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AN ASYMPTOTIC NUMERICAL METHOD FOR SINGULARLY PERTURBED CONVECTION-DIFFUSION PROBLEMS WITH A NEGATIVE SHIFT

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ABSTRACT. In this paper an asymptotic numerical method on Shishkin mesh is suggested to solve singularly perturbed boundary value problem for second order ordinary differential equations of convection-diffusion type with a negative shift (delay). An error estimate is derived by using the supremum norm . Numerical results are provided to illustrate the theoretical results.

Key words: Singularly perturbed problem, Maximum principle, Convection-diffusion problem, Boundary value problem, Shishkin mesh, Delay, Negative shift

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1. Introduction

In many applications, one assumes the system under consideration is governed by a principle of causality; that is, the future state of the system is independent of the past states and is determined solely by the present. However, under closer scrutiny, it becomes apparent that the principle of causality is often a first order approximation to the true situation and more realistic model would involve some of the past states of the system. This kind of systems are governed by differential equations with delay arguments.

A subclass of these equations consists of singularly perturbed ordinary differential equations with a delay, that is an ordinary differential equation in which the highest derivative is multiplied by a small parameter and involving at least one delay term. Such type of equations arise frequently in the mathematical modeling of various practical phenomena, for example, in the modeling of the human pupil-light reflex [4], the mathematical model of the determination of expected time for generation of action potentials in nerve cell by random synaptic inputs in the dendrites

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[3] and variational problems in control theory [8], etc. The well-posedneess of the problems for this type of equations has been well studied in the literature [13, 14]. It is well known that standard discretization methods for solving singularly perturbed differential equations are sometimes unstable and fail to give accurate results when the perturbation parameter ε is small. Therefore, it is important to develop suitable numerical methods to solve this type of equations, whose accuracy does not depend on the parameter ε, that is the methods are uniformly convergent with respect to the parameter. For more details of this type of numerical methods one may refer to [1, 5, 6, 9].

In the past, only very few people had worked in the area numerical methods to Singularly Perturbed Delay Differential Equation(SPDDE). But in the recent years, there has been growing interest in this area. In [10, 12] the authors considered second order linear ordinary differential equations with small delay in the reaction term. In [10] they solved the problem by using a finite difference method on a special type of mesh. Kadalbajoo and Sharma [11] considered second order nonlinear ordinary differential equation with small delay in the first derivative and obtained numerical solution by the Newton's method of quasi linearization. In [12] the authors solved the delay problem by expanding the delay term by Taylor's expansion in order to reduce the delay problem to a non-delay problem. Then they applied a standard uniform numerical method for non-delay second order singularly perturbed differential equation.

Lange and Miura [2] made a study of a class of BVPs for linear second order differential difference equations in which the highest order derivative is multiplied by a small parameter. Motivated by this work, we, in this paper, consider the following singularly perturbed boundary value problem (2.1) for second order ordinary differential equations of convection-diffusion type with a negative shift and suggest an asymptotic numerical method. It is proved that this method is convergent of order $O(\varepsilon + N^{-1} \ln N)$.

The present paper is organized as follows. In Section 2, the problem under study with continuous source term is stated. A maximum principle for the DDE is established in Section 3. Further a stability result is derived in the same section. Some analytical results are derived in Section 4. In Section 5 a mesh selection strategy is explained. Further the fourth order Runge-Kutta method with piecewise cubic Hermite interpolation on this mesh for a initial value problem for first order delay differential equation and an upwind finite difference scheme for non delay singularly perturbed second order ordinary differential equation are described. Also their error estimates are given. Section 6 presents the Asymptotic Numerical Method (ANM). The applicability of the ANM to partial differential equations is illustrated in Section 7. Section 8 presents numerical results. The paper concludes with a discussion.

2. Statement of the problem

Throughout the paper we assume that C and C_1 denote generic positive constants independent of the singular perturbation parameter ε and the discretization parameter N of the discrete problem. The supremum norm is used for studying the convergence of the numerical solution to the exact solution of a singular perturbation problem:

$$\|w\|_\Omega = \sup_{x\in\Omega}|w(x)|.$$

Motivated by the work of [2], we consider the following Boundary Value Problem (BVP) for SPDDE.

Find $u \in Y = C^0(\overline{\Omega}) \cap C^1(\Omega) \cap C^2(\Omega^- \cup \Omega^+)$ such that

$$\begin{cases} -\varepsilon u''(x) + a(x)u'(x) + b(x)u(x) + c(x)u(x-1) = f(x), \ x \in \Omega^- \cup \Omega^+, \\ u(x) = \phi(x), \ x \in [-1,0], \ u(2) = l, \end{cases} \tag{2.1}$$

where $a(x) \geq \alpha_1 > \alpha > 0$, $b(x) \geq \beta_0 \geq 0$, $\gamma_0 \leq c(x) \leq \gamma < 0$, $2\alpha_1 + 5\beta_0 + 5\gamma_0 \geq \eta > 0$, a, b, c, f, and ϕ are sufficiently smooth functions on $\overline{\Omega}$, $\Omega = (0,2)$, $\overline{\Omega} = [0,2]$, $\Omega^- = (0,1)$, $\Omega^+ = (1,2)$.

The above problem is equivalent to

(2.2)
$$Pu(x) := \begin{cases} -\varepsilon u''(x) + a(x)u'(x) + b(x)u(x) = f(x) - c(x)\phi(x-1), \ x \in \Omega^-, \\ -\varepsilon u''(x) + a(x)u'(x) + b(x)u(x) + c(x)u(x-1) = f(x), \ x \in \Omega^+, \end{cases}$$

$$u(0) = \phi(0), \ u(1-) = u(1+), \ u'(1-) = u'(1+), \ u(2) = l,$$

where $u(1-)$ and $u(1+)$ denote the left and right limits of u at $x = 1$ respectively. This BVP (2.1) exhibits a strong boundary layer at $x = 2$ [2].

3. Stability Result

Let

$$s(x) = \begin{cases} \frac{1}{8} + \frac{x}{2}, & x \in [0,1], \\ \frac{3}{8} + \frac{x}{4}, & x \in [1,2]. \end{cases} \tag{3.1}$$

The differential-difference operator P defined in (2.2) satisfies the following maximum principle.

Theorem 3.1 (Maximum principle)**.** *Let $w \in C^0(\overline{\Omega}) \cap C^2(\Omega^- \cup \Omega^+)$ be any function satisfying $w(0) \geq 0$, $w(2) \geq 0$, $Pw(x) \geq 0$, $\forall x \in \Omega^- \cup \Omega^+$ and $w'(1+) - w'(1-) = [w'](1) \leq 0$. Then $w(x) \geq 0$, $\forall x \in \overline{\Omega}$.*

Proof. See [15]. □

Corollary 3.2 (Stability Result). *For any $u \in Y$ we have*

$$|u(x)| \leq C \ \max\{|u(0)|, |u(2)|, \sup_{\xi \in \Omega^- \cup \Omega^+} |Pu(\xi)|\}, \ \forall \ x \in \overline{\Omega}. \tag{3.2}$$

Proof. See [15]. □

4. **Analytical results**

Let $u_0 \in C^0(\overline{\Omega}) \cap C^1(\Omega \cup \{2\})$ be the solution of the reduced problem of (2.1) given by

$$\begin{cases} a(x)u_0'(x) + b(x)u_0(x) + c(x)u_0(x-1) = f(x), \ x \in \Omega \cup \{2\}, \\ u_0(x) = \phi(x), \ x \in [-1, 0]. \end{cases} \tag{4.1}$$

Further, we assume that, $\| u_0'' \|_{\Omega^- \cup \Omega^+} \leq C$.

Theorem 4.1. *Let u be the solution of* (2.1) *and u_0 be its reduced problem solution as defined in* (4.1). *Then, $|u(x) - u_0(x)| \leq C\varepsilon + C\exp(\frac{-\alpha(2-x)}{\varepsilon}), \ x \in \overline{\Omega}$.*

Proof. Consider the barrier function

$$\varphi^{\pm}(x) = C_1 \varepsilon s(x) + C_1 \exp\left(\frac{-\alpha(2-x)}{\varepsilon}\right) \pm (u(x) - u_0(x)), \ x \in \overline{\Omega},$$

where s is defined by (3.1). It is easy to see that $\varphi^{\pm} \in C^0(\overline{\Omega}) \cap C^2(\Omega^- \cup \Omega^+)$. Further, $\varphi^{\pm}(0) \geq 0$ and $\varphi^{\pm}(2) \geq 0$ for a suitable choice of $C_1 > 0$. When $x \in \Omega^-$ we have

$$\begin{aligned} P\varphi^{\pm}(x) &= C_1 \Big[\Big[\frac{\alpha}{\varepsilon}(a(x) - \alpha) + b(x)\Big] \exp\left(\frac{-\alpha(2-x)}{\varepsilon}\right) + \varepsilon[a(x)s'(x) + b(x)s(x)]\Big] \\ &\quad \pm \varepsilon u_0''(x) \\ &\geq C_1 \Big[\Big[\frac{\alpha}{\varepsilon}(\alpha_1 - \alpha) + \beta_0\Big] \exp\left(\frac{-\alpha(2-x)}{\varepsilon}\right) + \varepsilon[\alpha_1/2 + \beta_0/8]\Big] \\ &\quad \mp C\varepsilon \ \geq 0, \end{aligned}$$

for a suitable choice of $C_1 > 0$. When $x \in \Omega^+$ we have

$$\begin{aligned} P\varphi^{\pm}(x) &= C_1 \Big[\Big[\frac{\alpha}{\varepsilon}(a(x) - \alpha) + b(x) + c(x)\exp\left(\frac{-\alpha}{\varepsilon}\right)\Big] \exp\left(\frac{-\alpha(2-x)}{\varepsilon}\right) \\ &\quad + \varepsilon[a(x)s'(x) + b(x)s(x) + c(x)s(x-1)]\Big] \pm \varepsilon u_0''(x) \\ &\geq C_1 \Big[\Big[\frac{\alpha}{\varepsilon}(\alpha_1 - \alpha) + \beta_0 + \gamma_0 \exp\left(\frac{-\alpha}{\varepsilon}\right)\Big] \exp\left(\frac{-\alpha(2-x)}{\varepsilon}\right) \\ &\quad + \varepsilon[\alpha_1/4 + 5\beta_0/8 + 5\gamma_0/8]\Big] \mp C\varepsilon \ \geq 0, \end{aligned}$$

for a suitable choice of $C_1 > 0$. Then by the Theorem 3.1 we have $\varphi^{\pm}(x) \geq 0, \ x \in \overline{\Omega}$, that is

$$|u(x) - u_0(x)| \leq C\varepsilon + C\exp\left(\frac{-\alpha(2-x)}{\varepsilon}\right), \ x \in \overline{\Omega}.$$

Hence the proof of the theorem. □

Note 4.2. From the above Theorem 4.1, it is clear that the solution u of the boundary value problem (2.1) exhibits a strong boundary layer at $x = 2$ and further, away from the boundary layer and in particular on $[0, 1]$, we have

$$|u(x) - u_0(x)| \le C\Big(\varepsilon + \exp\big(-\alpha/\varepsilon\big)\Big) \le C\varepsilon,\ x \in [0, 1].$$

We now define an auxiliary problem to (2.1). Find $u^* \in Y^* = C^0(\overline{\Omega}) \cap C^2(\Omega)$ such that

$$\begin{cases} P^*u^*(x):\ = -\varepsilon u^{*''}(x) + a(x)u^{*'}(x) + b(x)u^*(x) = f^*(x), \\ u^*(0) = u(0),\ u^*(2) = u(2), \end{cases} \tag{4.2}$$

where

$$f^*(x) = \begin{cases} f(x) - c(x)\phi(x-1),\ x \in \Omega^- \cup \{1\}, \\ f(x) - c(x)u_0(x-1),\ x \in \Omega^+. \end{cases}$$

We now state a maximum principle for this problem.

Theorem 4.3. *Let $w \in C^0(\overline{\Omega}) \cap C^2(\Omega^- \cup \Omega^+)$ be any function satisfying $w(0) \ge 0$, $w(2) \ge 0$, $P^*w(x) \ge 0$, $\forall x \in \Omega^- \cup \Omega^+$ and $[w'](1) \le 0$. Then, $w(x) \ge 0$, $\forall x \in \overline{\Omega}$.*

Proof. See [9]. □

Theorem 4.4. *Let u and u^* be the solutions of the problems* (2.1) *and* (4.2) *respectively. Then, $|u(x) - u^*(x)| \le C\varepsilon$, $x \in \overline{\Omega}$.*

Proof. Consider the barrier function

$$\varphi^{\pm}(x) = C_1\varepsilon\ s(x) \pm z(x),\ x \in \overline{\Omega},$$

where, $z(x) = u(x) - u^*(x)$. Note that $\varphi^{\pm} \in C^0(\overline{\Omega}) \cap C^2(\Omega^- \cup \Omega^+)$. Further, $\varphi^{\pm}(0) \ge 0$ and $\varphi^{\pm}(2) \ge 0$ for a suitable choice of $C_1 > 0$. Also $[\varphi^{\pm'}(1)] < 0$. When $x \in \Omega^-$ we have

$$\begin{aligned} P\varphi^{\pm}(x) &= C_1\varepsilon\Big[a(x)s'(x) + b(x)s(x)\Big] \pm 0 \\ &\ge C_1\varepsilon[\alpha_1/2 + \beta_0/8] \pm 0\ \ge 0, \end{aligned}$$

for a suitable choice of $C_1 > 0$. When $x \in \Omega^+$ we have

$$\begin{aligned} P\varphi^{\pm}(x) &= C_1\varepsilon[a(x)s'(x) + b(x)s(x)] \pm c(x)[u_0(x-1) - u(x-1)] \\ &\ge C_1\varepsilon[\alpha_1/4 + 5\beta_0/8] \mp C\varepsilon\ \ge 0, \end{aligned}$$

for a suitable choice of $C_1 > 0$.

Then by the Theorem 4.3 we have $\varphi^{\pm}(x) \ge 0$, $x \in \overline{\Omega}$, that is

$$|u(x) - u^*(x)| \le C\varepsilon,\ x \in \overline{\Omega}.$$

Hence the proof of the theorem. □

5. **Discrete problem**

In this section, first a mesh selection strategy is explained. Then the fourth order Runge-Kutta method with piecewise cubic Hermite interpolation on this mesh for initial value problem (4.1) and an upwind finite difference scheme for the BVP (4.2) are presented. Further results on error estimates of these methods are stated.

5.1. **Mesh selection strategy.** Since the BVP (2.1) and the auxiliary problem (4.2) exhibit a strong boundary layer at $x = 2$ [2] and the function f^* is not differentiable at $x = 1$, we choose a piece-wise uniform Shishkin mesh on $[0, 2]$. For this we divide the interval $[0, 2]$ into four subintervals, namely $\Omega_1 = [0, 1 - \tau]$, $\Omega_2 = [1 - \tau, 1]$, $\Omega_3 = [1, 2 - \tau]$, $\Omega_4 = [2 - \tau, 2]$, where $\tau = \min\{0.5, \frac{2\varepsilon \ln N}{\alpha}\}$. Let $h = 2N^{-1}\tau$ and $H = 2N^{-1}(1 - \tau)$. The mesh $\overline{\Omega}^{2N} = \{x_i\}_{i=1}^{2N}$ is defined by

$$x_0 = 0.0,\ x_i = x_0 + iH,\ i = 1(1)\frac{N}{2},\ x_{i+\frac{N}{2}} = x_{\frac{N}{2}} + ih,\ i = 1(1)\frac{N}{2},$$
$$x_{i+N} = x_N + iH,\ i = 1(1)\frac{N}{2},\ x_{i+\frac{3N}{2}} = x_{\frac{3N}{2}} + ih,\ i = 1(1)\frac{N}{2}.$$

5.2. **Numerical Method for (4.1).** In order to obtain a numerical solution for the problem (4.1), we apply the fourth order Runge-Kutta method with piecewise cubic Hermite interpolation on $\overline{\Omega}^{2N}$ [7]. In fact, the numerical solution is given by

$$U_0(x_0) = \phi(x_0),$$
$$U_0(x_{i+1}) = U_0(x_i) + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4),\ i = 0(1)2N - 1, \tag{5.1}$$

where

$$\begin{cases} K_1 = h^* \left[\frac{f(x_i)}{a(x_i)} - \frac{b(x_i)}{a(x_i)}U_0(x_i) - \frac{c(x_i)}{a(x_i)}U_0^{h^*}(x_i)\right], \\ K_2 = h^* \left[\frac{f(x_i+\frac{h^*}{2})}{a(x_i+\frac{h^*}{2})} - \frac{b(x_i+\frac{h^*}{2})}{a(x_i+\frac{h^*}{2})}(U_0(x_i) + \frac{K_1}{2}) - \frac{c(x_i+\frac{h^*}{2})}{a(x_i+\frac{h^*}{2})}U_0^{h^*}(x_i + \frac{h^*}{2})\right], \\ K_3 = h^* \left[\frac{f(x_i+\frac{h^*}{2})}{a(x_i+\frac{h^*}{2})} - \frac{b(x_i+\frac{h^*}{2})}{a(x_i+\frac{h^*}{2})}(U_0(x_i) + \frac{K_2}{2}) - \frac{c(x_i+\frac{h^*}{2})}{a(x_i+\frac{h^*}{2})}U_0^{h^*}(x_i + \frac{h^*}{2})\right], \\ K_4 = h^* \left[\frac{f(x_i+h^*)}{a(x_i+h^*)} - \frac{b(x_i+h^*)}{a(x_i+h^*)}(U_0(x_i) + K_3) - \frac{c(x_i+h^*)}{a(x_i+h^*)}U_0^{h^*}(x_i + h^*)\right], \end{cases}$$

$$h^* = \begin{cases} H,\ i = 0(1)\frac{N}{2} - 1,\ i = N(1)\frac{3N}{2} - 1, \\ h,\ i = \frac{N}{2}(1)N - 1,\ i = \frac{3N}{2}(1)2N - 1, \end{cases}$$

$$U_0^{h^*}(x) = \begin{cases} \phi(x - 1),\ x \in [x_i, x_{i+1}],\ i = 0(1)N - 1, \\ U_0(x_{i-N})A_{i-N}(x - 1) + U_0(x_{i-N+1})A_{i+1-N}(x - 1) + \\ B_{i-N}(x - 1)\tilde{f}(x_{i-N}) + B_{i+1-N}(x - 1)\tilde{f}(x_{i-N+1}), \\ \qquad x \in [x_i, x_{i+1}],\ i = N(1)2N - 1, \end{cases}$$

$$A_i(x) = \left[1 - \frac{2(x-x_i)}{x_i - x_{i+1}}\right]\frac{(x-x_{i+1})^2}{(x_i-x_{i+1})^2},\ A_{i+1}(x) = \left[1 - \frac{2(x-x_{i+1})}{x_{i+1}-x_i}\right]\frac{(x-x_i)^2}{(x_{i+1}-x_i)^2},$$
$$B_i(x) = \frac{(x-x_i)(x-x_{i+1})^2}{(x_i-x_{i+1})^2},\ B_{i+1}(x) = \frac{(x-x_{i+1})(x-x_i)^2}{(x_{i+1}-x_i)^2},$$
$$\tilde{f}(x_{i-N}) = \frac{f(x_{i-N})}{a(x_{i-N})} - \frac{b(x_{i-N})}{a(x_{i-N})}U_0(x_{i-N}) - \frac{c(x_{i-N})}{a(x_{i-N})}\phi(x_{i-N}-1),$$
$$\tilde{f}(x_{i-N+1}) = \frac{f(x_{i-N+1})}{a(x_{i-N+1})} - \frac{b(x_{i-N+1})}{a(x_{i-N+1})}U_0(x_{i-N+1}) - \frac{c(x_{i-N+1})}{a(x_{i-N+1})}\phi(x_{i-N+1}-1).$$

Theorem 5.1. *Let $u_0(x)$ be the solution of the problem* (4.1). *Further let $U_0(x_i)$ be its numerical solution defined by* (5.1). *Then, $\| u_0 - U_0 \|_{\overline{\Omega}^{2N}} \leq C\overline{h}^4$, where $\overline{h} = \max\{H, h\}$.*

Proof. See [7]. □

5.3. **A finite difference scheme for (4.2).** On $\overline{\Omega}^{2N}$, we define the following scheme for the BVP (4.2):

$$(5.2)\qquad \begin{cases} P^{*N}U^*(x_i) = -\varepsilon\delta^2 U^*(x_i) + a(x_i)D^- U^*(x_i) + b(x_i)U^*(x_i) = F^*(x_i), \\ \qquad\qquad i = 1(1)N-1, N+1(1)2N-1, \\ D^-U^*(x_N) = D^+U^*(x_N), \\ U^*(x_0) = u^*(0),\ U^*(x_{2N}) = u^*(2), \end{cases}$$

where

$$\delta^2 U^*(x_i) = \frac{2}{x_{i+1}-x_{i-1}}\left[D^+U^*(x_i) - D^-U^*(x_i)\right],$$
$$D^-U^*(x_i) = \frac{U^*(x_i) - U^*(x_{i-1})}{x_i - x_{i-1}},\ D^+U^*(x_i) = \frac{U^*(x_{i+1}) - U^*(x_i)}{x_{i+1}-x_i},$$

$$(5.3)\qquad F^*(x_i) = f^*(x_i),\ x_i \in \overline{\Omega}^{2N} \setminus \{x_0, x_N, x_{2N}\}$$

or

$$(5.4)\qquad F^*(x_i) = \begin{cases} f(x_i) - c(x_i)\phi(x_i - 1),\ x_i \in \Omega^- \cap \overline{\Omega}^{2N}, \\ f(x_i) - c(x_i)U_{0_{i-N}},\ x_i \in \Omega^+ \cap \overline{\Omega}^{2N}. \end{cases}$$

Theorem 5.2 (Discrete maximum principle)**.** *Suppose a mesh function $Z(x_i)$ satisfies $Z(x_0) \geq 0$, $Z(x_{2N}) \geq 0$, $P^{*N}Z(x_i) \geq 0$, $x_i \in \overline{\Omega}^{2N} \setminus \{x_0, x_N, x_{2N}\}$ and $[D]Z(x_N) = D^+Z(x_N) - D^-Z(x_N) \leq 0$. Then, $Z(x_i) \geq 0$, $\forall x_i \in \overline{\Omega}^{2N}$.*

Proof. See [9]. □

A consequence of this theorem is the following stability result.

Theorem 5.3. *Let $U^*(x_i)$ be a numerical solution of the problem* (5.2)*. Then,*

$$|U^*(x_i)| \leq C \max\{|U^*(x_0)|, |U^*(x_{2N})|, \max_{j\in J} P^{*N}U^*(x_j)\},$$
$$J = \{1, \dots, N-1, N+1, \dots, 2N-1\},\ i = 0(1)2N.$$

Theorem 5.4. *Let u^* be the solution of the auxiliary problem* (4.2) *and let $U^*(x_i)$ be the corresponding numerical solution defined by* (5.2) *and* (5.3)*. Then,*

$$|u^*(x_i) - U^*(x_i)| \leq CN^{-1}\ln N,\ x_i \in \overline{\Omega}^{2N}.$$

Proof. See [9]. □

Theorem 5.5. *Let u^* be the solution of the auxiliary problem* (4.2) *and let $U^*(x_i)$ be the corresponding numerical solution defined by* (5.2) *and* (5.4)*. Then,*

$$|u^*(x_i) - U^*(x_i)| \leq CN^{-1}\ln N,\ x_i \in \overline{\Omega}^{2N}.$$

Proof. Using the Theorem 5.3 and the results given in [9] one can derive the desired result. □

6. **Asymptotic numerical method**

We now explain how to obtain a numerical solution for the BVP (2.1) by the ANM. First we solve the reduced problem (4.1) either exactly or numerically. Then we solve numerically the auxiliary problem (4.2) by using the scheme (5.2) with either (5.3) or (5.4). This numerical solution is taken as an approximation to the exact solution of the BVP (2.1). An error estimate for this approximation is given in the following theorem.

Theorem 6.1. *Let u be the solution of the problem* (2.1) *and let $U^*(x_i)$ be a numerical solution defined by* (5.2) *with either* (5.3) *or* (5.4)*. Then,*

$$\| u - U^* \|_{\overline{\Omega}^{2N}} \leq C(\varepsilon + N^{-1}\ln N).$$

Proof. Then by the Theorems 4.4 and 5.4 or 5.5 we have,

$$|u(x_i) - U^*(x_i)| \leq |u(x_i) - u^*(x_i)| + |u^*(x_i) - U^*(x_i)|,\ x_i \in \overline{\Omega}^{2N}$$
$$\leq C\varepsilon + CN^{-1}\ln N = C(\varepsilon + N^{-1}\ln N).$$

Hence the proof of the theorem. □

7. An application to partial differential equations

To illustrate the applicability of the ANM to PDEs we now consider the following Initial Boundary Value Problem (IBVP):

Find $u \in C^0(\overline{G}) \cap C^1(G) \cap C^2(G^- \cup G^+)$ such that

$$(7.1)\qquad \begin{cases} -\varepsilon u_{xx}(x,t) + u_t + a(x)u_x(x,t) + b(x)u(x,t) + c(x)u(x-1,t) \\ \qquad\qquad = f(x,t),\ (x,t) \in G^- \cup G^+, \\ u(x,t) = \phi(x,t),\ (x,t) \in [-1,0] \times [0,T], \\ u(x,0) = g(x),\ x \in [0,2], \\ u(2,t) = g_1(t),\ t \in [0,T]. \end{cases}$$

where $0 < \varepsilon \ll 1$, $\alpha^* \geq a(x) \geq \alpha_1 > \alpha > 0$, $b(x) \geq \beta_0 \geq 0$, $\gamma_0 \leq c(x) \leq 0$, $2\alpha + 5\beta_0 + 5\gamma_0 \geq \eta > 0$, a, b, c and f are sufficiently smooth functions on $\bar{G}$, $G = (0,2) \times (0,T]$, $G_1 = (0,2] \times (0,T]$, $G^- = \Omega^- \times (0,T]$, $G^+ = \Omega^+ \times (0,T]$.

The above problem (7.1) is equivalent to

$$(7.2)\qquad \mathcal{P}u(x,t) := \begin{cases} -\varepsilon u_{xx}(x,t) + a(x)u_x(x,t) + b(x)u(x,t) \\ \qquad\qquad = f(x,t) - c(x)\phi(x-1,t),\ (x,t) \in G^-, \\ -\varepsilon u_{xx}(x,t) + a(x)u_x(x,t) + b(x)u(x,t) \\ \qquad\qquad + c(x)u(x-1,t) = f(x,t),\ (x,t) \in G^+, \end{cases}$$

$$\begin{aligned} &u(x,0) = g(x),\ x \in \Omega, \\ &u(0,t) = \phi(0,t),\ u(2,t) = g_1(t),\ t \in [0,T], \\ &u(1-,t) = u(1+,t),\ u_x(1-,t) = u_x(1+,t),\ t \in [0,T], \end{aligned}$$

where $u(1-,t)$ and $u(1+,t)$ denote the left and right limits of u at $(1,t)$, respectively.

Let $u_0(x,t)$ be the solution of the reduced problem of (7.1) given by

$$(7.3)\quad \begin{cases} \dfrac{\partial u_0(x,t)}{\partial t} + a(x)\dfrac{\partial u_0(x,t)}{\partial x} + b(x)u_0(x,t) + c(x)u_0(x-1,t) = f(x,t),\ (x,t) \in G_1, \\ u_0(x,t) = \phi(x,t),\ (x,t) \in [-1,0] \times [0,T], \\ u_0(x,0) = g(x),\ x \in [0,2]. \end{cases}$$

Further, it is assumed that, $\| \frac{\partial^2 u_0}{\partial x^2} \|_{G^- \cup G^+} \leq C$.

We now define an auxiliary problem to (7.1). Find $u^* \in Y^* = C^0(\overline{G}) \cap C^2(G)$ such that

(7.4)
$$\begin{cases} \mathcal{P}^* u^*(x,t) : = -\varepsilon u^*_{xx}(x,t) + a(x) u^*_x(x,t) + b(x) u^*(x,t) = f^*(x,t),\ (x,t) \in G, \\ u^*(x,0) = u(x,0), x \in [0,2] \\ u^*(0,t) = u(0,t),\ u^*(1,t) = u(1,t),\ t \in [0,T]. \end{cases}$$

where

$$f^*(x,t) = \begin{cases} f(x,t) - c(x)\phi(x-1,t),\ (x,t) \in G^- \cup \{(1,t) : t \in (0,T]\}, \\ f(x,t) - c(x)u_0(x-1,t),\ (x,t) \in G^+. \end{cases}$$

Step 1: Solve the reduced problem (7.3) of the IBVP (7.1) numerically by using the scheme suggested in [16]:

$$(7.5) \qquad \begin{cases} D_t^+ U_0(x_i,t_j) + a(x_i) D_x^- U_0(x_i,t_j) + b(x_i) U_0(x_i,t_j) \\ \qquad\qquad + c(x_i) U_0(x_i - 1, t_j) = f(x_i,t_j),\ (x_i,t_j) \in G_1 \cap \bar{\Omega}^{2N,\ M} \\ U_0(x_0,t_j) = \phi(x_0,t_j),\ j = 1(1)M, \\ U_0(x_i,t_0) = u(x_i,t_0)),\ i = 1(1)2N, \end{cases}$$

where

$$(7.6) \qquad \bar{\Omega}^{2N,\ M} = \bar{\Omega}^{2N} \times \{t_i : t_i = ih, i = 0(1)M,\ h = T/M\},$$

$\bar{\Omega}^{2N}$ is defined in Section 5.1.

Step 2: Solve the above IBVP (7.4) on the mesh $\bar{\Omega}^{2N,\ M}$, by using the finite difference scheme given in [17, 18] we get

(7.7)
$$\begin{cases} \mathcal{P}^{N,M} U^*(x_i,t_j) = -\varepsilon \delta_x^2 U^*(x_i,t_j) + D_t^- U^*(x_i,t_j) + a(x_i) D_x^- U^*(x_i,t_j) \\ + b(x_i) U^*(x_i,t_j) = F^*(x_i,t_j),\ i = 1(1)N-1, N+1(1)2N-1,\ j = 1(1)M, \\ D_x^- U^*(x_N,t_j) = D_x^+ U^*(x_N,t_j),\ j = 1(1)M \\ U^*(x_i,t_0) = u^*(x_i,t_0),\ i = 1(1)2N, \\ U^*(x_0,t_j) = u^*(0,t_j),\ U^*(x_{2N},t_j) = u^*(2,t_j),\ j = 1(1)M \end{cases}$$

where

$$\delta_x^2 U^*(x_i,t_j) = \frac{2}{x_{i+1} - x_{i-1}} \left[D_x^+ U^*(x_i,t_j) - D_x^- U^*(x_i,t_j) \right],$$
$$D_x^- U^*(x_i,t_j) = \frac{U^*(x_i,t_j) - U^*(x_{i-1},t_j)}{x_i - x_{i-1}},$$
$$D_x^+ U^*(x_i,t_j) = \frac{U^*(x_{i+1},t_j) - U^*(x_i,t_j)}{x_{i+1} - x_i},$$
$$D_t^- U^*(x_i,t_j) = \frac{U^*(x_i,t_j) - U^*(x_i,t_{j-1})}{t_j - t_{j-1}},$$

$$F^*(x_i,t_j)=\begin{cases} f(x_i,t_j)-c(x_i)\phi(x_i-1,t_j),\ (x_i,t_j)\in G^-\cap\overline{\Omega}^{2N,\ M}, \\ f(x_i,t_j)-c(x_i)u_0(x_i-1,t_j),\ (x_i,t_j)\in G^+\cap\overline{\Omega}^{2N,\ M} \end{cases} \tag{7.8}$$

or

$$F^*(x_i,t_j)=\begin{cases} f(x_i,t_j)-c(x_i)\phi(x_i-1,t_j),\ (x_i,t_j)\in G^-\cap\overline{\Omega}^{2N,\ M}, \\ f(x_i,t_j)-c(x_i)U_0(x_{i-N},t_j),\ (x_i,t_j)\in G^+\cap\overline{\Omega}^{2N,\ M}. \end{cases} \tag{7.9}$$

The numerical solution $U^*(x_i,t_j)$ will be taken as an approximation to the solution of the IBVP (7.1).

8. **Numerical results**

In this section, three examples are given to illustrate the numerical method discussed in this paper. The double mesh principle is used to estimate the error and compute the experiment rate of convergence in our computed solution. For this we put

$$D_\varepsilon^M=\max_{0\le i\le M}|U_i^M-U_{2i}^{2M}|,$$

where U_i^M and U_{2i}^{2M} are the i^{th} components of the numerical solutions on meshes of M and $2M$ points respectively, here $M=2N$. We compute the uniform error and rate of convergence as

$$D^M=\max_\varepsilon D_\varepsilon^M,\ p^M=\log_2\left(\frac{D^M}{D^{2M}}\right).$$

Example 8.1. (Constant coefficient problem)

$$\begin{cases} -\varepsilon u''(x)+3u'(x)-u(x-1)=0,\ x\in\Omega^-\cup\Omega^+ \\ u(x)=1,\ x\in[-1,0],\ u(2)=2. \end{cases} \tag{8.1}$$

The exact solution of this problem is given by

$$u(x)=\begin{cases} 1+c_1[e^{\frac{3x}{\varepsilon}}-1]+\frac{x}{3},\ x\in[0,1], \\ c_2+\frac{x}{3}+\frac{(x-1)^2}{18}+\frac{\varepsilon x}{27}-\frac{c_1x}{3}-\frac{c_1x}{3}e^{\frac{3(x-1)}{\varepsilon}} \\ +e^{\frac{3(x-2)}{\varepsilon}}[\frac{23}{18}-\frac{2\varepsilon}{27}-c_2+\frac{2c_1}{3}+\frac{2c_1}{3}e^{\frac{3}{\varepsilon}}],\ x\in[1,2], \end{cases}$$

where

$$c_1=e^{\frac{-6}{\varepsilon}}\left[\frac{\frac{4\varepsilon}{9}-\frac{\varepsilon^2}{27}-3}{3-4e^{\frac{-6}{\varepsilon}}+\frac{2\varepsilon}{3}[e^{\frac{-3}{\varepsilon}}-e^{\frac{-6}{\varepsilon}}]}\right],$$

$$c_2=\left[\frac{1-\frac{23}{18}e^{\frac{-3}{\varepsilon}}+\frac{2\varepsilon}{27}e^{\frac{-3}{\varepsilon}}-\frac{\varepsilon}{27}+c_1e^{\frac{3}{\varepsilon}}[1-e^{\frac{-3}{\varepsilon}}-\frac{2}{3}e^{\frac{-6}{\varepsilon}}]}{1-e^{\frac{-3}{\varepsilon}}}\right].$$

Table 1 presents the values of D^M and p^M for this problem and Figure 1 represents the numerical solution $U^*(x)$, exact solution $u(x)$ and the reduced problem solution $u_0(x)$.

Example 8.2 (Variable coefficient problem).

$$\begin{cases} -\varepsilon u''(x) + (x+10)u'(x) - u(x-1) = x, \ x \in \Omega^- \cup \Omega^+ \\ u(x) = x, \ x \in [-1,0], \ u(2) = 2. \end{cases} \tag{8.2}$$

Table 2 presents the values of D^M and p^M for this problem.

Example 8.3 (Partial differential equation).

$$-\varepsilon u_{xx}(x,t) + u_t(x,t) + 3u_x(x,t) + u(x,t) - u(x-1,t) = 1,$$
$$u(x,t) = 1, \ (x,t) \in [-1,0] \times [0,1],$$
$$u(x,0) = 1, \ x \in [0,2], \ u(2,t) = 1, \ t \in [0,1].$$

Table 3 presents the values of D^M and p^M for this problem and Figure 2 represents the numerical solution $U^*(x,t)$.

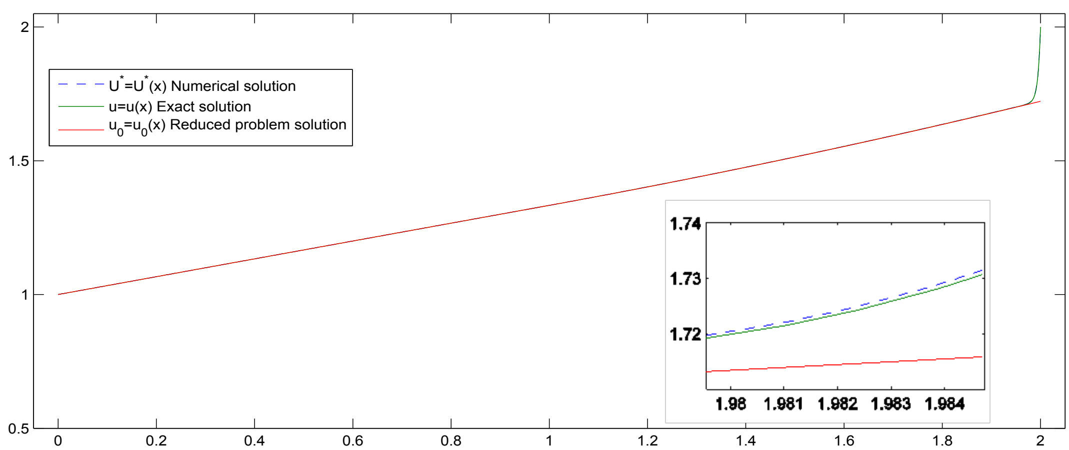

FIGURE 1. Numerical solution U^*, exact solution u and reduced problem solution u_0 of the Example 8.1.

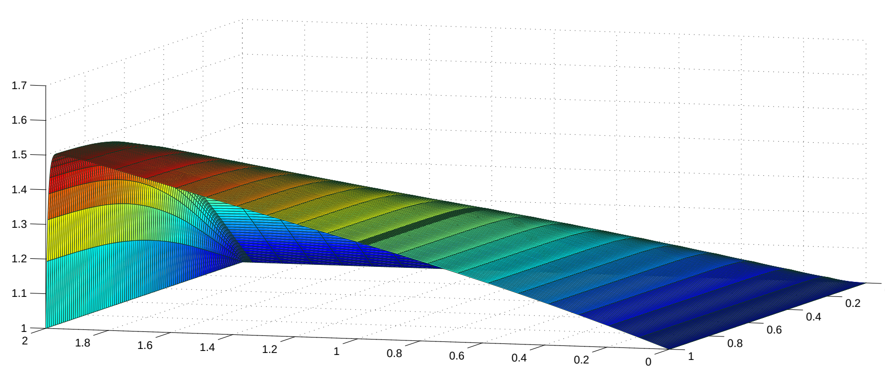

FIGURE 2. Numerical solution $U^*(x,t)$ of the Example 8.3.

TABLE 1. Numerical results for the Example 8.1

ε	M (Number of mesh points)						
$\downarrow$	64	128	256	512	1024	2048	4096
2^{-6}	7.6654e-3	4.4464e-3	2.5266e-3	1.4405e-3	8.0398e-4	4.4359e-4	2.4261e-4
2^{-7}	7.7791e-3	4.5252e-3	2.5677e-3	1.4614e-3	8.1282e-4	4.4716e-4	2.4403e-4
2^{-8}	7.8352e-3	4.5658e-3	2.5920e-3	1.4758e-3	8.2007e-4	4.5037e-4	2.4527e-4
2^{-9}	7.8625e-3	4.5856e-3	2.6044e-3	1.4839e-3	8.2507e-4	4.5310e-4	2.4648e-4
2^{-10}	7.8760e-3	4.5951e-3	2.6103e-3	1.4880e-3	8.2788e-4	4.5501e-4	2.4755e-4
2^{-11}	7.8826e-3	4.5998e-3	2.6131e-3	1.4899e-3	8.2925e-4	4.5607e-4	2.4831e-4
2^{-12}	7.8859e-3	4.6021e-3	2.6145e-3	1.4908e-3	8.2988e-4	4.5658e-4	2.4873e-4
2^{-13}	7.8875e-3	4.6033e-3	2.6151e-3	1.4912e-3	8.3016e-4	4.5681e-4	2.4893e-4
2^{-14}	7.8883e-3	4.6038e-3	2.6155e-3	1.4914e-3	8.3030e-4	4.5691e-4	2.4901e-4
2^{-15}	7.8887e-3	4.6041e-3	2.6156e-3	1.4915e-3	8.3036e-4	4.5696e-4	2.4905e-4
2^{-16}	7.8890e-3	4.6043e-3	2.6157e-3	1.4915e-3	8.3040e-4	4.5699e-4	2.4907e-4
2^{-17}	7.8891e-3	4.6043e-3	2.6157e-3	1.4916e-3	8.3041e-4	4.5700e-4	2.4908e-4
2^{-18}	7.8891e-3	4.6044e-3	2.6158e-3	1.4916e-3	8.3042e-4	4.5700e-4	2.4908e-4
2^{-19}	7.8891e-3	4.6044e-3	2.6158e-3	1.4916e-3	8.3042e-4	4.5700e-4	2.4908e-4
2^{-20}	7.8891e-3	4.6044e-3	2.6158e-3	1.4916e-3	8.3043e-4	4.5701e-4	2.4909e-4
2^{-21}	7.8892e-3	4.6044e-3	2.6158e-3	1.4916e-3	8.3043e-4	4.5701e-4	2.4909e-4
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2^{-25}	7.8892e-3	4.6044e-3	2.6158e-3	1.4916e-3	8.3043e-4	4.5701e-4	2.4909e-4
D^M	7.8892e-3	4.6044e-3	2.6158e-3	1.4916e-3	8.3043e-4	4.5701e-4	2.4909e-4
p^M	7.7685e-1	8.1578e-1	8.1039e-1	8.4492e-1	8.6164e-1	8.7557e-1	-

9. **Discussion**

A BVP for a class of SPDDEs is considered. To obtain an approximate solution for this type of problems, an asymptotic numerical method is presented. The method is shown to be convergent of order $O(\varepsilon + N^{-1}\ln N)$. From the Tables 1-2, we see that the convergent order is $O(N^{-1}\ln N)$. An application of the ANM to partial differential equations is given in Section 7.

Table 2. Numerical results for the Example 8.2

ε	M (Number of mesh points)						
$\downarrow$	64	128	256	512	1024	2048	4096
2^{-6}	7.7509e-2	6.5711e-2	4.7100e-2	3.1385e-2	1.8340e-2	1.0657e-2	6.0429e-3
2^{-7}	7.7559e-2	6.5749e-2	4.7124e-2	3.1399e-2	1.8348e-2	1.0662e-2	6.0455e-3
2^{-8}	7.7584e-2	6.5768e-2	4.7136e-2	3.1407e-2	1.8352e-2	1.0664e-2	6.0468e-3
2^{-9}	7.7597e-2	6.5778e-2	4.7142e-2	3.1410e-2	1.8354e-2	1.0665e-2	6.0474e-3
2^{-10}	7.7603e-2	6.5783e-2	4.7145e-2	3.1412e-2	1.8355e-2	1.0666e-2	6.0478e-3
2^{-11}	7.7606e-2	6.5785e-2	4.7146e-2	3.1413e-2	1.8356e-2	1.0666e-2	6.0479e-3
2^{-12}	7.7608e-2	6.5787e-2	4.7147e-2	3.1414e-2	1.8356e-2	1.0666e-2	6.0480e-3
2^{-13}	7.7608e-2	6.5787e-2	4.7147e-2	3.1414e-2	1.8356e-2	1.0667e-2	6.0481e-3
2^{-14}	7.7609e-2	6.5788e-2	4.7148e-2	3.1414e-2	1.8356e-2	1.0667e-2	6.0481e-3
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2^{-25}	7.7609e-2	6.5788e-2	4.7148e-2	3.1414e-2	1.8356e-2	1.0667e-2	6.0481e-3
D^M	7.7609e-2	6.5788e-2	4.7148e-2	3.1414e-2	1.8356e-2	1.0667e-2	6.0481e-3
p^M	2.3841e-1	4.8063e-1	5.8578e-1	7.7515e-1	7.8315e-1	8.1855e-1	-

Table 3. Numerical results for the Example 8.3

ε	M (Number of mesh points)				
$\downarrow$	16	32	64	128	256
2^{-6}	3.5276e-2	2.5136e-2	1.9006e-2	1.3578e-2	8.8928e-3
2^{-7}	3.5631e-2	2.5426e-2	1.9308e-2	1.3841e-2	9.1276e-3
2^{-8}	3.5809e-2	2.5573e-2	1.9464e-2	1.3985e-2	9.2834e-3
2^{-9}	3.5899e-2	2.5647e-2	1.9543e-2	1.4061e-2	9.3718e-3
2^{-10}	3.5943e-2	2.5685e-2	1.9584e-2	1.4100e-2	9.4186e-3
2^{-11}	3.5965e-2	2.5703e-2	1.9604e-2	1.4119e-2	9.4428e-3
2^{-12}	3.5977e-2	2.5713e-2	1.9614e-2	1.4129e-2	9.4550e-3
2^{-13}	3.5982e-2	2.5717e-2	1.9619e-2	1.4134e-2	9.4612e-3
2^{-14}	3.5985e-2	2.5720e-2	1.9621e-2	1.4137e-2	9.4643e-3
2^{-15}	3.5986e-2	2.5721e-2	1.9623e-2	1.4138e-2	9.4658e-3
D^M	3.5986e-2	2.5721e-2	1.9623e-2	1.4138e-2	9.4658e-3
p^M	4.8451e-1	3.9040e-1	4.7295e-1	5.7878e-1	-

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