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FINITE ELEMENT METHOD WITH QUADRATURE FOR PARABOLIC INTERFACE PROBLEMS

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ABSTRACT. The purpose of this paper is to establish some new a priori error estimates in finite element method with quadrature for parabolic interface problems. Due to low global regularity of the solutions, the error analysis of the standard finite element methods for parabolic problems is difficult to adopt for parabolic interface problems. In this paper, we fill a theoretical gap between standard error analysis technique of finite element method for non interface problems and parabolic interface problems. Optimal $L^\infty(H^1)$ and $L^\infty(L^2)$ norms error estimates have been derived for the semidiscrete case under practical regularity assumptions of the true solution for fitted finite element method with straight interface triangles. Further, the fully discrete backward Euler scheme is also considered and optimal $L^\infty(L^2)$ norm error estimate is established. The interface is assumed to be smooth for our purpose.

Key words. Parabolic, interface, finite element method, quadrature, semi discrete and fully discrete schemes, optimal, point-wise error estimates

AMS (MOS) Subject Classification. 65N15, 65N30

1. INTRODUCTION

Let Ω be a convex polygonal domain in $\mathbb{R}^2$ with boundary $\partial\Omega$ and $\Omega_1 \subset \Omega$ be an open domain with C^2 smooth boundary $\Gamma = \partial\Omega_1$. Let $\Omega_2 = \Omega\backslash\Omega_1$ be an another open domain contained in Ω with boundary $\Gamma\cup\partial\Omega$ (see Figure 1). In $\Omega = \Omega_1\cup\Gamma\cup\Omega_2$, we consider the following parabolic interface problem

$$u_t - \nabla\cdot(\beta(x)\nabla u) = f(x,t) \quad \text{in } \Omega\times(0,T] \tag{1.1}$$

with initial and boundary conditions

$$u(x,0) = u_0 \ \text{ in } \Omega; \ \ u(x,t) = 0 \quad \text{on } \partial\Omega\times(0,T] \tag{1.2}$$

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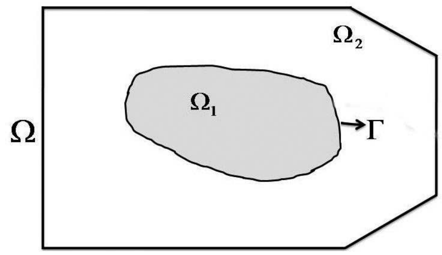

FIGURE 1. Domain Ω and its sub domains Ω_1, Ω_2 with interface Γ.

and jump conditions on the interface

$$[u] = 0, \quad \left[\beta \frac{\partial u}{\partial \mathbf{n}}\right] = g(x,t) \quad \text{along } \Gamma, \tag{1.3}$$

where the symbol $[v]$ is a jump of a quantity v across the interface Γ and $\mathbf{n}$ is the unit outward normal to the boundary $\partial\Omega_1$. The coefficient function β is positive and piecewise constant, i.e.

$$\beta(x) = \beta_i \quad \text{for } x \in \Omega_i, \quad i = 1, 2.$$

Many physical phenomena can be modeled by partial differential equations with singularities and interfaces. Interface problems are generally those problems or differential equations in which the input data are non smooth or discontinuous or singular across one or more interfaces in the solution domain. Parabolic equations (1.1) with discontinuous coefficients occur in many applications such as in material sciences and fluid dynamics. As a model, we consider non-stationary heat conduction problems in two dimensions with a conduction coefficient β which is discontinuous across a smooth interface. For a detailed references on models for interface problems, see [6, 7, 11, 15].

The standard finite difference and finite element methods may not be successful in giving satisfactory numerical results for such problems. Hence, many new methods have been developed. Some of them are developed with the modifications in the standard methods, so that they can deal with the discontinuities and the singularities. For the literature on the recent developments of the numerical methods for such problems, we refer to [2, 14] which includes extensive list of relevant literatures. Although a good number of articles is devoted to the finite element approximation of elliptic interface problems, the literature seems to lack concerning the convergence of finite element solutions to the true solutions of parabolic interface problems (1.1)–(1.3). For the backward Euler time discretization, Chen and Zou [4] have studied the convergence of fully discrete solution to the exact solution using fitted finite element method. They have proved almost optimal error estimates in $L^2(L^2)$ and $L^2(H^1)$ norms when global regularity of the solution is low. Then an essential improvement was made in [16]. The authors of [16] have used a finite element discretization where interface triangles are assumed to be curved triangles instead of straight triangles like classical finite element methods. Optimal order error estimates in $L^2(L^2)$ and $L^2(H^1)$ norms are shown to hold for both semi discrete and fully discrete scheme in [16]. More recently, for similar triangulation, Deka and Sinha ([9]) have studied the pointwise-in-time convergence in finite element method for parabolic interface problems. They have shown optimal error estimates in $L^\infty(H^1)$ and $L^\infty(L^2)$ norms under the assumption that grid line exactly follow the actual interface. This may causes some technical difficulties in practice for the evaluation of the integrals over those curved elements near the interface. In fact, in practice, the integrals appearing in finite element approximation are evaluated numerically by using some well known quadrature schemes. Therefore, quadrature based finite element method has been proposed and analyzed in this work. Quadrature based finite element method for elliptic interface problems can be found in [8, 12].

In this work, we are able to show that the standard error analysis technique of finite element method can be extended to parabolic interface problems. Optimal order pointwise-in-time error estimates in the L^2 and H^1 norms are established for the semidiscrete scheme. In addition, a fully discrete method based on backward Euler time-stepping scheme is analyzed and related optimal pointwise-in-time error bounds are derived. To the best of our knowledge, optimal point-wise in time error estimates for a finite element discretization based on [4] have not been established earlier for the parabolic interface problem. The achieved estimates are analogous to the case with a regular solution, however, due to low regularity, the proof requires a careful technical work coupled with a approximation result for the linear interpolant. Other technical tools used in this paper are Sobolev embedding inequality, approximation properties for elliptic projection, duality arguments and some known results on elliptic interface problems.

A brief outline of this paper is as follows. In Section 2, we introduce some notation, recall some basic results from the literature and obtain the a priori estimate for the solution. In Section 3, we describe a finite element discretization for the problem (1.1)–(1.3) and prove some approximation properties related to the auxiliary projection used in our analysis. While Section 4 is devoted to the error analysis for the semidiscrete finite element approximation, error estimates for the fully discrete backward Euler time stepping scheme are derived in Section 5. Finally, a numerical example is presented in Section 6 for the completeness of this work.

2. NOTATIONS AND PRELIMINARIES

In this section, we shall introduce the standard notation for Sobolev spaces and norms to be used in this paper.

For $m \geq 0$ and real p with $1 \leq p \leq \infty$, we use $W^{m,p}(\Omega)$ to denote Sobolev space of order m with norm $\|.\|_{H^m}$ and in particular for $p = 2$, we write $W^{m,2} = H^m$.

$H_0^m(\Omega)$ is a closed subspace of $H^m(\Omega)$, which is also closure of $C_0^\infty(\Omega)$ (the set of all C^∞ functions with compact support) with respect to the norm of $H^m(\Omega)$ (c.f. [1]).

We shall also need the following spaces:

$$X = H^1(\Omega) \cap H^2(\Omega_1) \cap H^2(\Omega_2) \quad \text{and} \quad Y = L^2(\Omega) \cap H^1(\Omega_1) \cap H^1(\Omega_2)$$

equipped with the norms

$$\begin{aligned} \|v\|_X &= \|v\|_{H^1(\Omega)} + \|v\|_{H^2(\Omega_1)} + \|v\|_{H^2(\Omega_2)} \\ \|v\|_Y &= \|v\|_{L^2(\Omega)} + \|v\|_{H^1(\Omega_1)} + \|v\|_{H^1(\Omega_2)}, \end{aligned}$$

respectively. For a given Banach space $\mathcal{B}$, we define, for $m = 0, 1$,

$$H^m(0,T;\mathcal{B}) = \left\{ u(t) \in \mathcal{B} \text{ for a.e. } t \in (0,T) \text{ and} \sum_{j=0}^{m} \int_0^T \left\| \frac{\partial^j u(t)}{\partial t^j} \right\|_{\mathcal{B}}^2 dt < \infty \right\}$$

equipped with the norm

$$\|u\|_{H^m(0,T;\mathcal{B})} = \left(\sum_{j=0}^{m} \int_0^T \left\| \frac{\partial^j u(t)}{\partial t^j} \right\|_{\mathcal{B}}^2 dt \right)^{\frac{1}{2}}.$$

We write $L^2(0,T;\mathcal{B}) = H^0(0,T;\mathcal{B})$. Throughout this paper, C denotes a generic positive constant which is independent of the mesh parameters h and k.

In order to introduce the weak formulation of the problem, we now define the local bilinear form $A^l(.,.) : H^1(\Omega_l) \times H^1(\Omega_l) \to \mathbf{R}$ by

$$A^l(w,v) = \int_{\Omega_l} \beta_l \nabla w \cdot \nabla v dx, \quad l = 1, 2.$$

Then the global bilinear map $A(\cdot,\cdot) : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbf{R}$ is defined by

$$\begin{aligned} A(w,v) &= \int_{\Omega} \beta(x) \nabla w \cdot \nabla v dx \\ &= A^1(w,v) + A^2(w,v) \;\; \forall\, w,\ v \in H_0^1(\Omega). \end{aligned} \tag{2.1}$$

The weak form for the problem (1.1)–(1.3) is defined as follows: Find $u : (0,T] \to H_0^1(\Omega)$ such that

$$(u_t, v) + A(u,v) = (f,v) + \langle g, v \rangle_\Gamma \;\; \forall v \in H_0^1(\Omega), \text{ a.e. } t \in (0,T] \tag{2.2}$$

with $u(x,0) = u_0(x)$. Here, $(\cdot,\cdot)$ and $\langle \cdot,\cdot \rangle_\Gamma$ are used to denote the inner products of $L^2(\Omega)$ and $L^2(\Gamma)$ spaces, respectively.

Regarding the regularity for the solution of the interface problem (1.1)–(1.3), we have the following result.

Theorem 2.1. *Let $f \in H^1(0,T;L^2(\Omega))$, $g \in H^1(0,T;H^{\frac{1}{2}}(\Gamma))$ and $u_0 \in H_0^1(\Omega)$. Then the problem* (1.1)–(1.3) *has a unique solution $u \in L^2(0,T;X) \cap H^1(0,T;Y)$. Further,* *for* $u_0 \in H^3(\Omega) \cap H_0^1(\Omega),\ f \in H^1(0,T;L^2(\Omega)),\ f(x,0) \in H^1(\Omega)$ *and* $g = 0$*, solution* u *satisfies the following a priori estimate*

$$\int_0^t \{\|u_t\|^2_{H^2(\Omega_1)} + \|u_t\|^2_{H^2(\Omega_2)}\} ds \le C \left\{ \|u_t(0)\|^2_{H^1(\Omega)} + \int_0^t \|f_t\|^2_{L^2(\Omega)} ds \right\}.$$

Proof. The existence of unique solution can be found in [4, 15].

Next, to obtain the a priori estimate we first transform the problem (1.1)–(1.3) to the following equivalent problem:

For a.e. $t \in (0,T]$, $u_t(x,t) \in H^2(\Omega_1) \cap H^2(\Omega_2)$ satisfies the following elliptic interface problem

$$-\nabla \cdot (\beta(x) \nabla u_t) = f_t - u_{tt} \quad \text{in } \Omega_i,\ i = 1,2 \tag{2.3}$$

along with boundary condition

$$u_t(x,t) = 0 \quad \text{on } \partial\Omega \times (0,T] \tag{2.4}$$

and jump conditions (cf. [13])

$$[u_t] = 0 \quad \text{and} \quad \left[\beta \frac{\partial u_t}{\partial \mathbf{n}}\right] = 0 \ \text{ along } \Gamma. \tag{2.5}$$

From the a priori estimate for elliptic interface problem (cf. [4]), it follows that

$$\|u_t\|_{H^2(\Omega_1)} + \|u_t\|_{H^2(\Omega_2)} \le C\{\|u_{tt}\|_{L^2(\Omega)} + \|f_t\|_{L^2(\Omega)}\}. \tag{2.6}$$

For any

$$v \in Y \cap \{\psi : \psi = 0 \ \text{ on } \partial\Omega\} \ \ \&\ [v] = 0 \text{ along } \Gamma,$$

we obtain

$$\begin{aligned}
&- \int_{\Omega_1} \nabla \cdot (\beta_1 \nabla u) v dx - \int_{\Omega_2} \nabla \cdot (\beta_2 \nabla u) v dx \\
&= - \int_\Gamma \beta_1 \frac{\partial u}{\partial \mathbf{n}} v ds + \int_{\Omega_1} \beta_1 \nabla u \cdot \nabla v dx \\
&\quad + \int_\Gamma \beta_2 \frac{\partial u}{\partial \mathbf{n}} v ds + \int_{\Omega_2} \beta_2 \nabla u \cdot \nabla v dx \\
&= \int_{\Omega_1} \beta_1 \nabla u \cdot \nabla v dx + \int_{\Omega_2} \beta_2 \nabla u \cdot \nabla v dx + \int_\Gamma \left[\beta \frac{\partial u}{\partial \mathbf{n}} v\right] ds \\
&= A^1(u,v) + A^2(u,v).
\end{aligned} \tag{2.7}$$

Since $[v] = 0$ and $[\beta \partial u/\partial \mathbf{n}] = 0$ along Γ. Then multiplying (2.3) by such v and integrating over Ω, we have

$$(u_t, v) + A^1(u,v) + A^2(u,v) = (f,v). \tag{2.8}$$

Again it follows from the arguments of [13] that $[u_{tt}] = 0$ along Γ and $u_{tt} = 0$ on $\partial\Omega$, and hence equation (2.8) leads to

$$(u_{tt}, u_{tt}) + A^1(u_t, u_{tt}) + A^2(u_t, u_{tt}) = (f_t, u_{tt}) \tag{2.9}$$

so that

$$\int_0^t \|u_{tt}\|^2_{L^2(\Omega)} ds + \frac{1}{2}A^1(u_t, u_t) + \frac{1}{2}A^2(u_t, u_t)$$
$$\leq \frac{1}{2}A^1(u_t(0), u_t(0)) + \frac{1}{2}A^2(u_t(0), u_t(0)) + C\int_0^t \|f_t\|^2_{L^2(\Omega)} ds.$$

Under the assumption that $u_0 \in H^3(\Omega)$ and $f(x,0) \in H^1(\Omega)$, we have $u_t(0) \in H^1(\Omega)$. Therefore u_{tt} satisfies the following a priori estimate

$$\int_0^t \|u_{tt}\|^2_{L^2(\Omega)} ds \leq C\left\{\|u_t(0)\|^2_{H^1(\Omega)} + \int_0^t \|f_t\|^2_{L^2(\Omega)} ds\right\}.$$

Finally, using above estimate in (2.6) we obtain

$$\int_0^t \{\|u_t\|^2_{H^2(\Omega_1)} + \|u_t\|^2_{H^2(\Omega_2)}\} ds \leq C\left\{\|u_t(0)\|^2_{H^1(\Omega)} + \int_0^t \|f_t\|^2_{L^2(\Omega)} ds\right\}.$$

□

Remark 2.2. Consider the following interface problems

$$\xi_t - \nabla\cdot(\beta(x)\nabla\xi) = f(x,t) \quad \text{in } \Omega\times(0,T]$$
$$\xi(x,0) = \frac{1}{2}u_0 \ \text{ in } \Omega; \ \ \xi(x,t) = 0 \quad \text{on } \partial\Omega\times(0,T]$$
$$[\xi] = 0, \quad \left[\beta\frac{\partial\xi}{\partial\mathbf{n}}\right] = 0 \quad \text{along } \Gamma,$$

and

$$\psi_t - \nabla\cdot(\beta(x)\nabla\psi) = 0 \quad \text{in } \Omega\times(0,T]$$
$$\psi(x,0) = \frac{1}{2}u_0 \ \text{ in } \Omega; \ \ \psi(x,t) = 0 \quad \text{on } \partial\Omega\times(0,T]$$
$$[\psi] = 0, \quad \left[\beta\frac{\partial\psi}{\partial\mathbf{n}}\right] = g(x,t) \quad \text{along } \Gamma.$$

Then, $\xi+\psi$ satisfies the following weak formulation

(2.10) $$(\xi_t + \psi_t, v) + A(\xi+\psi, v) = (f, v) + \langle g, v\rangle_\Gamma \quad \forall v \in H^1_0(\Omega).$$

Subtracting (2.10) from (2.2), we obtain

(2.11) $$(u_t - \xi_t - \psi_t, v) + A(u - \xi - \psi, v) = 0.$$

Setting $v = u - \xi - \psi$ in (2.11) and coercivity of $A(\cdot,\cdot)$ leads to

$$\|u - \xi - \psi\|^2_{L^2(\Omega)} \leq C\|u(0) - \xi(0) - \psi(0)\|^2_{L^2(\Omega)}.$$

Finally, use the fact $u(0) = \xi(0) + \psi(0)$ to have $u = \xi + \psi$ for a.e. $(x,t) \in \Omega\times(0,T]$.

For $g \in H^2(0,T; H^2(\Gamma))$, we assume that

$$\psi \in L^2(0,T; X\cap H^1_0(\Omega)) \cap H^1(0,T; L^2(\Omega)\cap H^2(\Omega_1)\cap H^2(\Omega_2))$$

so that $u \in H^1(0,T; L^2(\Omega)\cap H^2(\Omega_1)\cap H^2(\Omega_2))$.

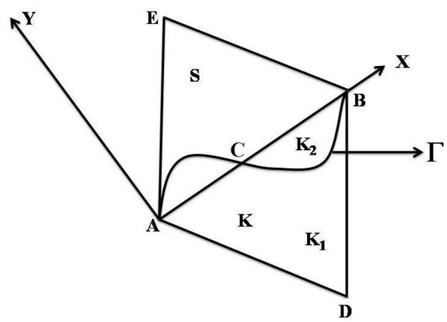

FIGURE 2. Interface triangles K and S along with interface Γ

Thus, under the assumptions $u_0 \in H^3(\Omega)\cap H_0^1(\Omega)$, $f \in H^1(0,T;L^2(\Omega))$, $f(x,0) \in H^1(\Omega)$ and $g \in H^2(0,T;H^2(\Gamma))$, solution u for the interface problem (1.1)–(1.3) is unique and $u \in L^2(0,T;X \cap H_0^1(\Omega)) \cap H^1(0,T;L^2(\Omega) \cap H^2(\Omega_1) \cap H^2(\Omega_2))$.

3. FINITE ELEMENT DISCRETIZATION AND SOME AUXILIARY RESULTS

For the purpose of finite element approximation of the problem (1.1)–(1.3), we now describe the triangulation $\mathcal{T}_h$ of Ω. We first approximate the domain Ω_1 by a domain Ω_1^h with the polygonal boundary Γ_h whose vertices all lie on the interface Γ. Let Ω_2^h be the approximation for the domain Ω_2 with polygonal exterior and interior boundaries as $\partial\Omega$ and Γ_h, respectively. The triangles with one or two vertices on Γ are called the interface triangles, the set of all interface triangles is denoted by $\mathcal{T}_\Gamma^*$ and we write $\Omega_\Gamma^* = \cup_{K\in\mathcal{T}_\Gamma^*} K$.

We assume that the triangulation $\mathcal{T}_h$ of the domain Ω satisfy the following conditions:

- **($\mathcal{A}1$):** $\overline{\Omega} = \cup_{K\in\mathcal{T}_h} K$.
- **($\mathcal{A}2$):** If $K_1,\ K_2 \in \mathcal{T}_h$ and $K_1 \neq K_2$, then either $K_1 \cap K_2 = \emptyset$ or $K_1 \cap K_2$ is a common vertex or edge of both triangles.
- **($\mathcal{A}3$):** Each triangle $K \in \mathcal{T}_h$ is either in Ω_1^h or Ω_2^h, and has at most two vertices lying on Γ_h.
- **($\mathcal{A}4$):** For each triangle $K \in \mathcal{T}_h$, let r_K, $\overline{r}_K$ be the radii of its inscribed and circumscribed circles, respectively. Let $h = \max\{\overline{r}_K : K \in \mathcal{T}_h\}$.

Let V_h be a family of finite dimensional subspaces of $H_0^1(\Omega)$ defined on $\mathcal{T}_h$ consisting of piecewise linear functions vanishing on the boundary $\partial\Omega$. Further, we assume the following inverse inequality

$$\|v_h\|_{H^1(K)} \leq Ch^{-1}\|v_h\|_{L^2(K)} \ \ \forall K \in \mathcal{T}_h, \ \ v_h \in V_h. \tag{3.1}$$

Examples of such finite element spaces can be found in [3] and [5].

In order to approximate $A(\cdot,\cdot)$, we now introduce approximate bilinear map $A_h(\cdot,\cdot) : H^1(\Omega) \times H^1(\Omega) \rightarrow \mathbf{R}$ defined as

$$A_h(w,v) = \sum_{K\in\mathcal{T}_h} \int_K \beta_K(x) \nabla w.\nabla v dx \ \forall w,v \in H^1(\Omega),$$

with $\beta_K(x) = \beta_i$ if $K \subset \Omega_i^h,\ i = 1,2$. To handle the L^2 inner product, we define the approximation on V_h and its induced norm by

$$(w,v)_h = \sum_{K\in\mathcal{T}_h} \left\{ \frac{1}{3}\text{meas(K)} \sum_{j=1}^{3} w(P_j^K)v(P_j^K) \right\}, \tag{3.2}$$

and $\|\phi\|_h = (\phi,\phi)_h^{\frac{1}{2}}$, where P_j^K are the vertices for the triangle K.

We now recall some existing results on the approximation A_h and the inner product which will be frequently used in our analysis. For a proof, we refer to [5] and [7].

Lemma 3.1. *For all $v_h, w_h \in V_h$, we have*

$$|A(v_h,w_h) - A_h(v_h,w_h)| \leq Ch \sum_{K\in\mathcal{T}_\Gamma^*} \|\nabla v_h\|_{L^2(K)} \|\nabla w_h\|_{L^2(K)}.$$

Lemma 3.2. *On V_h the norms $\|.\|_{L^2(\Omega)}$ and $\|.\|_h$ are equivalent. Further, for $w, v \in V_h$ and $f \in H^2(\Omega)$, we have*

$$\begin{aligned} |(w,v) - (w,v)_h| &\leq Ch^2 \|w\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)}, \\ |(f,v)_h - (f,v)| &\leq Ch^2 \|f\|_{H^2(\Omega)} \|v\|_{H^1(\Omega)}. \end{aligned}$$

Let $X^\star$ be the collection of all $v \in L^2(\Omega)$ with the property that $v \in H^2(\Omega_1) \cap H^2(\Omega_2) \cap \{\psi : \psi = 0 \text{ on } \partial\Omega\}$ and $[v] = 0$ along Γ. Since Γ is of class C^2, thus $v_i = v|_{\Omega_i}$, $i = 1,2$ can be extended to $\tilde{v}_i \in H^2(\Omega)$ such that

$$\|\tilde{v}_i\|_{H^2(\Omega)} \leq C\|v_i\|_{H^2(\Omega_i)}.$$

For the existence of such extensions, we refer to Stein [17]. Further, we have a C^2 function ϕ in $[C,B]$ such that (c.f. [10])

$$|\phi(x)| \leq Ch^2 \tag{3.3}$$

and hence

$$\text{meas}(K_2) = \int_C^B |\phi(x)| dx \leq Ch^2 \int_C^B dx \leq Ch^3.$$

Let $\Pi_h : C(\overline{\Omega}) \rightarrow V_h$ be the Lagrange interpolation operator corresponding to the space V_h. Then, for $K \in \mathcal{T}_h$ and $v \in X^\star$, we now define

$$v_I = \begin{cases} \Pi_h \tilde{v}_1 & \text{if } K \subseteq \Omega_1^h \\ \Pi_h \tilde{v}_2 & \text{if } K \subseteq \Omega_2^h. \end{cases} \tag{3.4}$$

Regarding v_I, we have the following approximation result

Lemma 3.3. *For any $v \in X^\star$, we have*

$$\|v - v_I\|_{H^1(\Omega_1)} + \|v - v_I\|_{H^1(\Omega_2)} \leq Ch(\|v\|_{H^2(\Omega_1)} + \|v\|_{H^2(\Omega_2)}).$$

Proof. For H^1 norm estimate, we have

$$\begin{aligned}
&\|v - v_I\|_{H^1(\Omega_1)} + \|v - v_I\|_{H^1(\Omega_2)} \\
&\leq \sum_{K\in\mathcal{T}_h\setminus\mathcal{T}_\Gamma^*} \|v - v_I\|_{H^1(K)} + \sum_{K\in\mathcal{T}_\Gamma^*} \{\|v - v_I\|_{H^1(K_1)} + \|v - v_I\|_{H^1(K_2)}\} \\
&\leq Ch\{\|v\|_{H^2(\Omega_1)} + \|v\|_{H^2(\Omega_2)}\} \\
&\quad + \sum_{K\in\mathcal{T}_\Gamma^*} \{\|v - v_I\|_{H^1(K_1)} + \|v - v_I\|_{H^1(K_2)}\}. \qquad (3.5)
\end{aligned}$$

Here, $K_1 = K \cap \Omega_1$ and $K_2 = K \cap \Omega_2$. Again, for any $K \in \mathcal{T}_h$ either $K \subseteq \Omega_1^h$ or $K \subseteq \Omega_2^h$. Let $K \subseteq \Omega_1^h$, then $v_I = \Pi_h\tilde{v}_1$ and hence, we have

$$\begin{aligned}
\|v - v_I\|_{H^1(K_1)} &= \|\tilde{v}_1 - \Pi_h\tilde{v}_1\|_{H^1(K_1)} \leq \|\tilde{v}_1 - \Pi_h\tilde{v}_1\|_{H^1(K)} \\
&\leq Ch\|\tilde{v}_1\|_{H^2(K)} \leq Ch\|v_1\|_{H^2(\Omega_1)}. \qquad (3.6)
\end{aligned}$$

Again, since $v \in H^2(\Omega_2)$ and $K_2 \subseteq \Omega_2$ with $\text{meas}(K_2) \leq Ch^3$, we have

$$\begin{aligned}
\|v - v_I\|_{H^1(K_2)} &\leq Ch^{\frac{3(p-2)}{2p}}\|v - v_I\|_{W^{1,p}(K_2)} \quad \forall p > 2 \\
&= Ch\|v - v_I\|_{W^{1,6}(K_2)} = Ch\|v_2 - \Pi_h\tilde{v}_1\|_{W^{1,6}(K_2)} \\
&\leq Ch\|\tilde{v}_2 - \tilde{v}_1\|_{W^{1,6}(K_2)} + Ch\|\tilde{v}_1 - \Pi_h\tilde{v}_1\|_{W^{1,6}(K_2)} \\
&\leq Ch\|\tilde{v}_2 - \tilde{v}_1\|_{W^{1,6}(K)} + Ch\|\tilde{v}_1 - \Pi_h\tilde{v}_1\|_{W^{1,6}(K)} \\
&\leq Ch\|\tilde{v}_2 - \tilde{v}_1\|_{H^2(\Omega)} + Ch\|\tilde{v}_1\|_{H^2(K)} \\
&\leq Ch\|\tilde{v}_1\|_{H^2(\Omega)} + Ch\|\tilde{v}_2\|_{H^2(\Omega)} \\
&\leq Ch(\|v\|_{H^2(\Omega_1)} + \|v\|_{H^2(\Omega_2)}). \qquad (3.7)
\end{aligned}$$

Then Lemma 3.3 follows immediately from the estimates (3.5)–(3.7). □

Let $Y^\star$ be the collection of all $w \in L^2(\Omega)$ such that $w \in H^1(\Omega_1) \cap H^1(\Omega_2) \cap \{\psi : \psi = 0 \text{ on } \partial\Omega\}$ with $[w] = 0$ along Γ. We now recall the elliptic projection $R_h : Y^\star \to V_h$ given by

$$A_h(R_hv, v_h) = A^1(v, v_h) + A^2(v, v_h) \quad \forall v_h \in V_h. \qquad (3.8)$$

Regarding the approximation properties of R_h operator defined by (3.8), we have the following results

Lemma 3.4. *Let R_h be defined by* (3.8)*, then for any $v \in X^\star$ there is a positive constant C independent of the mesh parameter h such that*

$$\|R_hv - v\|_{H^1(\Omega_1)} + \|R_hv - v\|_{H^1(\Omega_2)} \leq Ch(\|v\|_{H^2(\Omega_1)} + \|v\|_{H^2(\Omega_2)}).$$

Proof. Coercivity of each local bilinear map and the definition of R_h projection leads to

$$\begin{aligned}
&\|v-R_hv\|^2_{H^1(\Omega_1)}+\|v-R_hv\|^2_{H^1(\Omega_2)}\\
&\leq C\{A^1(v-R_hv,v-v_h)+A^2(v-R_hv,v-v_h)\}\\
&\quad +CA^1(v,v_h-R_hv)-CA^1(R_hv,v_h-R_hv)\\
&\quad +CA^2(v,v_h-R_hv)-CA^2(R_hv,v_h-R_hv)\\
&=C\{A^1(v-R_hv,v-v_h)+A^2(v-R_hv,v-v_h)\}\\
&\quad +C\{A^1_h(R_hv,v_h-R_hv)-A^1(R_hv,v_h-R_hv)\}\\
&\quad +C\{A^2_h(R_hv,v_h-R_hv)-A^2(R_hv,v_h-R_hv)\}\\
&=C\{A^1(v-R_hv,v-v_h)+A^2(v-R_hv,v-v_h)\}\\
&\quad +C\{A_h(R_hv,v_h-R_hv)-A(R_hv,v_h-R_hv)\}.
\end{aligned}$$

Then it follows from Lemma 3.1 and Young's inequality that

$$\begin{aligned}
&\|v-R_hv\|^2_{H^1(\Omega_1)}+\|v-R_hv\|^2_{H^1(\Omega_2)}\\
&\leq C\|v-R_hv\|_{H^1(\Omega_1)}\|v-v_h\|_{H^1(\Omega_1)}+C\|v-R_hv\|_{H^1(\Omega_2)}\|v-v_h\|_{H^1(\Omega_2)}\\
&\quad +Ch\|R_hv\|_{H^1(\Omega)}\|v_h-R_hv\|_{H^1(\Omega)}\\
&\leq \epsilon\|v-R_hv\|^2_{H^1(\Omega_1)}+\frac{C}{\epsilon}\|v-v_h\|^2_{H^1(\Omega_1)}+\epsilon\|v-R_hv\|^2_{H^1(\Omega_2)}\\
&\quad +\frac{C}{\epsilon}\|v-v_h\|^2_{H^1(\Omega_2)}+\frac{Ch^2}{\epsilon}\|R_hv\|^2_{H^1(\Omega)}+\epsilon\|v_h-R_hv\|^2_{H^1(\Omega)}.
\end{aligned}$$

Again applying the fact $\|R_hv\|_{H^1(\Omega)}\leq C(\|v\|_{H^1(\Omega_1)}+\|v\|_{H^1(\Omega_2)})$ and for suitable $\epsilon>0$, we have

$$\begin{aligned}
\|v-R_hv\|^2_{H^1(\Omega_1)}+\|v-R_hv\|^2_{H^1(\Omega_2)} \;\leq\; & C\|v-v_h\|^2_{H^1(\Omega_1)}+C\|v-v_h\|^2_{H^1(\Omega_2)}\\
&+Ch^2\{\|v\|^2_{H^1(\Omega_1)}+\|v\|^2_{H^1(\Omega_2)}\}.
\end{aligned}$$

Now, setting $v_h=v_I$ and then using Lemma 3.3, we have

$$\|v-R_hv\|_{H^1(\Omega_1)}+\|v-R_hv\|_{H^1(\Omega_2)}\leq Ch(\|v\|_{H^2(\Omega_1)}+\|v\|_{H^2(\Omega_2)}).$$

This completes the proof of Lemma 3.4. □

Corollary 3.5. *Let u be the exact solution of the interface problem* (1.1)–(1.3)*, then*

$$\begin{aligned}
&\|u-R_hu\|_{H^1(\Omega_1)}+\|u-R_hu\|_{H^1(\Omega_2)}\leq Ch(\|u\|_{H^2(\Omega_1)}+\|u\|_{H^2(\Omega_2)}),\\
&\|u_t-R_hu_t\|_{H^1(\Omega_1)}+\|u_t-R_hu_t\|_{H^1(\Omega_2)}\leq Ch(\|u_t\|_{H^2(\Omega_1)}+\|u_t\|_{H^2(\Omega_2)}).
\end{aligned}$$

Proof. As $u,\ u_t\in X^\star$, the result follows immediately from the previous result. □

Lemma 3.6. *Let R_h be defined by (3.8), then for any $v \in X^\star$ there is a positive constant C independent of the mesh size parameter h such that*

$$\|R_h v - v\|_{L^2(\Omega)} \le Ch^2(\|v\|_{H^2(\Omega_1)} + \|v\|_{H^2(\Omega_2)}).$$

Proof. For L^2 norm error estimate, we will use the duality argument. For this purpose, we consider the following interface problem

$$-\nabla \cdot (\beta \nabla \phi) = v - R_h v$$

with the boundary condition $\phi = 0$ on $\partial\Omega$ and interface conditions $[\phi] = 0$, $[\beta \frac{\partial \phi}{\partial \mathbf{n}}] = 0$ along Γ.

Now multiply the above equation by $w \in Y^\star$ and then integrate over Ω to have

$$A^1(\phi, w) + A^2(\phi, w) = (v - R_h v, w). \tag{3.9}$$

Let $\phi_h \in V_h$ be the finite element approximation to ϕ defined as: Find $\phi_h \in V_h$ such that

$$A_h(\phi_h, w_h) = (v - R_h v, w_h) \quad \forall w_h \in V_h. \tag{3.10}$$

Arguing as deriving Lemma 3.4, it can be concluded that

$$\begin{aligned}\|\phi - \phi_h\|_{H^1(\Omega_1)} \quad + \quad & \|\phi - \phi_h\|_{H^1(\Omega_2)} \\ & \le C(\|\phi - w_h\|_{H^1(\Omega_1)} + \|\phi - w_h\|_{H^1(\Omega_2)}) \\ & \quad + Ch(\|\phi\|_{H^2(\Omega_1)} + \|\phi\|_{H^2(\Omega_2)}) \quad \forall w_h \in V_h.\end{aligned}$$

Let ϕ_I be defined as in (3.4) and then set $w_h = \phi_I$ to have

$$\begin{aligned}\|\phi - \phi_h\|_{H^1(\Omega_1)} + \|\phi - \phi_h\|_{H^1(\Omega_2)} & \le Ch(\|\phi\|_{H^2(\Omega_1)} + \|\phi\|_{H^2(\Omega_2)}) \\ & \le Ch\|v - R_h v\|_{L^2(\Omega)}.\end{aligned}$$

In the last inequality, we used the elliptic regularity estimate $\|\phi\|_X \le C\|v - R_h v\|_{L^2(\Omega)}$ (cf. [4]). Thus, we have

$$\|\phi - \phi_h\|_{H^1(\Omega)} \le Ch\|v - R_h v\|_{L^2(\Omega)}. \tag{3.11}$$

Since $[v - R_h v] = 0$ along Γ and $v - R_h v \in L^2(\Omega) \cap H^1(\Omega_1) \cap H^1(\Omega_2) \cap \{\psi : \psi = 0 \text{ on } \partial\Omega\}$, therefore we can set $w = v - R_h v$ in (3.9) to have

$$\begin{aligned}\|v - R_h v\|^2_{L^2(\Omega)} &= A^1(\phi, v - R_h v) + A^2(\phi, v - R_h v) \\ &= A^1(\phi - \phi_h, v - R_h v) + A^2(\phi - \phi_h, v - R_h v) \\ &\quad + \{A^1(\phi_h, v - R_h v) + A^2(\phi_h, v - R_h v)\} \\ &\le C\|\phi - \phi_h\|_{H^1(\Omega_1)}\|v - R_h v\|_{H^1(\Omega_1)} \\ &\quad + C\|\phi - \phi_h\|_{H^1(\Omega_2)}\|v - R_h v\|_{H^1(\Omega_2)} \\ &\quad + \{A^1(\phi_h, v) + A^2(\phi_h, v)\} - \{A^1(\phi_h, R_h v) + A^2(\phi_h, R_h v)\}\end{aligned}$$

$$
\begin{aligned}
&\leq Ch\|v-R_hv\|_{L^2(\Omega)}\cdot Ch(\|v\|_{H^2(\Omega_1)}+\|v\|_{H^2(\Omega_2)})\\
&\quad +A_h(R_hv,\phi_h)-A(R_hv,\phi_h)\\
&= Ch^2\|v-R_hv\|_{L^2(\Omega)}(\|v\|_{H^2(\Omega_1)}+\|v\|_{H^2(\Omega_2)})\\
&\quad +\{A_h(R_hv,\phi_h)-A(R_hv,\phi_h)\}\\
&= Ch^2\|v-R_hv\|_{L^2(\Omega)}(\|v\|_{H^2(\Omega_1)}+\|v\|_{H^2(\Omega_2)})+(J). \qquad (3.12)
\end{aligned}
$$

Now, we apply Lemma 3.1 to have

$$
\begin{aligned}
|(J)| &\leq Ch\sum_{K\in\mathcal{T}_\Gamma^*}\|R_hv\|_{H^1(K)}\|\phi_h\|_{H^1(K)}\\
&\leq Ch\sum_{K\in\mathcal{T}_\Gamma^*}\|R_hv\|_{H^1(K_1)}\|\phi_h\|_{H^1(K_1)}\\
&\quad +Ch\sum_{K\in\mathcal{T}_\Gamma^*}\|R_hv\|_{H^1(K_2)}\|\phi_h\|_{H^1(K_2)}\\
&= (J)_1+(J)_2. \qquad (3.13)
\end{aligned}
$$

Again, using Corollary 3.5 and estimate (3.11), we have

$$
\begin{aligned}
&\|R_hv\|_{H^1(K_2)}\|\phi_h\|_{H^1(K_2)}\\
&\leq \{\|R_hv-v\|_{H^1(K_2)}+\|v\|_{H^1(K_2)}\}\{\|\phi_h-\phi\|_{H^1(K_2)}+\|\phi\|_{H^1(K_2)}\}\\
&\leq \{\|R_hv-v\|_{H^1(\Omega_2)}+\|\tilde{v}_2\|_{H^1(K_2)}\}\{\|\phi_h-\phi\|_{H^1(\Omega_2)}+\|\phi\|_{H^1(K_2)}\}\\
&\leq C\{h\|v\|_{H^2(\Omega_1)}+h\|v\|_{H^2(\Omega_2)}+\|\tilde{v}_2\|_{H^1(K)}\}\\
&\qquad \times\{h\|v-R_hv\|_{L^2(\Omega)}+\|\phi\|_{H^1(K)}\}. \qquad (3.14)
\end{aligned}
$$

Setting $p=4$ in the Sobolev embedding inequality (cf. [17, 18])

$$
\|v\|_{L^p(K_2)}\leq Cp^{\frac{1}{2}}\|v\|_{H^1(K_2)}\ \ \forall v\in H^1(K_2),\ \ p>2 \qquad (3.15)
$$

and further, using Hölder's inequality, we obtain

$$
\begin{aligned}
\|\tilde{v}_2\|_{H^1(K)} &= \|\tilde{v}_2\|_{L^2(K)}+\|\nabla\tilde{v}_2\|_{L^2(K)}\\
&\leq Ch^{\frac{1}{2}}\|\tilde{v}_2\|_{L^4(K)}+Ch^{\frac{1}{2}}\|\nabla\tilde{v}_2\|_{L^4(K)}\\
&\leq Ch^{\frac{1}{2}}\|\tilde{v}_2\|_{H^1(K)}+Ch^{\frac{1}{2}}\|\nabla\tilde{v}_2\|_{H^1(K)}\\
&\leq Ch^{\frac{1}{2}}\|\tilde{v}_2\|_{H^2(K)}\leq Ch^{\frac{1}{2}}\|v_2\|_{H^2(\Omega_2)}, \qquad (3.16)
\end{aligned}
$$

where we have used the fact that $\text{meas}(K)\leq Ch^2$. Similarly, for $\|\phi\|_{H^1(K)}$, we have

$$
\|\phi\|_{H^1(K)}\leq Ch^{\frac{1}{2}}\|\phi\|_X\leq Ch^{\frac{1}{2}}\|v-R_hv\|_{L^2(\Omega)}. \qquad (3.17)
$$

Combining (3.14)–(3.17), we have

$$
\begin{aligned}
&\|R_hv\|_{H^1(K_2)}\|\phi_h\|_{H^1(K_2)}\\
&\leq Ch\{\|v\|_{H^2(\Omega_1)}+\|v\|_{H^2(\Omega_2)}\}\|v-R_hv\|_{L^2(\Omega)}.
\end{aligned}
$$

Therefore, for $(J)_2$, we have

$$(J)_2 \le Ch^2\{\|v\|_{H^2(\Omega_1)} + \|v\|_{H^2(\Omega_2)}\}\|v - R_h v\|_{L^2(\Omega)}. \tag{3.18}$$

Similarly, for $(J)_1$, we have

$$(J)_1 \le Ch^2\{\|v\|_{H^2(\Omega_1)} + \|v\|_{H^2(\Omega_2)}\}\|v - R_h v\|_{L^2(\Omega)}. \tag{3.19}$$

Then, using the estimates (3.18) and (3.19) in (3.13), we have

$$|(J)| \le Ch^2\|v - R_h v\|_{L^2(\Omega)}(\|v\|_{H^2(\Omega_1)} + \|v\|_{H^2(\Omega_2)}). \tag{3.20}$$

Finally, (3.12) and (3.20) leads to the following optimal L^2 norm estimate

$$\|v - R_h v\|_{L^2(\Omega)} \le Ch^2(\|v\|_{H^2(\Omega_1)} + \|v\|_{H^2(\Omega_2)}).$$

This completes the rest of the proof. □

Corollary 3.7. *Let u be the exact solution of the interface problem* (1.1)–(1.3)*, then*

$$\|u - R_h u\|_{L^2(\Omega)} \le Ch^2\|u\|_X,$$

$$\|u_t - R_h u_t\|_{L^2(\Omega)} \le Ch^2(\|u_t\|_{H^2(\Omega_1)} + \|u_t\|_{H^2(\Omega_2)}).$$

Let $g_h \in V_h$ be the linear interpolant of g given by

$$g_h = \sum_{j=1}^{m_h} g(P_j)\Phi_j^h,$$

where $\{\Phi_j^h\}_{j=1}^{m_h}$ is the set of standard nodal basis functions corresponding to the nodes $\{P_j\}_{j=1}^{m_h}$ on the interface Γ. Following the argument of [4] it is possible to obtain the following approximation property of g_h to the interface function g.

Lemma 3.8. *Let $g \in H^2(\Gamma)$. If Ω_Γ^* is the union of all interface triangles then we have*

$$\left|\int_\Gamma g v_h ds - \int_{\Gamma_h} g_h v_h ds\right| \le Ch^2\|g\|_{H^2(\Gamma)}\|v_h\|_{H^1(\Omega_\Gamma^*)} \quad \forall v_h \in V_h.$$

Proof. It follows from [4, page 186] that

$$\left|\int_\Gamma g v_h ds - \int_{\Gamma_h} g_h v_h ds\right|$$
$$\le Ch^2\|g\|_{H^2(\Gamma)}\|v_h\|_{H^1(\Omega_\Gamma^*)} + Ch^{3/2}\|g\|_{H^2(\Gamma)}\|v_h\|_{L^2(\Omega_\Gamma^*)} \quad \forall v_h \in V_h.$$

Arguing as in (3.16), we obtain

$$\begin{aligned}\|v_h\|_{L^2(\Omega_\Gamma^*)} &= \sum_{K\in\mathcal{T}_\Gamma^*}\|v_h\|_{L^2(K)} \\ &\le Ch^{1/2}\sum_{K\in\mathcal{T}_\Gamma^*}\|v_h\|_{L^4(K)} \le Ch^{1/2}\|v_h\|_{H^1(\Omega_\Gamma^*)}.\end{aligned}$$

The desire result follows immediately from the above two estimates. This completes the proof. □

4. ERROR ANALYSIS FOR THE SEMIDISCRETE SCHEME

In this section, we discuss the semidiscrete finite element method for the problem (1.1)–(1.3) and derive optimal error estimates in L^2 and H^1 norms.

The continuous-time Galerkin finite element approximation to (2.2) is stated as follows: Find $u_h : [0,T] \to V_h$ such that $u_h(0) = R_h u_0$ and

$$(u_{ht}, v_h)_h + A_h(u_h, v_h) = (f, v_h)_h + \langle g_h, v_h\rangle_{\Gamma_h} \quad \forall v_h \in V_h, \quad t \in (0,T]. \tag{4.1}$$

Write the error $e(t) = u - u_h = u - R_h u + R_h u - u_h = \rho + \theta$, with $\rho = u - R_h u$ and $\theta = R_h u - u_h$. Again, using (3.8) for $v = u \in X^\star$ and further differentiating with respect to t, we have

$$A_h((R_h u)_t, v_h) = A^1(u_t, v_h) + A^2(u_t, v_h).$$

Also,

$$A_h(R_h u_t, v_h) = A^1(u_t, v_h) + A^2(u_t, v_h).$$

From the above two equations, we have

$$A_h((R_h u)_t - R_h u_t, v_h) = 0 \;\; \forall v_h \in V_h.$$

Setting $v_h = (R_h u)_t - R_h u_t$ in the above equation, we obtain $(R_h u)_t = R_h u_t$.

Now, by the definition R_h operator, (2.2) and (4.1), we obtain

$$\begin{aligned}
(\theta_t, v_h)_h + A_h(\theta, v_h) &= ((R_h u)_t - u_{ht}, v_h)_h + A_h(R_h u - u_h, v_h) \\
&= (R_h u_t, v_h)_h + A_h(R_h u, v_h) - (u_{ht}, v_h)_h - A_h(u_h, v_h) \\
&= (R_h u_t, v_h)_h + A(u, v_h) - (f, v_h)_h - \langle g_h, v_h\rangle_{\Gamma_h} \\
&= \{(R_h u_t, v_h)_h - (R_h u_t, v_h)\} + \{(f, v_h) - (f, v_h)_h\} \\
&\quad + \{\langle g, v_h\rangle_\Gamma - \langle g_h, v_h\rangle_{\Gamma_h}\} + (-\rho_t, v_h).
\end{aligned}$$

For $v_h = \theta$, we have

$$\begin{aligned}
(\theta_t, \theta)_h + C\|\theta\|^2_{H^1(\Omega)} &\le Ch^2\|R_h u_t\|_{H^1(\Omega)}\|\theta\|_{H^1(\Omega)} + Ch^2\|f\|_{H^2(\Omega)}\|\theta\|_{H^1(\Omega)} \\
&\quad + Ch^2\|g\|_{H^2(\Gamma)}\|\theta\|_{H^1(\Omega^\star_\Gamma)} + C\|\rho_t\|_{L^2(\Omega)}\|\theta\|_{L^2(\Omega)} \\
&\le C_\epsilon\Big(\|\rho_t\|^2_{L^2(\Omega)} + h^4\{\|R_h u_t\|^2_{H^1(\Omega)} + \|f\|^2_{H^2(\Omega)} \\
&\quad + \|g\|^2_{H^2(\Gamma)}\}\Big) + C(\epsilon)\|\theta\|^2_{H^1(\Omega)}.
\end{aligned}$$

Here, we have used Lemma 3.1 and Lemma 3.8. Integrating the above equation from 0 to t and using Corollary 3.7, we obtain

$$\|\theta(t)\|^2_{L^2(\Omega)} \leq Ch^4 \int_0^t \Big(\sum_{i=1}^2 \|u_t\|^2_{H^2(\Omega_i)} + \|f\|^2_{H^2(\Omega)} + \|g\|^2_{H^2(\Gamma)} \Big) ds. \tag{4.2}$$

Now, combining Corollary 3.7 and (4.2), we have the following optimal pointwise-in-time L^2-norm error estimates.

Theorem 4.1. *Let u and u_h be the solutions of the problem* (1.1)–(1.3) *and* (4.1), *respectively. Assume that $u_h(0) = R_h u_0$. Then there exists a constant C independent of h such that*

$$\begin{aligned} \|e(t)\|_{L^2(\Omega)} \leq \; & Ch^2 \Big[\|u\|_X + \Big(\int_0^t \Big\{ \sum_{i=1}^2 \|u_t\|^2_{H^2(\Omega_i)} \\ & + \|f\|^2_{H^2(\Omega)} + \|g\|^2_{H^2(\Gamma)} \Big\} ds \Big)^{\frac{1}{2}} \Big]. \end{aligned}$$

For H^1-norm estimate, we first use Corollary 3.5 to have

$$\sum_{i=1}^2 \|\rho(t)\|_{H^1(\Omega_i)} \leq Ch \sum_{i=1}^2 \|u\|_{H^2(\Omega_i)}. \tag{4.3}$$

Applying inverse estimate (3.1), we obtain

$$\begin{aligned} \|\theta(t)\|_{H^1(\Omega)} \leq \; & Ch^{-1} \|\theta(t)\|_{L^2(\Omega)} \\ \leq \; & Ch^{-1} h^2 \Big[\int_0^t \Big(\sum_{i=1}^2 \|u_t\|^2_{H^2(\Omega_i)} + \|f\|^2_{H^2(\Omega)} + \|g\|^2_{H^2(\Gamma)} \Big) ds \Big]^{\frac{1}{2}} \\ = \; & Ch \Big[\int_0^t \Big(\sum_{i=1}^2 \|u_t\|^2_{H^2(\Omega_i)} + \|f\|^2_{H^2(\Omega)} + \|g\|^2_{H^2(\Gamma)} \Big) ds \Big]^{\frac{1}{2}}. \end{aligned} \tag{4.4}$$

Combining (4.3) and (4.4), we have the following optimal pointwise-in-time H^1-norm error estimates.

Theorem 4.2. *Let u and u_h be the solutions of the problem* (1.1)–(1.3) *and* (4.1), *respectively. Assume that $u_h(0) = R_h u_0$. Then there exists a constant C independent of h such that*

$$\begin{aligned} \|e(t)\|_{H^1(\Omega)} \leq \; & Ch \Big[\|u\|_X + \Big(\int_0^t \Big\{ \sum_{i=1}^2 \|u_t\|^2_{H^2(\Omega_i)} \\ & + \|f\|^2_{H^2(\Omega)} + \|g\|^2_{H^2(\Gamma)} \Big\} ds \Big)^{\frac{1}{2}} \Big]. \end{aligned}$$

5. ERROR ANALYSIS FOR THE FULLY DISCRETE SCHEME

A fully discrete scheme based on backward Euler method is proposed and analyzed in this section. Optimal L^2 norm error estimate is obtained for fully discrete scheme.

We first partition the interval $[0,T]$ into M equally spaced subintervals by the following points

$$0 = t_0 < t_1 < \cdots < t_M = T$$

with $t_n = nk$, where $k = \frac{T}{M}$ be the time step. Let $I_n = (t_{n-1}, t_n]$ be the n-th subinterval and $\phi^n = \phi(t_n)$. For a given sequence $\{\phi^n\}_{n=0}^M \subset L^2(\Omega)$, we now introduce the backward difference quotient as

$$\Delta_k \phi^n = \frac{\phi^n - \phi^{n-1}}{k}.$$

The fully discrete finite element approximation to the problem (2.2) is defined as follows: For $n = 1, \ldots, M$, find $U^n \in V_h$ such that

$$(\Delta_k U^n, v_h)_h + A_h(U^n, v_h) = (f^n, v_h) + \langle g_h^n, v_h \rangle_{\Gamma_h} \quad \forall v_h \in V_h \tag{5.1}$$

with $U^0 = R_h u_0$. For each $n = 1, \ldots, M$, the existence of a unique solution to (5.1) can be found in [4]. We then define the fully discrete solution to be a piecewise constant function $U_h(x,t)$ in time and is given by

$$U_h(x,t) = U^n(x) \ \ \forall t \in I_n, \ \ 1 \le n \le M.$$

We now prove the main result of this section in the following theorem.

Theorem 5.1. *Let u and U be the solutions of the problem* (1.1)–(1.3) *and* (5.1), *respectively. Assume that $U^0 = R_h u_0$. Then there exists a constant C independent of h and k such that*

$$\|U(t_n) - u(t_n)\|_{L^2(\Omega)}$$
$$\le C(h^2 + k)\Big\{\|u^0\|_{H^2(\Omega)} + |||g^n||| + \|u_{tt}\|_{L^2(0,T;L^2(\Omega))} + \sum_{i=1}^{2} \|u_t\|_{L^2(0,T;H^2(\Omega_i))}\Big\}.$$

Proof. We write the error $U^n - u^n$ at time t_n as

$$U^n - u^n = (U^n - R_h u^n) + (R_h u^n - u^n) \equiv: \theta^n + \rho^n$$

where $\theta^n = U^n - R_h u^n$ and $\rho^n = R_h u^n - u^n$.

For θ^n, we have the following error equation

$$\begin{aligned}
&(\Delta_k \theta^n, v_h)_h + A_h(\theta^n, v_h)\\
&= (-\Delta_k R_h u^n + \Delta_k U^n, v_h)_h + A_h(-R_h u^n + U^n, v_h)\\
&= (\Delta_k U^n, v_h)_h + A_h(U^n, v_h) - (\Delta_k R_h u^n, v_h)_h - A_h(R_h u^n, v_h)
\end{aligned}$$

$$
\begin{aligned}
&= (f^n, v_h) + \langle g_h^n, v_h\rangle_{\Gamma_h} - (\Delta_k R_h u^n, v_h)_h - A(u^n, v_h) \\
&= (f^n, v_h) + \langle g_h^n, v_h\rangle_{\Gamma_h} - (\Delta_k R_h u^n, v_h)_h \\
&\quad +(u_t^n, v_h) - (f^n, v_h) - \langle g^n, v_h\rangle_\Gamma \\
&\equiv: -(w^n, v_h) + \{(\Delta_k R_h u^n, v_h) - (\Delta_k R_h u^n, v_h)_h\} \\
&\quad +\{\langle g_h^n, v_h\rangle_{\Gamma_h} - \langle g^n, v_h\rangle_\Gamma\}, \qquad (5.2)
\end{aligned}
$$

where $w^n = \Delta_k R_h u^n - u_t^n$. For simplicity of the exposition, we write $w^n = w_1^n + w_2^n$, where $w_1^n = R_h \Delta_k u^n - \Delta_k u^n$ and $w_2^n = \Delta_k u^n - u_t^n$.

Now, setting $v_h = \theta^n$ in (5.2), we have

$$
\begin{aligned}
(\Delta_k \theta^n, \theta^n)_h + A_h(\theta^n, \theta^n) \;&=\; -(w^n, \theta^n) + \{(\Delta_k R_h u^n, \theta^n) - (\Delta_k R_h u^n, \theta^n)_h\} \\
&\quad +\{\langle g_h^n, \theta^n\rangle_{\Gamma_h} - \langle g^n, \theta^n\rangle_\Gamma\}. \qquad (5.3)
\end{aligned}
$$

Since $A_h(\theta^n, \theta^n) \geq C\|\theta^n\|^2_{H^1(\Omega)}$, we have

$$
\begin{aligned}
\|\theta^n\|_{L^2(\Omega)} \;&\leq\; k\|w^n\|_{L^2(\Omega)} + \|\theta^{n-1}\|_{L^2(\Omega)} + Ch^2 k^{\frac12}\|R_h \Delta_k u^n\|_{H^1(\Omega)} \\
&\quad +Ch^2 k^{\frac12}\|g^n\|_{H^2(\Gamma)} \\
&\leq\; \|\theta^0\|_{L^2(\Omega)} + k\sum_{j=1}^n \|w_1^j\|_{L^2(\Omega)} + k\sum_{j=1}^n \|w_2^j\|_{L^2(\Omega)} \\
&\quad +Ch^2 k^{\frac12}\sum_{j=1}^n \|w_1^j\|_{H^1(\Omega)} + Ch^2 k^{\frac12}\sum_{j=1}^n \|\Delta_k u^j\|_{H^1(\Omega)} \\
&\quad +Ch^2 k^{\frac12}|||g^n|||, \qquad (5.4)
\end{aligned}
$$

with $|||g^n||| = \max_{1\leq j\leq n} |||g^j|||_{H^2(\Gamma)}$.

In Ω_1, the term w_1^j can be expressed as

$$
\begin{aligned}
w_1^j \;&=\; R_h \Delta_k u_1^j - \Delta_k u_1^j = (R_h - I)(\Delta_k u_1^j) \\
&=\; (R_h - I)\frac{1}{k}\int_{t_{j-1}}^{t^j} u_{1,t}dt = \frac{1}{k}\int_{t_{j-1}}^{t^j} (R_h u_{1,t} - u_{1,t})dt,
\end{aligned}
$$

where $u_i,\ i = 1, 2$ is the restriction of u in Ω_i and $u_{i,t} = \frac{\partial u_i}{\partial t}$.

An application of Corollary 3.7 leads to

$$
k\|w_1^j\|_{L^2(\Omega_1)} \leq Ch^2 \int_{t_{j-1}}^{t^j} \Big\{\sum_{i=1}^2 \|u_t\|_{H^2(\Omega_i)}\Big\}dt.
$$

Similarly, we obtain

$$
k\|w_1^j\|_{L^2(\Omega_2)} \leq Ch^2 \int_{t_{j-1}}^{t^j} \Big\{\sum_{i=1}^2 \|u_t\|_{H^2(\Omega_i)}\Big\}dt.
$$

Using above two estimates, we have

$$k\sum_{j=1}^{n}\|w_1^j\|_{L^2(\Omega)} \leq Ch^2\int_0^{t_n}\Big\{\sum_{i=1}^{2}\|u_t\|_{H^2(\Omega_i)}\Big\}dt. \tag{5.5}$$

Similarly, for the term w_2^n, we have

$$kw_2^j = u^j - u^{j-1} - ku_t^j = -\int_{t_{j-1}}^{t_j}(s-t_{j-1})u_{tt}ds$$

and hence

$$k\|w_2^j\|_{L^2(\Omega_i)} \leq k\int_{t_{j-1}}^{t_j}\|u_{tt}\|_{L^2(\Omega_i)}ds.$$

Summing over j from $j=1$ to $j=n$, we obtain

$$k\sum_{j=1}^{n}\|w_2^j\|_{L^2(\Omega)} \leq Ck\int_0^{t_n}\Big\{\sum_{i=1}^{2}\|u_{tt}\|_{L^2(\Omega_i)}\Big\}dt. \tag{5.6}$$

Arguing as in (5.5), we obtain

$$k\sum_{j=1}^{n}\|w_1^j\|_{H^1(\Omega)} \leq Ch\int_0^{t_n}\Big\{\sum_{i=1}^{2}\|u_t\|_{H^2(\Omega_i)}\Big\}dt. \tag{5.7}$$

Combining (5.4)–(5.7) and using the fact that

$$k\sum_{j=1}^{n}\|\Delta_k u^j\|^2_{H^1(\Omega)} \leq C\int_0^{t_n}\Big\{\sum_{i=1}^{2}\|u_t\|^2_{H^1(\Omega_i)}\Big\}dt,$$

we obtain

$$\begin{aligned}\|\theta^n\|_{L^2(\Omega)} &\leq C(h^2+k)\\ &\quad\times\left[\sum_{i=1}^{2}\Big\{\|u_t\|_{L^2(0,T;H^2(\Omega_i))}+\|u_{tt}\|_{L^2(0,T;L^2(\Omega_i))}\Big\}+|||g^n|||\right].\end{aligned} \tag{5.8}$$

An application of Corollary 3.7 for ρ^n yields

$$\|\rho^n\|_{L^2(\Omega)} \leq Ch^2\sum_{i=1}^{2}\|u^n\|_{H^2(\Omega_i)}.$$

Again, it is easy to verify that

$$\|u^n\|_{H^2(\Omega_i)} \leq \|u^0\|_{H^2(\Omega_i)} + \int_0^{t_n}\|u_t\|_{H^2(\Omega_i)}dt.$$

Thus, we have

$$\|\rho^n\|_{L^2(\Omega)} \leq Ch^2\Big\{\|u^0\|_{H^2(\Omega)}+\sum_{i=1}^{2}\|u_t\|_{L^2(0,T;H^2(\Omega_i))}\Big\}. \tag{5.9}$$

Combining (5.8) and (5.9) the desired estimate is easily obtained. This completes the proof. □

TABLE 6.1. Numerical results for the test problem (6.1)–(6.3).

h	$\|u - U_h\|_{L^2(\Omega)}$
1/8	2.06247×10^{-3}
1/16	5.28838×10^{-4}
1/32	1.36298×10^{-4}
1/64	3.47701×10^{-5}

6. NUMERICAL RESULTS

In this section, a numerical example is considered for the completeness of this work. We take for the domain the rectangle $\Omega = (0,2) \times (0,1)$. The interface occurs at $x = 1$ so that $\Omega_1 = (0,1) \times (0,1)$, $\Omega_2 = (1,2) \times (0,1)$ and the interface $\Gamma = \bar{\Omega}_1 \cap \bar{\Omega}_2$.

Consider the following parabolic boundary value problem in Ω:

$$u_t - \nabla \cdot (\beta \nabla u) = f \quad \text{in} \;\; \Omega \times (0,1], \; i = 1,2, \tag{6.1}$$

$$u(x,y,0) = u_0(x,y) \;\; \text{in } \Omega, \;\; u(x,y,t) = 0 \quad \text{on} \;\; \partial\Omega \times (0,1] \tag{6.2}$$

$$u_1|_\Gamma = u_2|_\Gamma, \quad (\beta_1 \nabla u_1 \cdot \mathbf{n}_1)|_\Gamma + (\beta_2 \nabla u_2 \cdot \mathbf{n}_2)|_\Gamma = 0, \tag{6.3}$$

where $\mathbf{n}_i$ denotes the unit outer normal vector on Ω_i, $i = 1,2$. For the exact solution, we choose

$$u_1(x,y) = e^{\sin t} \sin(\pi x) \sin(\pi y) \quad \text{in } \Omega_1 \times (0,1]$$

and

$$u_2(x,y) = -e^{\sin t} \sin(2\pi x) \sin(\pi y) \quad \text{in } \Omega_2 \times (0,1].$$

Then the source function f and the initial data u_0 are determined from the choice for u_1 and u_2 with $\beta_1 = 1$ and $\beta_2 = \frac{1}{2}$.

For our numerical results, globally continuous piecewise linear finite element functions based on the triangulations of Ω as stated in section 3 were used. The L^2-norm and H^1-norm errors at $t = 1/130$ for various step size h are presented in Table 6.1 for the fully discrete solution. The convergence rates are found to be within our expectation.

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