Neural, Parallel, and Scientific Computations 21 (2013) 411-418

DISCRETE DYNAMIC CONTROL OF AN IMPULSIVE SIR MODEL

N. G. MEDHIN AND M. SAMBANDHAM

Department of Mathematics, North Carolina State University
Raleigh, NC 27695-8205
Department of Mathematics, Morehouse College Atlanta, GA 30314

ABSTRACT. We consider discrete dynamic control of SIR epidemic model where susceptible and infected people are controlled by vaccination and medical intake at particular instants of time within the time horizon under consideration.

1. Introduction

Formulation of strategies to control or avoid the spread of epidemics are key components of the design of public health policy. Some of the strategies to control or block the spread of epidemics consist of health education campaigns, contact tracing and screening, and strategically timed mass vaccination, medical treatment and/or quarantine for those that are already infected. All these strategies have cost associated with them. Appropriate and timely strategies control cost ([1], [4], [9], [11], [18]).

In this paper we formulate a discrete dynamic impulsive control system where the objective is to reduce the number of infected people and cost by scheduled vaccinations and/or quarantine. The level of the application of resources depends on the size of the population and the state of the epidemic([2], [4], [9], [11]).

In general a model depends on population homogeneity and migration. The model could consider multiple groups, and the level of interaction between groups. The level of resources also depends on the size of population. The epidemic model also significantly differs depending on the particular epidemic under consideration. Influenza model is quite different from HIV model ([2], [4], [5], [8], [12], [13]).

Thus, control models should possibly consider multiple groups, the particular epidemic, time horizon, and control objectives, population size, state of the epidemic and available resources ([2], [4], [5], [8], [11], [18]). In this paper we consider a simple model following ([9], [10], [11]) where the controls are applied at distinct times in the planning/epidemic horizon. We start with a general mathematical model where the SIR models ([9], [10], [11]) are particular cases. We apply the results to the concrete model where numerical results are presented.

Received July 15, 2013

2. General Discrete Dynamic Problem

We consider a general problem given by the dynamics

$$
\begin{aligned}
\frac{dx_i}{dt} &= f_i(x_i(t)), \quad t_{i-1} < t < t_i,\ i = 2, \ldots, n \\
x_i(t_{i-1}) &= h_i(x_{i-1}(t_{i-1})) \cdot C_i + x_{i-1}(t_{i-1}) \\
\frac{dx_1}{dt} &= f_1(x_1(t)), \quad t_0 < t < t_1, \\
x_1(t_0) &= x_0
\end{aligned} \tag{1}
$$

The objective functional is given by

$$J(x_0, C_1, \ldots, C_n) = \sum_{i=1}^{n} \Phi_i(x_i(t_i)).$$

Assumptions

(1) The functions Φ_i are continuously differentiable and bounded below.
(2) The functions f_i, $i = 1, \ldots, n$ are continuously differentiable.
(3) The matrix functions h_i, $i = 1, \ldots, n$ are continuously differentiable

3. Sufficiency Theorem

Suppose, given that $(x_0, C_1, \ldots, C_n)$, ψ_i, $i = 1, \ldots, n$ are such that

$$
\begin{aligned}
\frac{d}{dt}\psi_1(t) &= -\psi_1(t)\frac{\partial}{\partial x} f_1(x_1(t)), \quad t_0 < t < t_1 \\
\frac{d}{dt}\psi_i(t) &= -\psi_i(t)\frac{\partial}{\partial x} f_i(x_i(t)), \quad t_{i-1} < t < t_i \\
\psi_{i-1}(t_{i-1}) &= \psi_i(t_{i-1})\left(\frac{\partial}{\partial x}(h_i(x_{i-1}(t_{i-1}))C_i) + I\right) \\
&\quad -\partial_x \Phi_{i-1}(x_{i-1}(t_{i-1})), \quad i = 1, \ldots, n \\
\psi_n(t_n) &= -\partial_x \Phi_n(x_n(t_n)) \\
\psi_i(t_{i-1}) \cdot h_i(x_{i-1}(t_{i-1})) &= 0 \\
\psi_1(x_1(t_0)) &= 0
\end{aligned} \tag{2}
$$

then $(x_0, C_1, \ldots, C_n)$ is a minimizer for J.

Remark 3.1. We remark that C_i is a minimizer for $\sum_{k=i}^{n} \Phi_k(x_k(t_k))$ with all the other decision variables $C_k, k = i+1, \ldots, n$ fixed. This is in line with dynamic programming. Note that ψ_i, $i = 1, \ldots, n$ are row vectors.

Proof. Consider

$$
\begin{aligned}
\frac{d}{dt}x_i &= f_i(x_i), \quad t_{i-1} < t < t_i, \\
x_i(t_{i-1}) &= h_i(x_{i-1}(t_{i-1}))C_i + x_{i-1}(t_{i-1})
\end{aligned} \tag{3}
$$

If we vary C_i by adding δC to it, then the corresponding variation in the trajectory x_i satisfies the equation

$$\begin{aligned} d/dt\,\delta x_i &= f_{i,x}(x_i(t))\delta x_i, \quad t_{i-1} < t < t_i, \\ \delta x_i(t_{i-1}) &= h_i(x_{i-1}(t_{i-1}))\delta C \end{aligned} \tag{4}$$

In (3), a variation of $\delta x_{i-1}(t_{i-1})$ in the x_{i-1} trajectory at t_{i-1}, leads to the variation in the state x_i given by

$$\begin{aligned} d/dt\,\delta x_i &= f_{i,x}(x_i(t))\delta x_i, \quad t_{i-1} < t < t_i, \\ \delta x_i(t_{i-1}) &= \left\{\frac{\partial}{\partial x}(h_i(x_{i-1}(t_{i-1}))C_i) + I\right\} \cdot \delta x_{i-1}(t_{i-1}) \end{aligned} \tag{5}$$

$$\begin{aligned} \int_{t_{i-1}}^{t_i} \psi_i(s) f_{i,x}(s)\delta x_i(s)ds &= \int_{t_{i-1}}^{t_i} \psi_i(s)\frac{d}{ds}\delta x_i(s)ds \\ &= \psi_i(s)\delta x_i(s)|_{t_{i-1}}^{t_i} \\ &\quad + \int_{t_{i-1}}^{t_i} \psi_i(s) f_{i,x}(s)\delta x_i(s)ds \end{aligned} \tag{6}$$

From (6) we get

$$\psi_i(t_i)\delta x_i(t_i) - \psi_{i-1}(t_{i-1})\delta x_{i-1}(t_{i-1}) = 0 \tag{7}$$

We can repeat (6) for $i, i+1, i+2, \ldots, n$. Adding the resulting equations as in ((7), and using the conditions stated in (2), a variation in C_i while leaving C_k, $k = 1, 2, \ldots, n$ fixed gives

$$\sum_{k=i}^{n} \partial_x \Phi_k(x_k(t_k)) = 0 \tag{8}$$

We remark that one can verify the above sufficiency conditions are also necessary. □

3.1. **Numerical Computation.** We use the above sufficiency theorem for computational purpose. The problem of minimizing the objective function following (1) is a mathematical programming problem. To solve the mathematical programming problem numerically we start with $(x_0, C_1, \ldots, C_n)$ and solve the system (1). Then, we solve the adjoint system backwards in time. However, for the gradients we use the expressions in (11) below

$$\begin{aligned} \frac{d}{dt}\psi_1(t) &= -\psi_1(t)\frac{\partial}{\partial x} f_1(x_1(t)), \quad t_0 < t < t_1 \\ \frac{d}{dt}\psi_i(t) &= -\psi_i(t)\frac{\partial}{\partial x} f_i(x_i(t)), \quad t_{i-1} < t < t_i \\ \psi_n(t_n) &= -\partial_x \Phi_n(x_n(t_n)) \\ \psi_1(x_1(t_0)) &= 0 \end{aligned} \tag{9}$$

This system comes from (2). We solve this adjoint system backwards in time. From (2) we have

$$\begin{aligned}\psi_{i-1}(t_{i-1}) &= \psi_i(t_{i-1})\left(\frac{\partial}{\partial x}(h_i(x_{i-1}(t_{i-1}))C_i)+I\right)\\ &\quad -\partial_x\Phi_{i-1}(x_{i-1}(t_{i-1})),\quad i=1,\dots,n\\ \psi_1(x_1(t_0)) &= 0\end{aligned} \tag{10}$$

Thus,

$$\partial_x\Phi_{i-1}(x_{i-1}(t_{i-1})) = \psi_i(t_{i-1})\left(\frac{\partial}{\partial x}(h_i(x_{i-1}(t_{i-1}))C_i)+I\right)-\psi_{i-1}(t_{i-1}), \tag{11}$$
$$i=1,\dots,n$$
$$\psi_n(t_n) = -\partial_x\Phi_n(x_n(t_n)).$$

Now we use (11) to calculate direction of descent for $C_n, C_{n-1}, \dots, C_1$.

4. SIR Model

Let x represent the susceptible population, y the infected, and z the recovered. Following([11]) we consider the dynamics given by

$$\begin{aligned}\frac{d}{dt}x &= -\beta\frac{xy}{x+y},\quad \beta>0\\ \frac{d}{dt}y &= \beta\frac{xy}{x+y}-\gamma y,\ \gamma>0\\ \frac{d}{dt}z &= \gamma y\\ x(0) &= x_0\\ y(0) &= y_0\end{aligned} \tag{12}$$

We remark that we could treat the model in [10] in the same way.

The impulsive version we consider is as follows. Consider $0 \le t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = t_f$. In the interval $[t_0, t_1]$ we consider the dynamics

$$\begin{aligned}\frac{d}{dt}x &= -\beta\frac{xy}{x+y},\\ \frac{d}{dt}y &= \beta\frac{xy}{x+y}-\gamma y,\quad \gamma>0\\ \frac{d}{dt}z &= \gamma y\\ x(0) &= c_{01}\\ y(0) &= c_{02}\end{aligned} \tag{13}$$

In the interval $[t_{i-1}, t_i]$, $i = 2, 3, \ldots, n$ we consider

$$
\begin{aligned}
\frac{d}{dt}x_i &= -\beta\frac{x_i y_i}{x_i + y_i} \\
\frac{d}{dt}y_i &= \beta\frac{x_i y_i}{x_i + y_i} - \gamma y_i,\ \gamma > 0 \\
\frac{d}{dt}z_i &= \gamma y_i \\
\begin{pmatrix} x_i(t_i) \\ y_i(t_i) \end{pmatrix} &= h_i(x_{i-1}(t_{i-1}), y_{i-1}(t_{i-1})) + C_i, \\
C_i &= \begin{pmatrix} c_{i1} \\ c_{i2} \end{pmatrix}
\end{aligned}
\tag{14}
$$

We remark that treatment strategies modeled here consider vaccination, and quarantine executed at scheduled times in the time horizon appropriate for the epidemic.

The i-th cost is given by

$$J_i = \Phi_i(x_i(t_i), y_i(t_i)), \quad i = 1, 2, \ldots, n.$$

We would like to minimize

$$\sum_{i=1}^{n} J_i$$

5. **Numerical Simulation of the SIR Model**

In this section we take the case $t_0 = 0,\ t_1 = 1,\ t_2 = 2,\ t_3 = 3 = t_f$ and numerically solve the mathematical problem defined by the dynamics (14) with the corresponding cost $\sum_{i=1}^{3} J_i$ where $\Phi_i(x_i(t_i), y_i(t_i)) = \frac{1}{2}y_i^2$. The decision variables are $C_0,\ C_1, \ldots, C_{n-1}$,

$$C_i = \begin{pmatrix} c_{i1} \\ c_{i2} \end{pmatrix}, \quad i = 0,\ 1, \ldots, n-1$$

We remark that $C_0 = x_0$ in (1). We note that we have a constrained mathematical programming problem. We start with x_0 and initialize C_1, C_2, C_3 and use direction of descent using (11) to minimize the objective function. Note that the system for the states (14) has to be solved forward in time, followed by the adjoint system solved backwards in time in each time interval using (11)). A change in C_3 is supposed to affect the objective function $\Phi_4(x_4(t_4), y_4(t_4))$, while a change in C_2 is supposed to affect $\Phi_4(x_4(t_4), y_4(t_4)) + \Phi_3(x_3(t_3), y_3(t_3))$ (see Remark 3.1).

6. **Conclusion**

We have presented a discrete dynamic impulsive control problem to deal with an epidemic model. The approach we have presented can be used to deal with more general models than the SIR model presented here. We note that the formulation here

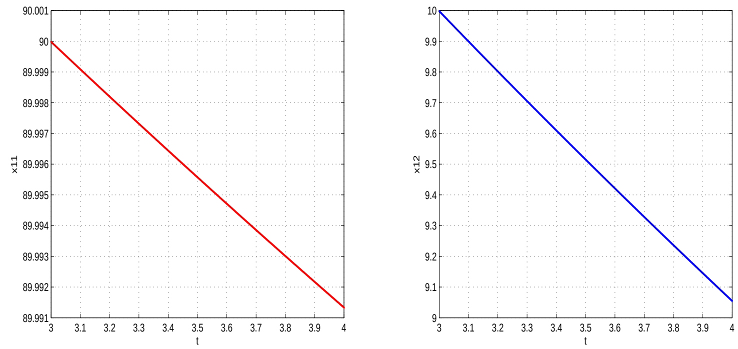

FIGURE 1. Susceptible & Infected Population in Time Interval 1.

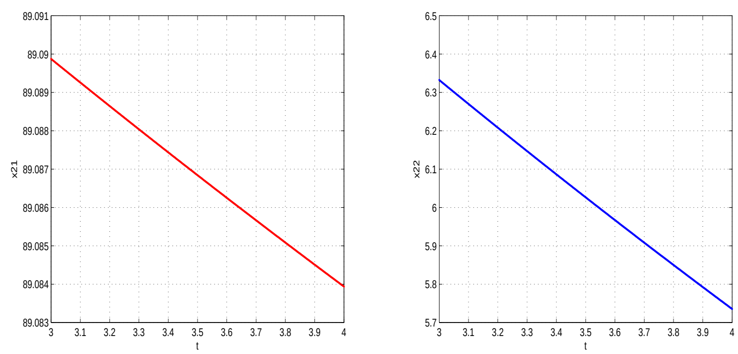

FIGURE 2. Susceptible & Infected Population in Time Interval 2.

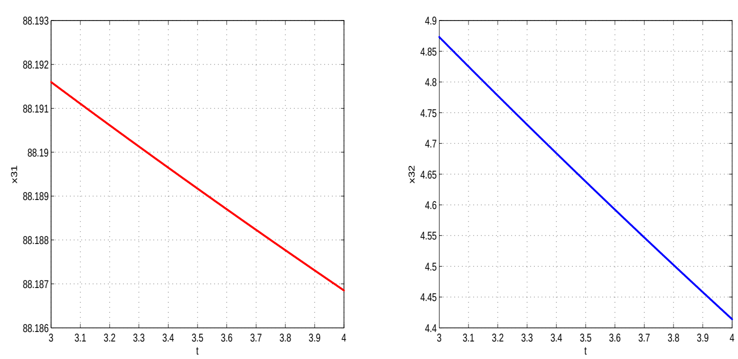

FIGURE 3. Susceptible & Infected Population in Time Interval 3.

considers application of treatments at distinct times with the prospect of reducing the susceptible and infected population and cost. The model is appropriate for a scheduled combination of vaccination and quarantine treatments. The numerical computation shows decreases in the number of susceptible and infected population steadily in each time interval. Since the cost depend on the final population count of the infected population we see the cost decreases from each time interval to the next.

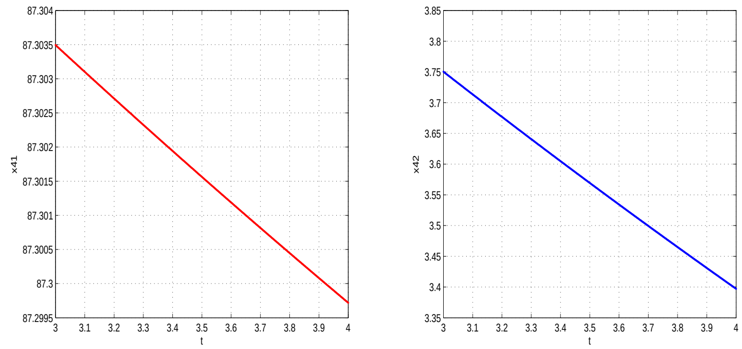

FIGURE 4. Susceptible Cost & Infected Cost in Time Interval 4.

REFERENCES

[1] N. G. Becker, D. N. Starczak, Optimal vaccination strategies for a community of household, Mathematical Biosciences, Vol. 139(1988), pp. 117–132.

[2] N. G. Beckerl, K. Glass, Z. Li, Controlling emerging infectious diseases like SARS, Mathematical Biosciences, Vol. 193 (2005) pp. 205–221.

[3] L. D. Berkovitz, D. N. G. Medhin, Nonlinear Optimal Control Theory, CRC Press 2012.

[4] M. Brandeau, G. S. Zaric, A. Richter, Resource allocation for control of infectious diseases in multiple independent populations: beyond cost effectiveness analysis, Jour. health economics, Vol. 22(2003) pp. 575–598.

[5] D. M. Edwards, R. D. Shachter, D. K. Owens, A Dynamic HIV-Transmission Model for Evaluating the Costs and Benefits of Vaccine Programs, Interfaces, Vol. 28, No. 3, Modeling AIDS(May-Jun., 1998), pp. 144–166 Published by INFORMS.

[6] K. Ganesh, M. Punniyamoorthyy, Optimization of continuous-time production planning using hybrid genetic algorithms-simulated annealing, The International Jour. of Advanced Manufacturing Technology, Vol. 26, No. 1–2, 2005, pp. 148–154.

[7] M. Garavello, B. Piccoli, Hybrid necessary principle, SIAM Jour. on Control and Optimization 43(2005), pp. 1867–1887.

[8] B. Gumel, S. M. Moghadas, R. E. Mickens, Effect of a preventive vaccine on the dynamics of HIV transmission, Communication in Nonlinear Science and Numerical Simulation, Vol. 9(2004) pp.649–659.

[9] S. H. Hou, K. H. Wong, Global Stability and Periodicity on SIS Epidemic Models with Backward Bifurcation, Computers and Mathematics with Applications 50(2005) pp. 1271–1290.

[10] J. Hui, D. Zhu, Optimal Impulsive Control Problem with Application to Human Immunodeficiency Virus Treatment, J. Optim. Theory Appl. 151 (2011) pp. 385–401.

[11] D. Iacoviello, G. Liuzzi, Optimal control for SIR epidemic model: a two treatment strategy, 16th Mediterranean Conf. on Control and Automation (2008) pp. 842–847.

[12] H. R. Joshi, Optimal control of an HIV immunology model, Optimal control applications and methods, Vol. 23 (2002) pp. 199–213.

[13] E. Jung, S. Lenhart, Z. Feng, Optimal control treatment of treatment in a two strain tuberculosis model, Discrete and continuous dynamical systems-SERIES B, Vol. 2 (2002) pp. 473–482.

[14] D. Kirschner. S. Lenhart, S. Serbin, Optimal control of the chemotherapy of HIV, Jour. Mathematical Biology, Vol. 35 (1997) pp. 775–792.

[15] N. G. Medhin, M. Sambandham, On the control of impulsive hybrid systems, Communications in Applied Analysis 16(2012) pp. 629–640. pp. 1587–1683.

[16] M. S. Shaikh, P. E. Caines, On the hybrid optimal control problem: Theory and algorithms, IEEE Transaction on Automatic Control 52(2007) pp. 1587–1683.

[17] G. N. Silva, R. B. Vinter, Necessary Conditions for Optimal Impulsive Control Problems, Proceedings of the 36th Conf. on Decision Control 1197, pp. 2086–2090.

[18] G. S. Zaric, M. Brandeau, Dynamic resource allocation for epidemic control in multiple populations, Jour. of Mathematics Applied in Medicine and Biology, Vo. 19(2002) pp. 235–255.