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ARITHMETIC AVERAGE GEOMETRIC MESH DISCRETIZATIONS FOR FOURTH AND SIXTH ORDER NONLINEAR TWO POINT BOUNDARY VALUE PROBLEMS

R. K. MOHANTY[1,2], NAVNIT JHA[1], AND VINOD CHAUHAN[2]

[1]Department of Mathematics, South Asian University, Akbar Bhawan,
Chanakya Puri, New Delhi 110021, India
rmohanty@sau.ac.in
[2]Department of Mathematics, Faculty of Mathematical Sciences,
University of Delhi, Delhi 110007, India

ABSTRACT. A third order accurate numerical method is developed for solving fourth and sixth order nonlinear ordinary differential equations with associated boundary conditions. Method is compact and uses one central and two off step geometric grids. Arithmetic average finite difference approximations have been applied for deriving new numerical scheme. The method can be easily extended to the high order (even) differential equations. The error analysis of the method has been analyzed briefly. The resulting difference equations leads to block tri-diagonal matrices and can be easily solved using block gauss-seidel algorithm. The numerical experiments with several singular and non-singular problems are conducted using proposed method. The computational results justify the reliability and efficiency of the method both in terms of order and accuracy.

Keywords. Arithmetic average, finite differences, geometric mesh, stiff problem, Navier-Stokes equation, root mean square errors

AMS (MOS) Subject Classification. 65L12, 65L10, 34K28

1. INTRODUCTION

We discuss an arithmetic average geometric mesh discretization for the numerical solution of general nonlinear fourth order boundary value problem

$$u^{iv}(r) = \varphi(r, u(r), u'(r), u''(r), u'''(r)), \quad \alpha < r < \beta \tag{1.1}$$

subject to the boundary conditions

$$u(\alpha) = \alpha_0, \quad u''(\alpha) = \alpha_1, \quad u(\beta) = \beta_0, \quad u''(\beta) = \beta_1 \tag{1.2}$$

where $\varphi \in C^4[\alpha, \beta]$ and α_0, α_1, β_0, $\beta_1$are real constants and $-\infty < \alpha \leq x \leq \beta < \infty$.

The equations (1.1)–(1.2) can be expressed in coupled form as

$$\begin{cases} u''(r) = v(r) \\ v''(r) = \varphi(r, u(r), u'(r), v(r), v'(r)), \quad \alpha < r < \beta \end{cases} \tag{1.3}$$

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subject to the natural boundary conditions

$$u(\alpha) = \alpha_0, \quad v(\alpha) = \alpha_1, \quad u(\beta) = \beta_0, \quad v(\beta) = \beta_1 \tag{1.4}$$

Higher order ordinary differential equations play an important role in various areas of mathematics and engineering. The mathematical modeling in beam theory [1], plate defection theory [2, 3], continuum mechanics [4], geological folding of rock layers [5], theory of plates and shell [6], waves on a suspension bridge [7, 8], reaction diffusion equation [9] etc. are some of the modeling problems interesting to mathematicians and physicists.

The analytical solution of (1.1)–(1.2) for arbitrary choice of φ is difficult and hence we resort to economical numerical method. The existence and uniqueness of the solutions of fourth and higher order boundary value problems has been established in [10, 11, 12]. Zahra [13] developed exponential spline basis for the numerical solution of the problems $u^{(4)} = \varphi(r, u)$ that shows improvement in the finite difference method discussed by Usmani [14]. The nonpolynomial spline method for $u^{(2m)} = \varphi(r)u + \psi(r)$ has been examined by Ramadan et al. [15] and superiority over usual spline (Siddiqi [16]) has been established. Usmani et al. [17] discussed the approximations of numerical solution to the self adjoint fourth order differential equations using finite difference scheme. Recently Rashidinia et al. [18] developed B-spline collocation function for the numerical solution to nonlinear two point boundary value problems of order up to six.

The geometric mesh technique gains its importance from the theory of electrochemical reaction-convection-diffusion problems in one-dimensional space geometry [19]. Jain et al. [20] formulated the finite difference variable mesh approximations for two point singular perturbation problems. The application of geometric mesh in the context of second order ordinary differential equations with Dirichlet's boundary conditions was studied extensively in [21, 22, 23, 24]. In this article, we derive a geometric mesh finite difference method using arithmetic average discretizations for the numerical solution of fourth and sixth order two point boundary value problems. The simplicity of the proposed method lies in its three point discretization without any use of fictitious nodes. The scheme is inherently compact and hence no special treatment is required for singular problems. The resulting systems of algebraic equations are solved using block gauss-seidel method obtained from the discretizations of linear differential equations. The classical Newton's method has been applied to nonlinear coupled difference equations.

The article is organized as follows: Section 2 is devoted to the derivation of the method for coupled nonlinear equation; In section 3, we extend the method to sixth order two point boundary value problems and their algorithmic details are given for the computer implementations. The error analysis for the canonical form of the fourth order two point boundary value problems has been discussed briefly in section 4. The computational tests based on geometric mesh and uniform mesh is provided in section 5. Finally the article is concluded in the last with observations and findings.

2. ARITHMETIC AVERAGE GEOMETRIC MESH DISCRETIZATION

We introduce a finite set of non-uniform grid points $\alpha = r_0 < r_1 < \cdots < r_n < r_{n+1} = \beta$ for the solution region $[\alpha, \beta]$. Let $h_k = r_k - r_{k-1}$, $k = 1(1)n+1$ be the non-uniform step size and $\delta_k = h_{k+1}/h_k > 0$ be the geometric mesh ratio. Let $U_k = u(r_k)$ and $V_k = v(r_k)$ be the exact solution values of u and v at the mesh r_k and u_k and v_k be their approximate solution respectively. Consider the following three point geometric mesh discretizations for $u^{(2)}(r) = v(r)$ and $v^{(2)}(r) = \varphi(r)$:

$$U_{k+1} - (1+\delta_k)U_k + \delta_k U_{k-1} - \frac{h_k^2\delta_k}{6}(2\delta_k V_{k+1/2} + (1+\delta_k)V_k + 2V_{k-1/2}) = \mathrm{O}(h_k^5), \tag{2.1}$$

$$V_{k+1} - (1+\delta_k)V_k + \delta_k V_{k-1} - \frac{h_k^2\delta_k}{6}(2\delta_k \varphi_{k+1/2} + (1+\delta_k)\varphi_k + 2\varphi_{k-1/2}) = E_k, \tag{2.2}$$

where

$$E_k = \frac{h_k^5}{720}\delta_k(\delta_k^4 - 1)\varphi_k^v(\zeta) + \mathrm{O}(h_k^6), \quad r_{k-1} < \zeta < r_{k+1}, \quad k = 1(1)n$$

We define the following arithmetic average approximations:

$$\widetilde{U}_{k\pm1/2} = \frac{1}{2}(U_{k\pm1} + U_k), \tag{2.3}$$

$$\widetilde{V}_{k\pm1/2} = \frac{1}{2}(V_{k\pm1} + V_k), \tag{2.4}$$

$$\widetilde{U}'_{k+1/2} = \frac{1}{h_{k+1}}(U_{k+1} - U_k), \tag{2.5}$$

$$\widetilde{V}'_{k+1/2} = \frac{1}{h_{k+1}}(V_{k+1} - V_k), \tag{2.6}$$

$$\widetilde{U}'_{k-1/2} = \frac{1}{h_k}(U_k - U_{k-1}), \tag{2.7}$$

$$\widetilde{V}'_{k-1/2} = \frac{1}{h_k}(V_k - V_{k-1}), \tag{2.8}$$

$$\widetilde{U}'_k = \frac{1}{h_k\delta_k(1+\delta_k)}(U_{k+1} - (1-\delta_k^2)U_k - \delta_k^2 U_{k-1}), \tag{2.9}$$

$$\widetilde{V}'_k = \frac{1}{h_k\delta_k(1+\delta_k)}(V_{k+1} - (1-\delta_k^2)V_k - \delta_k^2 V_{k-1}), \tag{2.10}$$

$$\widetilde{\varphi}_{k\pm1/2} = \varphi(r_{k\pm1/2}, U_{k\pm1/2}, \widetilde{U}'_{k\pm1/2}, V_{k\pm1/2}, \widetilde{V}'_{k\pm1/2}), \tag{2.11}$$

It is easy to verify that

$$\widetilde{\varphi}_{k+1/2} = \varphi_{k+1/2} + \frac{h_k^2\delta_k^2}{24}(3A_kU_k'' + B_kU_k''' + 3C_kV_k'' + D_kV_k''') + \mathrm{O}(h_k^3), \tag{2.12}$$

$$\widetilde{\varphi}_{k-1/2} = \varphi_{k-1/2} + \frac{h_k^2}{24}(3A_kU_k'' + B_kU_k''' + 3C_kU_k'' + D_kU_k''') + \mathrm{O}(h_k^3), \tag{2.13}$$

where $A_k = \left.\frac{\partial\varphi}{\partial u}\right|_{r_k}$, $B_k = \left.\frac{\partial\varphi}{\partial u'}\right|_{r_k}$, $C_k = \left.\frac{\partial\varphi}{\partial v}\right|_{r_k}$, $D_k = \left.\frac{\partial\varphi}{\partial v'}\right|_{r_k}$, etc.

Now, define

$$\widehat{U}_k = U_k + \mu_1 h_k^2(\widetilde{V}_{k+1/2} + \widetilde{V}_{k-1/2}), \tag{2.14}$$

$$\widehat{V}_k = V_k + \mu_2 h_k^2(\widetilde{\varphi}_{k+1/2} + \widetilde{\varphi}_{k-1/2}), \tag{2.15}$$

$$\widehat{U}_k' = \widetilde{U}_k' + \mu_3 h_k(\widetilde{V}_{k+1/2} - \widetilde{V}_{k-1/2}), \tag{2.16}$$

$$\widehat{V}_k' = \widetilde{V}_k' + \mu_4 h_k(\widetilde{\varphi}_{k+1/2} - \widetilde{\varphi}_{k-1/2}), \tag{2.17}$$

where μ_l, $l = 1(1)4$ are free parameters to be determined.

With the help of equations (2.14)–(2.17) and (2.3)–(2.4), it follows that

$$\widehat{U}_k = U_k + 2\mu_1 h_k^2 U_k'' + \frac{h_k^3}{2}(\delta_k - 1)\mu_1 U_k''' + \mathrm{O}(h_k^4) \tag{2.18}$$

$$\widehat{V}_k = V_k + 2\mu_2 h_k^2 V_k'' + \frac{h_k^3}{2}(\delta_k - 1)\mu_2 V_k''' + \mathrm{O}(h_k^4) \tag{2.19}$$

$$\widehat{U}_k' = U_k' + \frac{h_k^2}{6}(\delta_k + 3(\delta_k + 1)\mu_3)U_k''' + \mathrm{O}(h_k^3) \tag{2.20}$$

$$\widehat{V}_k' = V_k' + \frac{h_k^2}{6}(\delta_k + 3(\delta_k + 1)\mu_4)V_k''' + \mathrm{O}(h_k^3) \tag{2.21}$$

Further, we define

$$\widehat{\varphi}_k = \varphi(r_k, \widehat{U}_k, \widehat{U}_k', \widehat{V}_k, \widehat{V}_k'), \tag{2.22}$$

With the help of equations (2.18)–(2.21), it follows that

$$\begin{aligned}\widehat{\varphi}_k = \varphi_k + \frac{h_k^2}{6}\big((\sigma_k + 3\gamma_3(\sigma_k + 1))B_kU_k''' + (\sigma_k + 3\gamma_4(\sigma_k + 1))D_kV_k''' \\ + 2(A_k\gamma_1U_k'' + C_k\gamma_2V_k'')\big) + \mathrm{O}(h_k^3)\end{aligned} \tag{2.23}$$

Then, at each internal grid r_k, $k = 1(1)n$, the differential equations (1.3)–(1.4) are approximated by

$$U_{k+1} - (1+\delta_k)U_k + \delta_kU_{k-1} - \frac{h_k^2\delta_k}{6}(2\delta_k\widetilde{V}_{k+1/2} + (1+\delta_k)\widehat{V}_k + 2\widetilde{V}_{k-1/2}) = \mathrm{O}(h_k^5), \tag{2.24}$$

$$V_{k+1} - (1+\delta_k)V_k + \delta_kV_{k-1} - \frac{h_k^2\delta_k}{6}(2\delta_k\widetilde{\varphi}_{k+1/2} + (1+\delta_k)\widehat{\varphi}_k + 2\widetilde{\varphi}_{k-1/2}) = \widetilde{E}_k, \tag{2.25}$$

$U_0 = \alpha_0$, $U_{n+1} = \beta_0$, $V_0 = \alpha_1$, $V_{n+1} = \beta_1$, $k = 1(1)n$.

With the help of approximation (2.12), (2.13), (2.23) and the local truncation error terms E_k in (2.2), we obtain

$$\begin{aligned}\widetilde{E}_k = &-\frac{h_k^4}{24}\delta_k(1+\delta_k)((\delta_k^2-\delta_k+8\mu_1+1)A_kU_k^{(2)}+(\delta_k^2-\delta_k+8\mu_2+1)C_kV_k^{(2)})\\ &-\frac{h_k^4}{72}\delta_k(1+\delta_k)\big((\delta_k^2+\delta_k+6\mu_3\delta_k+6\mu_3+1)B_kU_k^{(2)}\\ &+(\delta_k^2+\delta_k+6\mu_4\delta_k+6\mu_4+1)D_kV_k^{(2)}\big)+(\delta_k-1)\mathrm{O}(h_k^5),\end{aligned}$$

The difference scheme (2.24) to be of $\mathrm{O}(h_k^3)$, the coefficients of h_k^4 in $\widetilde{E}_k$ must be zero and hence we obtain

$$\delta_k(\delta_k-1)+8\mu_l+1=0,\quad l=1,2, \tag{2.26}$$

$$(\delta_k+6\mu_l)(\delta_k+1)+1=0,\quad l=3,4, \tag{2.27}$$

Thus, the values of free parameters associated with equations (2.26)–(2.27) are given by

$$\mu_1=\mu_2=-\frac{1}{8}(\delta_k^2-\delta_k+1)\ \text{ and }\ \mu_3=\mu_4=-\frac{\delta_k^2+\delta_k+1}{6(\delta_k+1)},$$

and hence the local truncation errors reduces to $\widetilde{E}_k=\mathrm{O}(h_k^5)$, for $\sigma_k\neq 1$. Note that, if $\sigma_k=1$ i.e. for uniform mesh discretizations, the error becomes $\widetilde{E}_k=\mathrm{O}(h_k^6)$.

3. ALGORITHM FOR SOLVING SIXTH ORDER DIFFERENTIAL EQUATIONS

The proposed method can be easily be extended to the non linear sixth order differential equations

$$u^{vi}(r)=\varphi(r,u(r),u'(r),u''(r),u'''(r),u^{iv}(r),u^{v}(r)),\quad \alpha<r<\beta \tag{3.1}$$

subject to the necessary boundary conditions

$$u(\alpha)=\alpha_0,\ \ u''(\alpha)=\alpha_1,\ \ u^{iv}(\alpha)=\alpha_2,\quad u(\beta)=\beta_0,\ \ u''(\beta)=\beta_1,\ \ u^{iv}(\beta)=\beta_2 \tag{3.2}$$

or equivalently,

$$\begin{cases}u''(r)=v(r)\\ v''(r)=w(r)\\ w''(r)=\varphi(r,u,u'(r),v(r),v'(r),w(r),w'(r)),\quad \alpha<r<\beta\end{cases} \tag{3.3}$$

subject to the natural boundary conditions

$$u(\alpha)=\alpha_0,\ \ v(\alpha)=\alpha_1,\ \ w(\alpha)=\alpha_2,\ \ u(\beta)=\beta_0,\ \ v(\beta)=\beta_1,\ \ w(\beta)=\beta_2 \tag{3.4}$$

The derivation of the geometric mesh arithmetic average scheme for sixth order problem is similar to the fourth order two point boundary value problems discussed in section 2. Here, we only outline the algorithmic details for the equations (3.1)–(3.2):

$$\widetilde{U}_{k\pm1/2} = \frac{1}{2}(U_{k\pm1} + U_k), \tag{3.5}$$

$$\widetilde{V}_{k\pm1/2} = \frac{1}{2}(V_{k\pm1} + V_k), \tag{3.6}$$

$$\widetilde{W}_{k\pm1/2} = \frac{1}{2}(W_{k\pm1} + W_k), \tag{3.7}$$

$$\widetilde{U}'_{k+1/2} = \frac{1}{h_{k+1}}(U_{k+1} - U_k), \tag{3.8}$$

$$\widetilde{V}'_{k+1/2} = \frac{1}{h_{k+1}}(V_{k+1} - V_k), \tag{3.9}$$

$$\widetilde{W}'_{k+1/2} = \frac{1}{h_{k+1}}(W_{k+1} - W_k), \tag{3.10}$$

$$\widetilde{U}'_{k-1/2} = \frac{1}{h_k}(U_k - U_{k-1}), \tag{3.11}$$

$$\widetilde{V}'_{k-1/2} = \frac{1}{h_k}(V_k - V_{k-1}), \tag{3.12}$$

$$\widetilde{W}'_{k-1/2} = \frac{1}{h_k}(W_k - W_{k-1}), \tag{3.13}$$

$$\widetilde{U}'_k = \frac{1}{h_k\delta_k(1+\delta_k)}(U_{k+1} - (1-\delta_k^2)U_k - \delta_k^2 U_{k-1}), \tag{3.14}$$

$$\widetilde{V}'_k = \frac{1}{h_k\delta_k(1+\delta_k)}(V_{k+1} - (1-\delta_k^2)V_k - \delta_k^2 V_{k-1}), \tag{3.15}$$

$$\widetilde{W}'_k = \frac{1}{h_k\delta_k(1+\delta_k)}(W_{k+1} - (1-\delta_k^2)W_k - \delta_k^2 W_{k-1}), \tag{3.16}$$

$$\widetilde{\varphi}_{k\pm1/2} = \varphi(r_{k\pm1/2}, U_{k\pm1/2}, \widetilde{U}'_{k\pm1/2}, V_{k\pm1/2}, \widetilde{V}'_{k\pm1/2}, W_{k\pm1/2}, \widetilde{W}'_{k\pm1/2}), \tag{3.17}$$

$$\widehat{U}_k = U_k - \frac{h_k^2}{8}(\delta_k^2 - \delta_k + 1)(\widetilde{V}_{k+1/2} - \widetilde{V}_{k-1/2}), \tag{3.18}$$

$$\widehat{V}_k = V_k - \frac{h_k^2}{8}(\delta_k^2 - \delta_k + 1)(\widetilde{W}_{k+1/2} - \widetilde{W}_{k-1/2}), \tag{3.19}$$

$$\widehat{W}_k = W_k - \frac{h_k^2}{8}(\delta_k^2 - \delta_k + 1)(\widetilde{\varphi}_{k+1/2} - \widetilde{\varphi}_{k-1/2}), \tag{3.20}$$

$$\widehat{U}'_k = \widetilde{U}'_k - \frac{h_k(\delta_k^2+\delta_k+1)}{6(\delta_k+1)}(\widetilde{V}_{k+1/2} - \widetilde{V}_{k-1/2}), \tag{3.21}$$

$$\widehat{V}'_k = \widetilde{V}'_k - \frac{h_k(\delta_k^2+\delta_k+1)}{6(\delta_k+1)}(\widetilde{W}_{k+1/2} - \widetilde{W}_{k-1/2}), \tag{3.22}$$

$$\widehat{W}'_k = \widetilde{W}'_k - \frac{h_k(\delta_k^2+\delta_k+1)}{6(\delta_k+1)}(\widetilde{\varphi}_{k+1/2} - \widetilde{\varphi}_{k-1/2}), \tag{3.23}$$

$$\widehat{\varphi}_k = \varphi(r_k, \widehat{U}_k, \widehat{U}'_k, \widehat{V}_k, \widehat{V}'_k, \widehat{W}_k, \widehat{W}'_k), \tag{3.24}$$

Then, for $k = 1(1)n$, the $\mathrm{O}(h_k^3)$-approximations for (3.1)–(3.2) or (3.3)–(3.4) can be obtained by the following relations:

$$U_{k+1} - (1+\delta_k)U_k + \delta_k U_{k-1} - \frac{h_k^2\delta_k}{6}(2\delta_k\widetilde{V}_{k+1/2} + (1+\delta_k)\widehat{V}_k + 2\widetilde{V}_{k-1/2}) = \mathrm{O}(h_k^5), \tag{3.25}$$

$$V_{k+1} - (1+\delta_k)V_k + \delta_k V_{k-1} - \frac{h_k^2\delta_k}{6}(2\delta_k\widetilde{W}_{k+1/2} + (1+\delta_k)\widehat{W}_k + 2\widetilde{W}_{k-1/2}) = \mathrm{O}(h_k^5), \tag{3.26}$$

$$W_{k+1} - (1+\delta_k)W_k + \delta_k W_{k-1} - \frac{h_k^2\delta_k}{6}(2\delta_k\widetilde{\varphi}_{k+1/2} + (1+\delta_k)\widehat{\varphi}_k + 2\widetilde{\varphi}_{k-1/2}) = \mathrm{O}(h_k^5), \tag{3.27}$$

The boundary conditions (3.2) or (3.4) are used to obtain values at $k \pm 1$ for $k = 1$ and n respectively. The numerical scheme may be implemented by neglecting $\mathrm{O}(h_k^5)$ terms from the equations (3.25)–(3.27). The resulting difference equations in case of linear boundary value problems gives a $3n \times 3n$ linear block tri-diagonal system of equations for the unknowns U_k, V_k, W_k, $k = 1(1)n$ and can be easily solved using block gauss seidel method. For the convergence, δ_k must be positive (Chawla et al. [25]).

4. CONVERGENCE THEORY

In this section, we derive the difference scheme of the model problem and investigate the convergence property of the proposed method. Consider the problem

$$u^{iv}(r) = a(r)u(r) + g(r), \quad \alpha < r < \beta \tag{4.1}$$

along with the appropriate boundary conditions (1.2).

Now applying the method (2.24)–(2.25) to the equation (4.1), we obtain the following system of difference equations in matrix vector notations

$$P_k Z_{k-1} + Q_k Z_k + R_k Z_{k+1} = S_k + T_k(h_k), \quad k = 1(1)n \tag{4.2}$$

where

$$P_k = \begin{bmatrix} -\delta_k - \dfrac{h_k^4}{96}\delta_k(1+\delta_k^3)a_{k-1/2} & \dfrac{h_k^2}{6}\delta_k \\ \dfrac{h_k^2}{6}\delta_k a_{k-1/2} & -\delta_k - \dfrac{h_k^4}{96}\delta_k(1+\delta_k^3)a_k \end{bmatrix}$$

$$Q_k = \begin{bmatrix} 1+\delta_k - \dfrac{h_k^4}{96}\delta_k(1+\delta_k^3)(a_{k+1/2}+a_{k-1/2}) & \dfrac{h_k^2}{3}\delta_k(1+\delta_k) \\ \dfrac{h_k^2}{6}(\delta_k^2 a_{k+1/2} + \delta_k a_{k-1/2} + \delta_k(1+\delta_k)a_k) & 1+\delta_k - \dfrac{h_k^4}{48}\delta_k(1+\delta_k^3)a_k \end{bmatrix}$$

$$R_k = \begin{bmatrix} -1 - \dfrac{h_k^4}{96}(1+\delta_k^3)a_{k+1/2} & \dfrac{h_k^2}{6}\delta_k^2 \\ \dfrac{h_k^2}{6}\delta_k^2 a_{k+1/2} & -1 - \dfrac{h_k^4}{96}\delta_k(1+\delta_k^3)a_k \end{bmatrix}$$

$$S_k = \begin{bmatrix} \dfrac{h_k^4}{48}\delta_k(1+\delta_k^3)(g_{k+1/2}+g_{k-1/2}) \\ -\dfrac{h_k^2}{6}((2g_{k+1/2}+g_k)\delta_k^2 + (2g_{k-1/2}+g_k)\delta_k) \end{bmatrix}$$

$$Z_k = [U_k, V_k]^{\mathrm{T}} \quad \text{and} \quad T_k(h_k) = \mathrm{O}(h_k^5)$$

Incorporating the boundary values $U_0 = \alpha_0$, $V_0 = \alpha_1$, $U_{n+1} = \beta_0$ and $V_{n+1} = \beta_1$, the system of difference equations (4.2) in the matrix-vector form can be written as

$$MZ = J + T_k(h_k), \tag{4.3}$$

where $M = \begin{bmatrix} P_k & Q_k & R_k \end{bmatrix}$ is the block tri-diagonal matrix, $J = [S_1 - P_1\alpha, S_2, \dots, S_{n-1}, S_n - R_n\beta]^{\mathrm{T}}$, $\alpha = [\alpha_0, \alpha_1]^{\mathrm{T}}$ and $\beta = [\beta_0, \beta_1]^{\mathrm{T}}$.

Let $z_k = [u_k, v_k]^{\mathrm{T}}$, $k = 1(1)n$ and $z = [z_1, \dots, z_n] \cong Z$, which satisfies

$$Mz = J \tag{4.4}$$

Let $\varepsilon = [\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n]^{\mathrm{T}}$ be the discretization error vector and $\varepsilon_k = Z_k - z_k$, $k = 1(1)n$ be the discretization errors at the node r_k. Subtracting (4.4) from (4.3), we obtain the error equation

$$M\varepsilon = T_k(h_k) \tag{4.5}$$

Let

$$\widetilde{a} = \max_k\{a_k, a_{k+1/2}, a_{k-1/2}\}, \qquad \underset{\sim}{a} = \min_k\{a_k, a_{k+1/2}, a_{k-1/2}\},$$
$$\widetilde{g} = \max_k\{g_k, g_{k+1/2}, g_{k-1/2}\}, \qquad \underset{\sim}{g} = \min_k\{g_k, g_{k+1/2}, g_{k-1/2}\},$$

Then, we obtain

$$\|P_k\|_\infty \le \max_k \begin{cases} \delta_k + \dfrac{h_k^2}{6}\delta_k + \dfrac{h_k^4}{96}\delta_k(1+\delta_k^3)|\widetilde{a}| \\ \delta_k + \dfrac{h_k^2}{6}\delta_k|\widetilde{a}| + \dfrac{h_k^4}{96}\delta_k(1+\delta_k^3)|\widetilde{a}| \end{cases}$$

$$\|R_k\|_\infty \le \max_k \begin{cases} 1 + \dfrac{h_k^2}{6}\delta_k^2|\widetilde{a}| + \dfrac{h_k^4}{96}\delta_k(1+\delta_k^3)|\widetilde{a}| \\ 1 + \dfrac{h_k^2}{6}\delta_k^2 + \dfrac{h_k^4}{96}\delta_k(1+\delta_k^3)|\widetilde{a}| \end{cases}$$

Thus for sufficiently small h_k, or equivalently as $h_k \to 0$, we obtain the relations $\|P_k\|_\infty \le \delta_k$, $k = 2(1)n$ and $\|R_k\|_\infty \le 1$, $k = 1(1)n-1$. Moreover, the graph of matrix M is strongly connected and hence the matrix M is irreducible and monotone (Varga [26]). Consequently M^{-1} exists and $M^{-1} \ge 0$.

Further, let Σ_l be the sum of the l^{th} row sum of the matrix M, then

For $l = 1$

$$\Sigma_l \ge \delta_l + \frac{h_l^2}{6}\delta_l(3\delta_l + 2) - \frac{h_l^4}{32}\delta_l(1+\delta_l^3)\underset{\sim}{a},$$

$$\Sigma_{l+1} \ge \delta_l + \frac{h_l^2}{6}\delta_l(3\delta_l + 2)\underset{\sim}{a} - \frac{h_l^4}{32}\delta_l(1+\delta_l^3)\underset{\sim}{a},$$

For $l = 3(2)2n-3$

$$\Sigma_l \ge \frac{h_l^2}{2}\delta_l(1+\delta_l) - \frac{h_l^2}{24}\delta_l(1+\delta_l^3)\underset{\sim}{a},$$

$$\Sigma_{l+1} \ge \frac{h_l^2}{2}\delta_l(1+\delta_l)\underset{\sim}{a} - \frac{h_l^4}{24}\delta_l(1+\delta_l^3)\underset{\sim}{a},$$

For $l = 2n-1$

$$\Sigma_l \ge 1 + \frac{h_l^2}{6}\delta_l(3+2\delta_l) - \frac{h_l^4}{32}\delta_l(1+\delta_l^3)\underset{\sim}{a},$$

$$\Sigma_{l+1} \ge 1 + \frac{h_l^2}{6}\delta_l(3+2\delta_l)\underset{\sim}{a} - \frac{h_l^4}{32}\delta_l(1+\delta_l^3)\underset{\sim}{a},$$

Let $M_{i,l}^{-1}$ be the $(i,l)^{\text{th}}$ element of M^{-1}, and we define

$$\|M^{-1}\| = \max_{1\le l\le 2n} \sum_{l=1}^{2n} |M_{i,l}^{-1}| \quad \text{and} \quad \|T\| = \max_{1\le l\le 2n} \sum_{l=1}^{2n} |T_l(h_l)|$$

From the theory of matrix, we know that

$$\sum_{l=1}^{2n} M_{i,l}^{-1}\Sigma_l = 1, \quad 1 \le i \le 2n, \tag{4.6}$$

Thus the following bounds can be estimated with the help of series expansion

For $l = 1$

$$M_{i,l}^{-1} \le \Sigma_l^{-1} \le \frac{1}{\delta_l} - \frac{h_l^2}{6\delta_l}(3\delta_l + 2) + \frac{h_l^4}{288\delta_l}(9(1+\delta_l^3)\widetilde{a} + 8(3\delta_l+2)^2) + \mathrm{O}(h_l^6),$$

$$M_{i,l+1}^{-1} \le \Sigma_{l+1}^{-1} \le \frac{1}{\delta_k} - \frac{h_k^2\widetilde{a}}{6\delta_k}(3\delta_k + 2) + \frac{h_k^4\widetilde{a}}{288\delta_k}(9(1+\delta_k^3) + 8(3\delta_k+2)^2\widetilde{a}) + \mathrm{O}(h_l^6),$$

For $l = 3(2)2n-3$

$$M_{i,l}^{-1} \le \min_l \Sigma_l^{-1} \le \frac{2}{\delta_l(1+\delta_l)h_l^2} + \frac{(1-\delta_l+\delta_l^2)\widetilde{a}}{6\delta_l(1+\delta_l)} + \frac{h_l^2}{72}\frac{(1-\delta_l+\delta_l^2)^2\widetilde{a}^2}{\delta_l(1+\delta_l)} + \frac{h_l^4}{864}\frac{(1-\delta_k+\delta_k^2)^3\widetilde{a}^3}{\delta_k(1+\delta_k)} + \mathrm{O}(h_l^6)$$

$$M_{i,l+1}^{-1} \le \min_l \Sigma_{l+1}^{-1} \le \frac{2}{h_l^2\delta_l(1+\delta_l)\underset{\sim}{a}} + \frac{1-\delta_l+\delta_l^2}{6\delta_l(1+\delta_l)\underset{\sim}{a}} + \frac{h_l^2(1-\delta_l+\delta_l^2)^2}{72\delta_l(1+\delta_l)\underset{\sim}{a}} + \frac{h_l^4(1-\delta_l+\delta_l^2)^3}{864\delta_l(1+\delta_l)\underset{\sim}{a}} + \mathrm{O}(h_l^6),$$

For $l = 2n-1$

$$M_{i,l}^{-1} \le \Sigma_l^{-1} \le 1 - \frac{h_l^2}{6}\delta_l(3+2\delta_l) + \frac{h_l^4}{288}\delta_k(9(\delta_l^3+1)\widetilde{a} + 8\delta_l(2\delta_l+3)^2) + \mathrm{O}(h_l^6),$$

$$M_{i,l+1}^{-1} \le \Sigma_{l+1}^{-1} \le 1 - \frac{h_l^2}{6}\delta_k(3+2\delta_k)\widetilde{a} + \frac{h_l^4}{288}\delta_k(9(\delta_l^3+1) + 8\delta_l(2\delta_l+3)^2\widetilde{a})\widetilde{a} + \mathrm{O}(h_l^6),$$

Consequently, we obtain the following bounds

$$\|M^{-1}\| \le \frac{2(1+|\widetilde{a}|)}{h_l^2\delta_l(1+\delta_l)|\underset{\sim}{a}|} + \frac{12(1+\delta_l)^2|\widetilde{a}| + (1-\delta_l+\delta_l^2)(1+|\widetilde{a}|^2)}{6\delta_l(1+\delta_l)|\underset{\sim}{a}|} + \mathrm{O}(h_l^2), \tag{4.7}$$

With the help of equations (4.5) and (4.7), we obtain the bounds of error

$$\|\varepsilon\| \le \|M^{-1}\| \cdot \|T_l(h_l)\| \le \mathrm{O}(h_l^3), \tag{4.8}$$

This proves the third order convergence of the proposed method. In a similar manner, we can establish the third order convergence for canonical form of sixth order two point boundary value problems. We generalize the above results in the following theorem

Theorem 4.1. *The method given by* (2.24)–(2.25) *for the numerical solution of fourth order two point boundary value problems* (1.1)–(1.2) *with sufficiently small* h_k *and* $0 < \delta_k \neq 1$, *gives a third order convergent solution.*

5. COMPUTATIONAL ILLUSTRATIONS

In order to examine the utility and to corroborate the order of convergence obtained by the arithmetic mean geometric mesh finite difference method, we have solved some linear and nonlinear boundary value problems with associated boundary conditions and the results are reported in Tables 1-14. The boundary conditions and unknown function $g(r)$ may be obtained from the exact solution as a test procedure. The initial guess is considered as zero for solving nonlinear problems and the error tolerance is $\leq 10^{-15}$ (Atkinson [27]). The numerical accuracy of results are tested using maximum absolute error ($\varepsilon^{(\infty)}_{u^{(m)}}$), root mean square errors ($\varepsilon^{(2)}_{u^{(m)}}$) and computational order of convergence (Θ_m) for interpolating m^{th} order derivative of $u(r)$.

$$\varepsilon^{(\infty)}_{u^{(m)}} = \max_{1\leq k\leq n} |u^{(m)}_k - u^{(m)}(r_k)|, \quad \varepsilon^{(2)}_{u^{(m)}} = \frac{1}{n}\left(\sum_{k=1}^{n} |u^{(m)}_k - u^{(m)}(r_k)|^2\right)^{1/2},$$

$$\Theta_m = \log_2\left(\frac{\varepsilon^{(2)}_{u^{(m)}}|_{n \text{ grid points}}}{\varepsilon^{(2)}_{u^{(m)}}|_{2n \text{ grid points}}}\right).$$

For the simplicity in computation, we choose $\delta_k = \delta =$ constant, for $k = 1(1)n$ and define the geometric mesh as follows (Kadalbajoo et al. [21])

$$x_0 = \alpha$$

$$h_1 = \begin{cases} (\beta-\alpha)(1-\delta)/(1-\delta^{n+1}), & \delta < 1 \\ (\beta-\alpha)(\delta-1)/(\delta^{n+1}-1), & \delta > 1 \end{cases}$$

The subsequent mesh spacing is determined by $h_{k+1} = \delta h_k$, $k = 1(1)n$. The numerical solution of stiff problems was also tested for varying geometric mesh ratio parameter referred as optimum geometric mesh ratio. All the algorithms were tested with long double length arithmetic in C programming under Linux operating system with Intel Core 2Duo 2GHz CPU and 2GB of RAM.

Example 5.1 (Talwar et al. [28])**.** Consider the fourth order linear singular problem in polar form

$$\Delta^4 u(r) \equiv \left(\frac{d^2}{dr^2} + \frac{\mu}{r}\frac{d}{dr}\right)^2 u(r) = g(r), \quad 0 < r < 1$$

Above problem represents polar and spherical symmetry for $\mu = 1$ and 2 respectively. The exact solution is $u(r) = \exp(r)$. The numerical accuracy of solutions are obtained in Table 1–4 for various values of n with uniform mesh ($\delta = 1$) and geometric mesh ($\delta \neq 1$).

Example 5.2 (Scott et al. [29], and Conte [30])**.** Consider the stiff two point boundary value problem

$$u^{iv}(r) - (1+\lambda)u''(r) + \lambda u(r) = \frac{\lambda}{2}r^2 + 1, \quad 0 < r < 1$$

The exact solution is given by $u(r) = 1 + \frac{r^2}{2} + \sinh(r)$. We know that ± 1 and $\pm\lambda$ are the eigenvalues of this equation and hence the problem is stiff for large values of λ. We have solved the problem for small as well as large values of λ and the behavior of solution is sufficiently smooth for $\lambda < 10^8$ both in case of uniform and geometric mesh. Some improvement in the numerical solution for $\lambda = 10^8$ has been observed in Table 5–6 with geometric mesh comparing with uniform grids.

Example 5.3 (Elcrat [31])**.** Consider the boundary value problem arises from time dependent Navier-Stokes equation for axis symmetric flow of an incompressible fluid contained between infinite disks

$$u^{iv}(r) = \tau u(r)u''(r) + g(r), \quad 0 < r < 1$$

The exact solution is $u(r) = (1 - r^2)\exp(r)$. The errors estimates of exact and approximate solutions are reported in Table 7–8 with uniform and geometric mesh respectively for various values of n and $\tau = 10^3$. The sufficiently smoothness in the solution has been observed for $\tau < 10^3$.

Example 5.4. Consider the sixth order linear singular problem in spherical coordinate

$$\Delta^6 u(r) \equiv \left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr}\right)^3 u(r) = g(r), \quad 0 < r < 1$$

The exact solution is $u(r) = \sinh(r)$. The accuracy of numerical and exact solutions is computed in Table 9–10 for various values of n using uniform and geometric mesh. The accuracy remains similar for $u''(r)$ and $u^{iv}(r)$ in both the cases $\delta = 1$ and $\delta \neq 1$. The superiority of considering $\delta \neq 1$ is evident from the maximum absolute errors observed for $u(r)$. The computational order of convergence shows close resemblance with the theoretical error estimates.

Example 5.5 (Noor et al. [32])**.** Consider the sixth order stiff problem

$$u^{vi}(r) - (1+\lambda)u^{iv}(r) + \lambda u''(r) = \lambda r, \quad 0 < r < 1$$

The exact solution is given by $u(r) = 1 + \dfrac{r^3}{3} + \sinh(r)$. We have computed the solution for various small as well as large values of λ and observed high smoothness in the solution. The error estimate for uniform mesh is encouraging. However with very large values of λ, geometric mesh found to be more accurate compared with the uniform mesh ($\delta = 1$). The results are enumerated in Table 11–12 for $n = 8, 16, 32$ and 64 corresponding to optimum geometric mesh ratio parameters.

Example 5.6. Consider the non-linear problem

$$u^{vi}(r) = \mu(u(r)u''(r) + u''(r)u^{iv}(r) + u(r)u^{iv}(r)) + g(r), \quad 0 < r < 1$$

The exact solution is $u(r) = \sinh(r)$. The numerical accuracy of exact and approximate solution along with its second and fourth order derivates are reported in Table 13–14 for various values of n and $\mu = 2^{13}$. Refer to Table 13, in case of small number of grids, for example $n = 20$, the solution oscillates rapidly with uniform mesh. The occurrence of week layer near the right boundary gives rise to oscillatory solution while using uniform mesh. The oscillatory behaviour of solution can be easily overcome using geometric mesh approaches which concentrate more grids near the right boundary. Moreover, improvement in the errors are observed for $\delta \neq 1$. The uniform order of computational convergence can be easily achieved with optimum geometric mesh ratio parameters.

6. CONCLUSION

The computational illustrations show that the proposed arithmetic average geometric mesh finite difference scheme is convergent. The scheme is compact using evaluations at two off step nodes and one central node, consequently the method is directly applicable to both singular and non-singular differential equations. The theoretical order of accuracy is three for geometric mesh, whereas it comes out to be four for uniform mesh. Although, practically we have observed that the numerical accuracy in terms of maximum absolute errors or root mean square errors computed for geometric mesh shows superiority over corresponding uniform mesh. The essence of geometric mesh lies in the fact that often nonlinear or singular problems might not exhibit smooth solution inside the domain of consideration. The geometric meshes resolve thin layers that occur inside and on the boundary region. The optimum mesh ratio parameter within the specified convergent region may be obtained using simulations. We have employed block gauss seidel method to solve the block matrix systems. The method can be extended to general even order nonlinear ordinary differential equations. Applications of the proposed scheme to nonlinear singular partial differential equation is an open problem.

TABLE 1. The error estimates for Example 5.1 with uniform mesh $\delta = 1.0$ and $\mu = 1$

n	$\varepsilon_u^{(\infty)}$	$\varepsilon_{u''}^{(\infty)}$	$\varepsilon_u^{(2)}$	$\varepsilon_{u''}^{(2)}$	Θ_0	Θ_2
10	1.03e-05	2.10e-04	7.79e-06	7.86e-05	—	—
20	1.80e-06	8.54e-05	1.32e-06	2.26e-05	2.6	1.8
40	2.83e-07	3.77e-05	2.05e-07	6.93e-06	2.7	1.7
80	4.17e-08	1.77e-05	3.01e-08	2.28e-06	2.8	1.6

TABLE 2. The error estimates for Example 5.1 with geometric mesh $\delta = 1.032$ and $\mu = 1$

n	$\varepsilon_u^{(\infty)}$	$\varepsilon_{u''}^{(\infty)}$	$\varepsilon_u^{(2)}$	$\varepsilon_{u''}^{(2)}$	Θ_0	Θ_2
10	6.30e-06	9.50e-05	4.89e-06	3.77e-05	—	—
20	6.53e-07	2.57e-05	5.14e-07	7.38e-06	3.2	2.4
40	3.52e-08	6.77e-06	3.01e-08	1.37e-06	4.1	2.4
80	3.59e-09	1.35e-06	1.79e-09	1.94e-07	4.1	2.8

TABLE 3. The error estimates for Example 5.1 with uniform mesh $\delta = 1.0$ and $\mu = 2$

n	$\varepsilon_u^{(\infty)}$	$\varepsilon_{u''}^{(\infty)}$	$\varepsilon_u^{(2)}$	$\varepsilon_{u''}^{(2)}$	Θ_0	Θ_2
10	3.32e-07	1.11e-05	2.57e-07	3.84e-06	—	—
20	2.49e-08	1.69e-06	1.87e-08	4.29e-07	3.8	3.2
40	1.72e-09	2.34e-07	1.27e-09	4.31e-08	3.9	3.3
80	1.13e-10	3.09e-08	8.27e-11	4.09e-09	3.9	3.4

TABLE 4. The error estimates for Example 5.1 with geometric mesh $\delta = 0.9946$ and $\mu = 2$

n	$\varepsilon_u^{(\infty)}$	$\varepsilon_{u''}^{(\infty)}$	$\varepsilon_u^{(2)}$	$\varepsilon_{u''}^{(2)}$	Θ_0	Θ_2
10	2.76e-07	1.19e-05	2.13e-07	4.11e-06	—	—
20	1.68e-08	1.95e-06	1.26e-08	4.92e-07	4.1	3.1
40	6.63e-10	3.14e-07	4.80e-10	5.73e-08	4.7	3.1
80	2.75e-11	5.59e-08	1.83e-11	7.28e-09	4.7	3.0

TABLE 5. The error estimates for Example 5.2 with uniform mesh $\delta = 1.0$ and $\lambda = 10^8$

n	$\varepsilon_u^{(\infty)}$	$\varepsilon_{u''}^{(\infty)}$	$\varepsilon_u^{(2)}$	$\varepsilon_{u''}^{(2)}$	Θ_0	Θ_2
8	2.70e-07	2.74e-03	2.10e-07	1.88e-03	—	—
16	2.12e-08	8.14e-04	1.60e-08	5.14e-04	3.7	2.0
32	1.49e-09	2.23e-04	1.11e-09	1.35e-04	3.8	2.0
64	9.95e-11	5.85e-05	7.34e-11	3.49e-05	3.9	2.0

TABLE 6. The error estimates for Example 5.2 with optimum geometric mesh and $\lambda = 10^8$

n	δ	$\varepsilon_u^{(\infty)}$	$\varepsilon_{u''}^{(\infty)}$	$\varepsilon_u^{(2)}$	$\varepsilon_{u''}^{(2)}$	Θ_0
8	1.027592	8.96e-09	3.47e-03	6.93e-09	1.95e-03	—
16	1.014520	7.18e-10	1.03e-03	5.29e-10	5.34e-04	3.6
32	1.007457	5.07e-11	2.83e-04	3.67e-11	1.40e-04	3.8
64	1.003798	3.55e-12	7.45e-05	2.46e-12	3.67e-05	3.8

TABLE 7. The error estimates for Example 5.3 with uniform mesh $\delta = 1.0$ and $\tau = 10^3$

n	$\varepsilon_u^{(\infty)}$	$\varepsilon_{u''}^{(\infty)}$	$\varepsilon_u^{(2)}$	$\varepsilon_{u''}^{(2)}$	Θ_0	Θ_2
20	2.21e-05	7.34e-03	1.17e-05	3.18e-03	—	—
40	1.58e-06	4.63e-04	8.46e-07	1.94e-04	3.8	4.0
80	1.05e-07	2.97e-05	5.61e-08	1.23e-05	3.9	4.0

TABLE 8. The error estimates for Example 5.3 with geometric mesh $\delta = 0.985$ and $\tau = 10^3$

n	$\varepsilon_u^{(\infty)}$	$\varepsilon_{u''}^{(\infty)}$	$\varepsilon_u^{(2)}$	$\varepsilon_{u''}^{(2)}$	Θ_0	Θ_2
20	1.16e-05	4.51e-03	6.47e-06	2.27e-03	—	—
40	4.16e-07	1.91e-04	2.33e-07	1.14e-04	4.8	4.3
80	1.69e-08	1.94e-05	7.54e-09	8.58e-06	4.9	3.7

TABLE 9. The error estimates for Example 5.4 with uniform mesh $\delta = 1.0$

n	$\varepsilon_u^{(\infty)}$	$\varepsilon_{u''}^{(\infty)}$	$\varepsilon_{u^{iv}}^{(\infty)}$	$\varepsilon_u^{(2)}$	$\varepsilon_{u''}^{(2)}$	$\varepsilon_{u^{iv}}^{(2)}$	Θ_0	Θ_2	Θ_4
10	7.35e-08	2.75e-07	1.64e-05	5.54e-08	2.13e-07	5.79e-06	—	—	—
20	5.53e-09	2.08e-08	2.46e-06	4.08e-09	1.56e-08	6.38e-07	3.8	3.8	3.2
40	3.82e-10	1.43e-09	3.37e-07	2.78e-10	1.06e-09	6.37e-08	3.9	3.9	3.3
80	2.51e-11	9.39e-11	4.43e-08	1.81e-11	6.90e-11	6.02e-09	3.9	3.9	3.4

TABLE 10. The error estimates for Example 5.4 with uniform mesh $\delta = 1.00246$

n	$\varepsilon_u^{(\infty)}$	$\varepsilon_{u''}^{(\infty)}$	$\varepsilon_{u^{iv}}^{(\infty)}$	$\varepsilon_u^{(2)}$	$\varepsilon_{u''}^{(2)}$	$\varepsilon_{u^{iv}}^{(2)}$	Θ_0	Θ_2	Θ_4
10	6.08e-08	2.94e-07	1.59e-05	4.56e-08	2.27e-07	5.63e-06	—	—	—
20	3.73e-09	2.36e-08	2.30e-06	2.71e-09	1.76e-08	5.99e-07	4.1	3.7	3.2
40	1.38e-10	1.81e-09	2.94e-07	9.53e-11	1.33e-09	5.59e-08	4.8	3.7	3.4
80	7.43e-12	1.46e-10	3.32e-08	5.72e-12	1.06e-10	4.57e-09	4.1	3.6	3.6

TABLE 11. The error estimates for Example 5.5 with uniform mesh $\delta = 1.0$ and $\lambda = 10^{16}$

n	$\varepsilon_u^{(\infty)}$	$\varepsilon_{u''}^{(\infty)}$	$\varepsilon_{u^{iv}}^{(\infty)}$	$\varepsilon_u^{(2)}$	$\varepsilon_{u''}^{(2)}$	$\varepsilon_{u^{iv}}^{(2)}$	Θ_0	Θ_2	Θ_4
8	2.45e-07	3.56e-07	3.75e-07	1.76e-07	2.63e-07	2.62e-07	—	—	—
16	2.41e-08	4.14e-08	5.63e-08	1.63e-08	3.02e-08	3.21e-08	3.4	3.1	3.0
32	2.40e-09	1.71e-09	2.48e-09	1.82e-09	1.01e-09	1.08e-09	3.2	4.9	4.9
64	7.00e-10	1.78e-09	1.89e-09	5.20e-10	1.35e-09	1.35e-09	1.8	0.4	0.3

TABLE 12. The error estimates for Example 5.5 with optimum geometric mesh and $\lambda = 10^{16}$

n	δ	$\varepsilon_u^{(\infty)}$	$\varepsilon_{u''}^{(\infty)}$	$\varepsilon_{u^{iv}}^{(\infty)}$	$\varepsilon_u^{(2)}$	$\varepsilon_{u''}^{(2)}$	$\varepsilon_{u^{iv}}^{(2)}$	Θ_0
8	1.00106	1.83e-08	2.52e-07	2.69e-07	1.11e-08	1.88e-07	1.89e-07	—
16	1.00290	1.53e-09	1.65e-08	3.18e-08	8.98e-10	1.15e-08	1.61e-08	3.6
32	1.00006	3.25e-10	1.96e-09	3.36e-09	1.76e-10	1.14e-09	1.41e-09	2.2
64	1.00098	7.42e-11	1.43e-09	1.52e-09	3.51e-11	9.89e-10	9.91e-10	2.1

TABLE 13. The error estimates for Example 5.6 with uniform mesh $\delta = 1.0$ and $\mu = 2^{13}$

n	$\varepsilon_u^{(\infty)}$	$\varepsilon_{u''}^{(\infty)}$	$\varepsilon_{u^{iv}}^{(\infty)}$	$\varepsilon_u^{(2)}$	$\varepsilon_{u''}^{(2)}$	$\varepsilon_{u^{iv}}^{(2)}$	Θ_0	Θ_2	Θ_4
20	Overflow	Overflow	Overflow	Overflow	Overflow	Overflow	—	—	—
40	1.17e-09	8.66e-08	1.51e-03	8.44e-10	3.39e-08	5.04e-04	—	—	—
80	4.89e-11	1.09e-09	1.93e-05	3.52e-11	6.48e-10	8.18e-06	4.6	5.7	5.9
160	2.98e-12	5.73e-11	1.01e-06	2.14e-12	3.56e-11	4.41e-07	4.0	4.2	4.2

TABLE 14. The error estimates for Example 5.6 with geometric mesh $\delta = 0.999$ and $\mu = 2^{13}$

n	$\varepsilon_u^{(\infty)}$	$\varepsilon_{u''}^{(\infty)}$	$\varepsilon_{u^{iv}}^{(\infty)}$	$\varepsilon_u^{(2)}$	$\varepsilon_{u''}^{(2)}$	$\varepsilon_{u^{iv}}^{(2)}$	Θ_0	Θ_2	Θ_4
20	6.51e-09	7.44e-07	5.52e-03	4.91e-09	3.48e-07	2.86e-03	—	—	—
40	9.44e-10	6.78e-08	1.18e-03	6.81e-10	2.85e-08	4.15e-04	2.9	3.6	2.8
80	2.80e-11	9.17e-10	1.62e-05	1.99e-11	5.77e-10	7.32e-06	5.1	5.6	5.8
160	4.50e-13	4.16e-11	7.41e-07	2.77e-13	2.84e-11	3.64e-07	6.2	4.3	4.3

REFERENCES

[1] E. H. Twizell and S. I. A. Tirmizi, Multi-derivative methods for nonlinear beam problems, *Commun. Appl. Numer. Meth.*, 4:43–50, 1988.

[2] R. P. Agawal and G. Akrivis, Boundary value problems occurring in plate defection theory, *J. Comput. Appl. Math.*, 8:145–154, 1982.

[3] D. B. Tien and R. A. Usmani, Solving boundary value problems in plate deflection theory, *Simulation.*, 37:195–206, 1981.

[4] A. Leizarowitz and V. J. Mizel, One dimensional infinite-horizon variational problems arising in continuum mechanics, *Arch. Ration. Mech. Anal.*, 106:161–194, 1989.

[5] C. J. Budd, G. W. Hunt and M. A. Peletier, Self-similar fold evolution under prescribed end shortening, *Math. Geol.*, 31:989–1005, 1999.

[6] S. Timoshenko and S. W. Krieger, *Theory of Plates and Shell*, McGraw-Hill, New York, 1959.

[7] Y. Chen and P. J. McKenna, Traveling waves in a nonlinearly suspended beam: theoretical results and numerical observations, *J. Differ. Equations*, 136:325–335, 1991.

[8] A. C. Lazer and P. J. McKenna, Large-amplitude oscillations in suspension bridges: some new connections with nonlinear analysis, *SIAM Review*, 32:537–578, 1990.

[9] A. Doelman and V. Rottschafer, Singularly perturbed and nonlocal modulation equations for systems with interacting instability mechanisms, *J. Nonlinear Sci.*, 7:371–410, 1997.

[10] R. P. Agarwal and P. R. Krishnamoorthy, Boundary value problems for nth order ordinary differential equations, *Bull. Institute Math. Academia Sinica*, 7:211–230, 1979.

[11] A. R. Aftabizadeh, Existence and uniqueness theorems for fourth-order boundary value problems, *J. Math. Anal. Appl.*, 116:415–426, 1986

[12] D. O' Regan, Solvability of some fourth (and higher) order singular boundary value problems, *J. Math. Anal. Appl.*, 161:78–116, 1991.

[13] W. K. Zahra, A smooth approximation based on exponential spline solutions for nonlinear fourth order two point boundary value problems, *Appl. Math. Comput.*, 217:8447–8457, 2011.

[14] R. A. Usmani, Finite difference methods for a certain two point boundary value problem, *Indian J. Pure Appl. Math.*, 14:398–411, 1983.

[15] M. A. Ramadan, I. F. Lashien and W. K. Zahra, High order accuracy nonpolynomial spline solution for 2μth order two point boundary value problems, *Appl. Math. Comput.*, 204:920–927, 2008.

[16] S. S. Siddiqi and E. H. Twizell, Spline solution of linear sixth order boundary value problems, *Int. J. Comput. Math.*, 60:295–304, 1996.

[17] R. A. Usmani and P. J. Taylor, Finite difference methods for solving $(p(x)y'')'' + q(x)y = r(x)$, *Int. J. Comput. Math*, 14:277–293, 1983.

[18] J. Rashidinia and M. Ghasemia, B-spline collocation for solution of two-point boundary value problems, *J. Comput. Appl. Math.*, 23:2325–2342, 2011.

[19] D. Britz, Digital simulation in electrochemistry, *Lect. Notes Phys.*, Vol. 66, Springer, Berlin, 2005.

[20] M. K. Jain, S. R. K. Iyengar and G. S. Subramanyam, Variable mesh method for the numerical solution of two point singular perturbation problems, *Comput. Meth. Appl. Mech. Eng.*, 42:273–286, 1984.

[21] M. K. Kadalbajoo and D. Kumar, Geometric mesh FDM for self-adjoint singular perturbation boundary value problems, *Appl. Math. Comput.*, 190:1646–1656, 2007.

[22] R. K. Mohanty, A class of non-uniform mesh three point arithmetic average discretization for $y'' = f(x, y, y')$ and the estimates of y', *Appl. Math. Comput.*, 183:477–485, 2006.

[23] R. K. Mohanty, A family of variable mesh methods for the estimates of (du/dr) and the solution of non-linear two point boundary value problems with singularity, *J. Comput. Appl. Math.*, 182:173–187, 2005.

[24] J. Navnit, A fifth order accurate geometric mesh finite difference method for general nonlinear two point boundary value problems, *Appl. Math. Comput.*, 219:8425–8434, 2013.

[25] M. M. Chawla and P. N. Shivkumar, An efficient finite difference method for two point boundary value problems, *Neural Parallel Sci. Comput.*, 4:387–396, 1996.

[26] R. S. Varga, Matrix Iterative Analysis, *Springer Series in Computational Mathematics*, Germany, 2000.

[27] K. E. Atkinson, W. Han and D. E. Stewart, *Numerical Solution of Ordinary Differential Equations*, John Wiley & Sons, 2009.

[28] J. Talwar and R. K. Mohanty, A class of numerical method for the solution of fourth order ordinary differential equations in polar coordinates, *Advances in Numerical Analysis*, Vol. 2012, doi:10.1155/2012/626419, 2012.

[29] M. R. Scott and H. A. Watts(1975), SUPPORT — A computer code for two point boundary value problems via orthonormalization, *Sandia National Laboratories Report SAND*, 75–0198.

[30] S. D. Conte, The numerical solution of linear boundary value problems, *SIAM Rev.*, 8:309–321, 1966.

[31] A. R. Elcrat, On the radial flow of a viscous fluid between porous disks, *Arch. Ration. Mech. Anal*, 61:91–96, 1976.

[32] M. A. Noor and S. T. Mohyud-Din, Homotopy perturbation method for solving sixth order boundary value problems, *Comput. Math. Appl.*, 55:953–2972, 2008.