal $\{\phi_{0,m}, m \geq N\}$ constitute an orthonormal basis for V_0^{left}. Note that $\phi_{0,k}^{left}, k = 0, 1, \ldots, N-1$, have staggered supports and supp $\phi_{0,k}^{left} = [0, N+k]$.

The constructed orthonormal $\phi_{0,k}^{left}$ satisfy the following recursion relation

$$\phi_{-j,k}^{left} = \sum_{l=0}^{N-1} H_{k,l}^{left} \phi_{-j-1,l}^{left} + \sum_{m=N}^{N+2k} h_{k,m}^{left} \phi_{-j-1,m}. \tag{2.8}$$

where $H_{k,l}^{left}, h_{k,m}^{left}$ are constants.

Proposition 2.2. *Define the functions $\tilde{\psi}^k$, $k = 0, 1, \ldots, N-1$, by*

$$\tilde{\psi}^k = \phi_{-1,k}^{left} - \sum_{m=0}^{N-1} < \phi_{-1,k}^{left}, \phi_{0,m}^{left} > \phi_{0,m}^{left}.$$

Then the $\tilde{\psi}^k$ are N independent functions in W_0^{left}, orthogonal to the $\psi_{0,m}$, $m \geq N$.

Similar to equation (2.8) there exists constants $G^{left}_{k,l}, g^{left}_{k,m}$ such that

$$\psi^{left}_{-j,k} = \sum_{l=0}^{N-1} G^{left}_{k,l}\phi^{left}_{-j-1,l} + \sum_{m=N}^{N+2k} g^{left}_{k,m}\phi_{-j-1,m}. \tag{2.9}$$

The wavelet series expansion of $f \in L^2([0,\infty)$ in terms of the left edge scaling functions and left edge wavelets is given by

$$f(x) = \sum_{k=0}^{1} \tilde{c}^l_k \phi^{left}_{0,k}(x) + \sum_{k=2}^{\infty} \tilde{c}_k\phi_{0,k}(x) + \sum_{j\geq 0}(\sum_{k=0}^{1} \tilde{d}^l_{j,k}\psi^{left}_{j,k}(x) + \sum_{k=2}^{\infty} \tilde{d}_{j,k}\psi_{j,k}(x)). \tag{2.9}$$

where $\tilde{c}^l_k = (f, \phi^{left}_{0,k})$, $d^{left}_{j,k} = (f, \psi^{left}_{j,k})$,$\phi^{left}_{0,k} = \phi^{left}(x-k)$, $\psi^{left}_{j,k} = 2^{j/2}\psi^{left}(2^j x - k)$ for $k = 0, 1$, and $c^l_k = (f, \phi_{0,k})$, $d^l_{j,k} = (f, \psi_{j,k})$ for $k \geq 2$ and $j \geq 0$. The convegence of the above sum is in the L^2- sense.

3. **The Heatlet Decomposition**

In this section, we construct the heatlets, the fundamental solutions of heat equation using wavelets, corresponding to Initial-Boundary value problems associated with a diffusion equation. These fundamental solutions are used as building blocks in the construction of the approximate solution to the inhomogeneous and homogeneous Initial-Boundary value problems of diffusion equation on $[0,\infty)$.

Let $\phi(x)$ and $\psi(x)$ be the scaling function and the wavelet associated to a MRA of $L^2(\mathbb{R})$ respectively. Consider the initial boundary value problem

$$\begin{array}{ll} u_t = u_{xx}, & x > 0, t > 0. \\ u(x,0) = \phi(x-n), & x > 0, \\ u(0,t) = 0, & t > 0, \\ u \to 0 \text{ as } x \to 0. & \end{array} \tag{3.1}$$

We call the fundamental solution $\Phi(x,n,t)$ of the above problem as a heatlet.

Let $h(x)$ be the odd extension of $\phi(x-n)$, given by

$$h(x) = \begin{cases} \phi(x-n), & x \geq 0, \\ -\phi(x-n), & x < 0. \end{cases} \tag{3.2}$$

and the heat kernel $K(x,t)$ is defined by,

$$K(x,t) = \frac{e^{-x^2/4t}}{\sqrt{4\pi t}}.$$

For $x > 0$, $t > 0$, $\Phi(x,n,t)$ is given by,

(3.3)

$$\begin{aligned} \Phi(x,n,t) &= \frac{1}{2\sqrt{\pi t}}\int_{-\infty}^{\infty} K(x-y,t)h(y)\,dy, \\ &= \frac{1}{2\sqrt{\pi t}}\int_{0}^{\infty} [e^{-(x-n-y)^2/4t} - e^{-(x+n+y)^2/4t}]\phi(y)\,dy,\ n = 0,1,2,\ldots. \end{aligned}$$

For $n = 0$, $\Phi(x, 0, t)$ is called a refinable heat and is given by

$$\Phi(x,0,t) = \frac{1}{2\sqrt{\pi t}} \int_0^\infty [e^{-(x-y)^2/4t} - e^{-(x+y)^2/4t}]\phi(y)\, dy,\ x > 0, t > 0. \tag{3.4}$$

When the intial function is a dilated version of $\phi(x - n)$, that is $\phi(2x - n)$, the fundamental solution is given by

$$\Phi(2x,n,4t) = \frac{1}{2\sqrt{\pi t}} \int_0^\infty [e^{-(x-y)^2/4t} - e^{-(x+y)^2/4t}]\phi(2y-n)\, dy,\ x > 0, t > 0. \tag{3.5}$$

Similarly, $\Psi^h(x, n, t)$, $\Psi^h(2x, n, 4t)$ denote the fundamental solutions corresponding to the initial functions $\psi(x - n)$, and $\psi(2x - n)$.

The following proposition shows the refinement relations satisfied by the refinable heat and the heatlets.

Proposition 3.1. *Suppose that $\phi(x)$ and $\psi(x)$ satisfy the following two scale relations:*

$$\phi(x) = 2\sum_{k\in\mathbb{Z}} h_k\phi(2x-k),$$
$$\psi(x) = 2\sum_{k\in\mathbb{Z}} g_k\phi(2x-k).$$

Then the refinable heat $\Phi(x, 0, t)$ and heatlet $\Psi(x, 0, t)$ satisfy:

$$\Phi(x,0,t) = 2\sum_{k\in\mathbb{Z}} h_k\Phi(2x,k,4t), \tag{3.6}$$

$$\Psi(x,0,t) = 2\sum_{k\in\mathbb{Z}} g_k\Phi(2x,k,4t). \tag{3.7}$$

Proof.

$$\begin{aligned}\Phi(x,0,t) &= \frac{1}{2\sqrt{\pi t}} \int_0^\infty [e^{-(x-y)^2/4t} - e^{-(x+y)^2/4t}]\phi(y)dy,\\ &= 2\sum_{k\in\mathbb{Z}} h_k \frac{1}{2\sqrt{\pi t}} \int_0^\infty [e^{-(x-y)^2/4t} - e^{-(x+y)^2/4t}]\phi(2y-k)dy,\\ &= 2\sum_{k\in\mathbb{Z}} h_k\Phi(2x,k,4t).\end{aligned}$$

Similarly, we can show that

$$\Psi(x,0,t) = 2\sum_{k\in\mathbb{Z}} g_k\Phi(2x,k,4t).$$

Next, we consider the initial boundary value problem associated with the non-homogeneous Heat equation :

$$\begin{aligned}&u_t = u_{xx} + \phi(x-n), && x>0, t>0, n\geq 0.\\ &u(x,0) = 0, && x>0,\\ &u(0,t) = 0, && t>0,\\ &u\to 0 \text{ as } x\to 0.\end{aligned} \tag{3.8}$$

The solution of (3.8) is given by,

$$\tilde{\Phi}(x,n,t) = \int_0^t \int_0^\infty \frac{e^{-(x-n-y)^2/4(t-\tau)} - e^{-(x+n+y)^2/4(t-\tau)}}{2\sqrt{\pi(t-\tau)}} \phi(y)\, dy\, d\tau. \tag{3.9}$$

The dilated version $\frac{1}{4}\tilde{\Phi}(2x,n,4t)$ turns out to be a heatlet corresponding to the forcing term $\phi(2x-n)$. In fact,

$$\begin{aligned}\tilde{\Phi}(2x,n,4t) &= \int_0^{4t} \int_0^\infty \frac{e^{-(2x-n-y)^2/4(4t-\tau)} - e^{-(2x+n+y)^2/4(4t-\tau)}}{2\sqrt{\pi(4t-\tau)}} \phi(y)\, dy\, d\tau, \\ &= 4\int_0^{t} \int_0^\infty \frac{e^{-(x-y)^2/4(t-v)} - e^{-(x+y)^2/4(t-v)}}{2\sqrt{\pi(t-v)}} \phi(2y-n)\, dy\, dv.\end{aligned}$$

Similarly, $\tilde{\Psi}(x,n,t), \frac{1}{4}\Psi(2x,n,4t)$ are the heatlet solutions corresponding to the forcing term $\psi(x-n)$ abd $\psi(2x-n)$ respectively. The following proposition gives the two scale relation satisfied by the refinable heat $\tilde{\Phi}$ and the heatlet $\tilde{\Psi}$. □

Proposition 3.2. *Suppose that $\phi(x)$ and $\psi(x)$ satisfy*

$$\phi(x) = 2\sum_{k\in\mathbb{Z}} h_k \phi(2x-k),$$

$$\psi(x) = 2\sum_{k\in\mathbb{Z}} g_k \phi(2x-k),$$

then the refinable heat $\tilde{\Phi}$ and heatlet $\tilde{\Psi}$ satisfy :

$$\tilde{\Phi}(x,0,t) = \frac{1}{2}\sum_{k\in\mathbb{Z}} h_k \Phi(2x,k,4t), \tag{3.10}$$

$$\tilde{\Psi}(x,0,t) = \frac{1}{2}\sum_{k\in\mathbb{Z}} g_k \Phi(2x,k,4t). \tag{3.11}$$

Proof.

$$\begin{aligned}\tilde{\Phi}(x,0,t) &= \int_0^t \int_0^\infty \frac{e^{-(x-y)^2/4(t-\tau)} - e^{-(x+y)^2/4(t-\tau)}}{2\sqrt{\pi(t-\tau)}} \phi(y)\, dy\, d\tau \\ &= 2\sum_{k\in\mathbb{Z}} h_k \int_0^t \int_0^\infty \frac{e^{-(x-y)^2/4(t-\tau)} - e^{-(x+y)^2/4(t-\tau)}}{2\sqrt{\pi(t-\tau)}} \phi(2y-k)\, dy\, d\tau, \\ &= \frac{1}{2}\sum_{k\in\mathbb{Z}} h_k \tilde{\Phi}(2x,k,4t).\end{aligned}$$

(3.11) can be obtained analogously.

We also consider the initial boundary value problems with scaling function and wavelets as the functions prescribing the boundary data :

$$\begin{aligned} &u_t = u_{xx} && x>0, t>0, n\geq 0. \\ &u(x,0) = 0, && x>0, \\ &u(0,t) = \phi(t-n), && t>0, \\ &u\to 0 \text{ as } x\to 0. \end{aligned} \tag{3.12}$$

The solution of (3.12) is given by

$$\begin{aligned}\Phi^*(x,t,n) &= \frac{1}{2\sqrt{\pi}}\int_0^t \frac{x}{(t-s)^{3/2}} e^{-\frac{x^2}{4(t-s)}}\phi(s-n)\,ds,\\ &= \frac{1}{2\sqrt{\pi}}\int_0^{t-n} \frac{xe^{-x^2/4(t-n-s)}}{(t-n-s)^{3/2}}\phi(s)\,ds.\end{aligned} \tag{3.13}$$

Then, we can immediately see that $\Phi^*(x,t,0)$ is the solution of (3.12) with the boundary term as $\phi(x)$.

Now, consider the dilated version $\Phi^*(\sqrt{2}x, 2t, n)$ of $\Phi^*(x,t,n)$. We have,

$$\begin{aligned}\Phi^*(\sqrt{2}x,2t,n) &= \frac{1}{2\sqrt{\pi}}\int_0^{2t-n} \frac{\sqrt{2}xe^{-(\sqrt{2}x)^2/4(2t-n-s)}}{(2t-n-s)^{3/2}}\phi(s)\,ds,\\ &= \frac{1}{2\sqrt{\pi}}\int_0^{t} \frac{xe^{-x^2/4(t-s)}}{(t-s)^{3/2}}\phi(2s-n)\,ds.\end{aligned}$$

Thus, $\Phi^*(\sqrt{2}x, 2t, n)$ is seen to be the solution of (3.12) with the boundary term as $\phi(2t-n)$. □

The following proposition illustrates the refinement relations for Φ^* and Ψ^*.

Proposition 3.3. *Suppose that $\phi(x)$ and $\psi(x)$ satisfy*

$$\phi(t) = 2\sum_{k\in\mathbb{Z}} h_k\phi(2t-k),$$

$$\psi(t) = 2\sum_{k\in\mathbb{Z}} g_k\phi(2t-k),$$

then the refinable heat Φ^* and heatlet Ψ^* satisfy:

$$\Phi^*(x,t,0) = 2\sum_{k\in\mathbb{Z}} h_k\Phi^*(\sqrt{2}x,2t,k), \tag{3.14}$$

$$\Psi^*(x,t,0) = 2\sum_{k\in\mathbb{Z}} g_k\Phi^*(\sqrt{2}x,2t,k). \tag{3.15}$$

Proof.

$$\begin{aligned}\Phi^*(x,t,0) &= \frac{1}{2\sqrt{\pi}}\int_0^t \frac{xe^{-x^2/4(t-s)}}{(t-s)^{3/2}}\phi(s)\,ds,\\ &= 2\sum_{k\in\mathbb{Z}} h_k\frac{1}{2\sqrt{\pi}}\int_0^t \frac{xe^{-x^2/4(t-s)}}{(t-s)^{3/2}}\phi(2s-n)\,ds,\\ &= 2\sum_{k\in\mathbb{Z}} h_k\Phi^*(\sqrt{2}x,2t,k).\end{aligned}$$

Similarly, we can prove that

$$\Psi^*(x,t,0) = 2\sum_{k\in\mathbb{Z}} g_k\Phi^*(\sqrt{2}x,2t,k).$$

□

Remark 3.4. The fundamental solution of the initial boundary value problem when the edge scaling functions $\phi_{0,k}^{left}$ and edge wavelets $\psi_{j,k}^{left}$ are involved, may also be obtained analogously.

Remark 3.5. We compute the heatlets of the problems (3.1),(3.8) and (3.12). Once we compute the wavelet coefficients of the forcing term, initial function and boundary function using wavelet expansion of the form (2.9), we may evaluate the heatlet solution using the linear combination of wavlet coefficients and heatlets.

4. **Numerical Experiments**

Let $\Omega = \{(x,t)/a < x < \infty, 0 < t \le T\}$, and $T = 1$. Let $m(x) = \begin{cases} x, & 0 \le x < 1, \\ 2-x, & 1 \le x < 2, \\ 0, & otherwise. \end{cases}$

Consider the following initial value problems:

$$\begin{aligned} &\frac{\partial u}{\partial t}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t) + f(x), && (x,t) \in \Omega, \\ &u(x,0) = g(x), && a < x < \infty, \\ &u|_{x=a} = h(t), && 0 < t \le T, \\ &\lim_{x\to\infty} u(x,t) = 0. && \end{aligned} \tag{4.1}$$

Example 4.1. We let $a = -1, f(x) = 0, g(x) = 0$ and $h(t) = erfc(1/2\sqrt{t})$ with

$$erfc(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-u^2} du.$$

The exact solution is given by

$$u(x,t) = erfc\left(\frac{x+2}{2\sqrt{t}}\right)$$

Example 4.2. We let $a = 0, f(x) = 0, g(x) = m(x)$ and $h(t) = 0$. The analytical solution is

$$u(x,t) = \frac{1}{2\sqrt{\pi t}} \int_0^2 [e^{-(x-y)^2/4t} - e^{-(x+y)^2/4t}] g(y)\, dy.$$

To obtain the numerical solution of examples (4.1) and (4.2) by using the heatlet decomposition method proposed in Section 3, We begin by decomposing the forcing term $f(x)$, the initial term $g(x)$ and the boundary term $h(t)$ in terms of compactly supported wavelets, namely Haar wavelet and Daubechies wavelet using [1].

Using (3.9), the heatlets for (4.1) with $g(x) = 0$ and $h(t) = 0$ can be written in the form

$$\tilde{\Phi}_k(x,n,t) = \int_0^t \int_{a_k}^{b_k} \frac{e^{-(x-n-y)^2/4(t-\tau)} - e^{-(x+n+y)^2/4(t-\tau)}}{2\sqrt{\pi(t-\tau)}} \phi_k(y)\, dy\, d\tau, \tag{4.6a}$$

and

$$\tilde{\Psi}_{jk}(x,n,t) = \int_0^t \int_{a_{jk}}^{b_{jk}} \frac{e^{-(x-n-y)^2/4(t-\tau)} - e^{-(x+n+y)^2/4(t-\tau)}}{2\sqrt{\pi(t-\tau)}} \psi_{jk}(y)\, dy\, d\tau, \tag{4.6b}$$

where we consider supp $\phi_k(x) = [a_k, b_k]$ and supp $\psi_{jk}(x) = [a_{jk}, b_{jk}]$.

Using (3.3), the heatlets for (4.1) with $f(x) = 0$ and $h(t) = 0$ can be written in the form

$$\Phi_k(x,n,t) = \int_{a_k}^{b_k} \frac{e^{-(x-n-y)^2/4(t)} - e^{-(x+n+y)^2/4(t)}}{2\sqrt{\pi(t)}} \phi_k(y)\, dy, \tag{4.7a}$$

and

$$\Psi_{jk}(x,n,t) = \int_{a_{jk}}^{b_{jk}} \frac{e^{-(x-n-y)^2/4(t)} - e^{-(x+n+y)^2/4(t)}}{2\sqrt{\pi(t)}} \psi_{jk}(y)\, dy, \tag{4.7b}$$

where we consider supp $\phi_k(x) = [a_k, b_k]$ and supp $\psi_{jk}(x) = [a_{jk}, b_{jk}]$.

Using (3.13), the heatlets for (4.1) with $f(x) = 0$ and $g(x) = 0$ can be written in the form using

$$\Phi^*_k(x,t,n) = \frac{1}{2\sqrt{\pi}} \int_0^{t-n} \frac{x e^{-x^2/4(t-n-s)}}{(t-n-s)^{3/2}} \phi_k(s)\, ds \tag{4.8a}$$

$$\Psi^*_{jk}(x,t,n) = \frac{1}{2\sqrt{\pi}} \int_0^{t-n} \frac{x e^{-x^2/4(t-n-s)}}{(t-n-s)^{3/2}} \psi_{jk}(s)\, ds \tag{4.8b}$$

where $\phi_k(s)$ and $\psi_{jk}(s)$ are compactly supported. Analogously, we can obtain the fundamental solutions using edge scaling functions and wavelets.

Using (4.8a) and (4.8b), the approximate solution of example 4.1 at level J is given by

$$\begin{aligned} U_h(x,t) = {} & \sum_{k=0}^{1} c_k^{*l} \Phi_k^{*left}(x,t) \\ & + \sum_{k=2}^{\infty} c_k^* \Phi_k^*(x,t) + \sum_{j\geq 0} \sum_{k=0}^{1} d_{j,k}^{*l} \Psi_{jk}^{*left}(x,t) + \sum_{k=2}^{\infty} d_{j,k}^* \Psi_{jk}^*(x,t). \end{aligned} \tag{4.9}$$

where $c_k^{*l}, c_k^*, d_{j,k}^{*l}$ and $d_{j,k}^*$ are the coefficients of the wavelet expansion of $h(t) \in L^2([0,\infty))$ given by

$$h(t) = \sum_{k=0}^{1} c_k^{*l} \phi_{0,k}^{left}(t) + \sum_{k=2}^{\infty} c_k^* \phi_{0,k}(t) + \sum_{j\geq 0} \sum_{k=0}^{1} d_{j,k}^{*l} \psi_{jk}^{left}(t) + \sum_{k=2}^{\infty} d_{j,k}^* \psi_{jk}(t).$$

Similarly, the approximate solution of example 4.2 at level J is given by

$$\begin{aligned} U_h(x,t) = {} & \sum_{k=0}^{1} c_k^{l} \Phi_k^{left}(x,t) + \sum_{k=2}^{\infty} c_k \Phi_k(x,t) \\ & + \sum_{j\geq 0} \sum_{k=0}^{1} d_{j,k}^{l} \Psi_{jk}^{left}(x,t) + \sum_{k=2}^{\infty} d_{j,k} \Psi_{jk}(x,t). \end{aligned} \tag{4.10}$$

Table 1. The relative error between the analytical solution and the numerical solution of (4.1) using finite difference and finite element methods.

M	FEM $\|u(x,1)-u_h(x,1)\|_{\infty,\Omega_B}$	FDM $\|u(x,1)-u_h(x,1)\|_{\infty,\Omega_B}$
8	6.4759E-02	1.0610E-01
16	3.2323E-02	5.2624E-02
32	1.6158E-02	2.6650E-02
64	8.0788E-03	1.3338E-02

where $c_k^l, c_k, d_{j,k}^l$ and $d_{j,k}$ are the coefficients of the wavelet expansion of $g(x) \in L^2([0,\infty))$ given by

$$g(x) = \sum_{k=0}^{1} c_k^l \phi_{0,k}^{left}(x) + \sum_{k=2}^{\infty} c_k \phi_{0,k}(x) + \sum_{j\geq 0}\sum_{k=0}^{1} d_{j,k}^l \psi_{jk}^{left}(x) + \sum_{k=2}^{\infty} d_{j,k}\psi_{jk}(x).$$

We now compute the simplest heatlets namely, the Haar heatlets and the Daubechies heatlets for the homogeneous and the non-homogeneous problem.

Remark 4.3. Haar heatlets are easy to compute in semi-infinite domain problems because, they are adaptable to $L^2([0,\infty))$. The closed form solution exist in initial boundary value problems. To compute Haar heatlets, we use Haar wavelets as the forcing term, the initial condition and the boundary condition. In example 4.1 we replace $h(t)$ by the Haar scaling function and Haar wavelet and its translated and dilated versions to evaluate the heatlets. In example 4.2 we replace $g(x)$ by ϕ_k's and ψ_{jk}'s and compute the corresponding heatlets. The basic Haar heatlets namely, the fundamental solution of example 4.2 with $g(x) = \phi(x)$ and $g(x) = \phi(x)$ are given by,

$$\Phi(x,t) = \frac{1}{2}\left[2erf\left(\frac{x}{2\sqrt{t}}\right) - erf\left(\frac{x-1}{2\sqrt{t}}\right) - erf\left(\frac{x+1}{2\sqrt{t}}\right)\right]. \tag{4.11}$$

$$\Psi(x,t) = \frac{1}{2}\Bigg[2erf\left(\frac{x}{2\sqrt{t}}\right) + erf\left(\frac{x-1}{2\sqrt{t}}\right) + erf\left(\frac{x+1}{2\sqrt{t}}\right) - 2erf\left(\frac{x-\frac{1}{2}}{2\sqrt{t}}\right) - 2erf\left(\frac{x+\frac{1}{2}}{2\sqrt{t}}\right)\Bigg]. \tag{4.12}$$

The Relative error and L^2 errors of examples 4.1 and 4.2 are tabulated in the following tables. The results guarantee the theoretical findings of heatlet approach.

Remark 4.4. To compute Daubechies heatlet, we use the edge wavelet function defined in Section 2. First, we generate the values of $\phi(x)$ and $\psi(x)$ for Daubecheis scaling function and wavelets. Using this, we generate the values of the edge wavelet

TABLE 2. The relative error between the analytical solution and the numerical solution of example (4.2) using finite difference and finite element methods

M	**FEM** $\|u(x,1) - u_h(x,1)\|_{\infty,\Omega_B}$	**FDM** $\|u(x,1) - u_h(x,1)\|_{\infty,\Omega_B}$
8	0.030357674	0.099744704
16	0.009942447	0.079978112
32	0.00332749	0.052703447
64	0.001207538	0.030688943
128	0.00048942	0.016636544

TABLE 3. The Relative error between the analytical solution and the numerical solution of example 4.1 using heatlets (Haar Wavelet).

M	**Level 1**	**Level 4**	**Level 7**	**Level 10**
8	0.018331328	0.000388596	1.86168E-06	2.578E-07
16	0.022194574	0.000652724	1.11096E-05	3.27685E-07
32	0.024385093	0.000824447	2.40819E-05	4.16652E-07
64	0.026198414	0.000944114	3.40535E-05	1.09474E-06
128	0.026154747	0.000971748	3.84792E-05	1.57021E-06

TABLE 4. The Relative error between the analytical solution and the numerical solution of example 4.2 using heatlets (Haar Wavelet).

M	**Level 1**	**Level 2**	**Level 4**	**Level 8**
8	0.004596907	0.001147958	7.17522E-05	3.12224E-07
16	0.004672466	0.001166804	7.2931E-05	3.18728E-07
32	0.004908645	0.001225769	7.66171E-05	3.35793E-07
64	0.004728118	0.001180684	7.37992E-05	3.23531E-07
128	0.004737323	0.00118298	7.39428E-05	3.24326E-07

TABLE 5. The Relative error between the analytical solution and the numerical solution of example 4.2 using heatlets (Daubechies Wavelet)

M	**Level 1**	**Level 2**	**Level 4**	**Level 8**
8	0.004201508	0.004201508	0.004198458	0.004198447
16	0.004227985	0.00417866	0.004175508	0.004175496
32	0.004217501	0.004167406	0.004164203	0.004164191
64	0.004212293	0.004161815	0.004158586	0.004158574
128	0.004209697	0.004159027	0.004155786	0.004155774

TABLE 6. The L^2 error using Heatlets (Haar wavelet) of example 4.1.

M	Level 1	Level 4	Level 7	Level 10
8	0.000192005	2.61975E-07	4.75682E-12	1.93542E-14
16	0.000231043	6.62851E-07	4.14124E-10	3.84362E-14
32	0.000250719	9.07421E-07	2.51848E-09	1.22386E-13
64	0.000242268	8.31242E-07	2.32117E-09	1.90351E-12
128	0.000265491	1.09592E-06	5.23552E-09	1.91286E-11

TABLE 7. L^2 error using Heatlets (Haar wavelet) for the example 4.2.

M	Level 1	Level 2	Level 4	Level 8
8	9.70527E-07	6.05202E-08	2.36446E-10	4.54869E-15
16	9.70464E-07	6.05162E-08	2.36431E-10	4.54851E-15
32	9.50883E-07	5.92936E-08	2.31658E-10	4.48175E-15
64	9.7046E-07	6.0516E-08	2.3643E-10	4.54849E-15
128	9.7046E-07	6.0516E-08	2.3643E-10	4.54849E-15

TABLE 8. L^2 error using Heatlets (Daubechies wavelet) for the example 4.2.

M	Level 1	Level 2	Level 4	Level 8
8	7.95005E-07	7.95005E-07	7.93901E-07	7.93897E-07
16	8.12442E-07	7.94989E-07	7.93885E-07	7.93881E-07
32	7.66668E-07	7.49521E-07	7.48436E-07	7.48431E-07
64	8.12441E-07	7.94988E-07	7.93884E-07	7.9388E-07
128	8.12441E-07	7.94988E-07	7.93884E-07	7.9388E-07

TABLE 9. The approximate Heatlet solution using Daubechies wavelet at $x = 0$ and $T = 1$ of example 4.1.

t	Anal. soln.	Level 1	Level 2	Level 3	Level 4
1	0.157299207	0.1571457	0.157271851	0.157271827	0.157271467
1/2	0.045500264	0.045001518	0.045474475	0.045499666	0.04550116
1/4	0.004677735	0.005392516	0.00475447	0.00469328	0.004681107
1/8	6.33E-05	-3.68E-04	9.54E-06	3.88E-05	6.26E-05

functions $\phi_k^{left}, \psi_{jk}^{left}$ as mentioned in Section 2. Using the translated and dilated versions of generated values, we evaluate the heatlet solutions at different levels. In Table 5, we tabulated the relative errors of example 4.2 at different levels. In Table 9, we tabulated the heatlet solutions of example 4.1 at different times using Daubechies heatlets.

Remark 4.5. We compute the approximate solutions of the problems given in Examples 4.1 and 4.2, using the finite difference methods (with crank nicolson scheme) and finite element methods. The procedure followed to obtain the numerical solutions is analogus to [6]. We compare the results obtained with the heatlet solutions.

From the tables, we can clearly observe that the Haar heatlet solutions have the order of convergence of 2^{-J} at level J, which is in agreement with the theoretical order of convergence. The order of convergence of Daubechies heatlet is not as good as the Haar heatlet, because of the numerical evaluation of the integral representation using simpson's rule. We can improve the solution by using suitable quadrature rule for Daubechies heatlets.

The advantage of using the heatlet approach is, we can evaluate the numerical solution at any given level without computing the previous level solutions. We also observe that the error decreases as we increase the level. Also, the solution is independent of the number of grid points we consider in space variable as well as the time variable. This allows us to compute the solution at any given point and at any given time without computing the solution at previous grid points. The same heatlets are used for problems with different forcing terms and different initial conditions. Therefore, one needs to have only the wavelet decomposition of the correspoinding initial function and forcing term. Once such a library of heatlets is built, the only computational cost involved in finding the numerical solution of the given problem is the computation of wavelet coefficients corresponding to the forcing and initial function.

The convergence of wavelet expansion of f and g are in L^2 sense, So, we computed the L^2 errors of the solutions and from Table 6 to Table 8 we can see that the L^2 error decreases as we increase the level.

5. ACKNOWLEDGMENTS

This work was initiated during the visit of Dr. Neela Nataraj to the Department of Mathematical Sciences, Florida Institute of Technology and Dr. Nataraj thanks the Department of Mathematical Sciences for the hospitality provided to her.

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