Neural, Parallel, and Scientific Computations 21 (2013) 497-516

OPERATOR THEORETIC APPROACH TO OPTIMAL CONTROL PROBLEMS DESCRIBED BY NONLINEAR DIFFERENTIAL EQUATIONS

RAJU K. GEORGE AND MOHAN C. JOSHI

Department of Mathematics, Indian Institute of Space Science and Technology
Thiruvananthapuram 695547
Department of Mathematics, Indian Institute of Technology
Gandhinagar, Gujarat

ABSTRACT We consider a class of control systems characterized by nonlinear differential equation of the form

$$\begin{aligned} \frac{dx}{dt} &= A(t)x + B(t)u + F(t,x), 0 \leq t_0 \leq t \leq t_1 < \infty. \\ x(t_0) &= x_0 \end{aligned}$$

where u denotes the control lying in a suitable Banach space and x denotes the state in another separable reflexive Banach space.We are interested in finding a control, u which minimizes a certain cost functional $J(u) = \phi(x, u)$. We provide conditions on $A(t), B(t), F(t, x)$ and $\phi(t, u)$ which gurantee the existence of an optimal control. We first reduce the system governed by the differential equation into an equivalent Hammerstein operator equation of the form

$$x = KNx + Hu$$

in suitable space. Subsequently we give sets of sufficient conditions on operators K, N and H which guarantee the existence of an optimal control. We use the theory of monotone operators and operators of type (M) in our analysis. Our results apply to both Lipschitzian and non-Lipschitzian (monotone) nonlinearities. The systems described by standard finite and infinite dimensional nonlinear differential equations are special cases of the general operator equation formulation. From the general results obtained for the operator equation we deduce results for the system described by differential equations as special cases. Also, we relate 'optimality system' to Hamiltonion system in the Minimum Principle of Pontriagin and Riccati Equations for systems governed by differential equations.

AMS (MOS) Subject Classification. 39A10.

1. INTRODUCTION

Let Y_1 be a Hilbert space and Y a dense linear subspace of Y_1 carrying the structure of a separable reflexive Banach space with continuous injection into Y_1 and

Received July 15, 2013

U be a separable reflexive Banach space. In this paper we consider a class of control systems governed by the nonlinear differential equation

$$\begin{aligned} \frac{dx}{dt} &= A(t)x + B(t)u + F(t,x), 0 \leq t_0 \leq t \leq t_1 < \infty. \\ x(t_0) &= x_0 \end{aligned} \tag{1.1}$$

where x_0, $x(t) \in Y$ and $u(t) \in U$. For each $t \in [t_0, t_1]$, let $A(t) : Y \to Y^*$ be a closed linear, not necessarily bounded, operator and $\{A(t)\}$, $t \in [t_0, t_1]$ generates an almost strong evolution operator $\Phi : \triangle \to BL(Y_1)$ where $\triangle = \{(s,t) \in [t_0, t_1] \times [t_0, t_1] : 0 \leq t_0 \leq s \leq t \leq t_1\}$, $BL(Y_1) =$: space of all bounded linear operators $\Phi(t,s)$ from Y_1 into itself, with $\|\Phi(t,s)\| \leq M$.

In the optimal control problem we are looking for a control u which minimizes a certain cost functional.Let the associated cost functional to be minimized be given by

$$J(u) = \phi(u,x) = \int_{t_0}^{t_1} g(t, u(t), x(t))\, dt$$

where g is a mapping from $[t_0, t_1] \times U \times Y$ into $\bar{\mathbb{R}}_+$ satisfying Caratheodory conditions with respect to t, u and x.

Further we make the following assumption:

Assumption (a_0) : The linear homogeneous evolution equation

$$\begin{aligned} \frac{dx}{dt} &= A(t)x + g \\ x(t_0) &= x_0 \end{aligned}$$

has a unique strong solution for all $g \in L^2([t_0, t_1], Y)$. See Tanabe [17] for sufficient condition for Assumption (a_0). Also, the nonlinear operator $F : [t_0, t_1] \times Y \to Y^*$ is such that for every $x \in Y$, $t \to F(t,x)$ is measurable and $x \to F(t,x)$ is continuous for almost all t in $[t_0, t_1]$ (Caratheodory conditions). Assume that for all $t \in [t_0, t_1]$, $B(t)$ is a bounded linear operator maps U into Y and let $A(t)$ be such that for all $t \in [t_0, t_1]$ domain of $A(t) =: D$ and $x_0 \in D$.

For a given $u \in L^2([t_0, t_1], U)$, by a solution of (1.1) we mean a mild solution $x \in L^2([t_0, t_1], Y)$ satisfying

$$x(t) = \Phi(t, t_0)x_0 + \int_{t_0}^{t} \Phi(t,\tau)B(\tau)u(\tau)d\tau + \int_{t_0}^{t} \Phi(t,\tau)F(\tau, x(\tau))d\tau \tag{1.2}$$

We now define the following spaces and operators defined on it:

Let $X = L^2([t_0, t_1], Y)$, $X_1 = L^2([t_0, t_1], Y_1)$ and $Z = L^2([t_0, t_1], U)$.

Define the operators $K : X^* \to X$, $N : X \to X^*$ and $H : Z \to X$ as under

$$(Kx)(t) = \int_{t_0}^{t} \Phi(t,\tau)x(\tau)d\tau$$

$$
\begin{aligned}
(Nx)(t) &= F(t, x(t)) \\
(Hu)(t) &= \Phi(t, t_0)x_0 + \int_{t_0}^{t} \Phi(t, \tau)B(\tau)u(\tau)d\tau
\end{aligned} \tag{1.3}
$$

By using the above definitions it is easy to see that the equation (1.2) can be reduced in to the abstract operator equation of Hammerstein type

$$x = KNx + Hu \tag{1.4}$$

in X for a fixed $u \in Z$. In the following section we formulate the problem in the abstract setup. In Section 3, we give preliminary definitions and results and in Section 4, we prove existence results for optimal pair both for constrained and unconstrained problems. The optimality system is derived for the abstract system in Section 5. As an application of our abstract we provide conditions for the existence of optimal pair for control systems described by partial differential equations. The paper ends with an Example to illustrate our abstract results.

2. OPTIMALITY PROBLEM FOR ABSTRACT SYSTEM

In this section we formulate two control problems for the abstract system characterized by the Hammerstein operator equation.

Let X be a real Banach space with dual X^*. Let Z be another real Banach space. Let $K : X^* \to X$, $N : X \to X^*$ and $H : Z \to X$ be operators, not necessarily linear. We consider a control system characterized by the nonlinear operator equation of the Hammerstein type

$$x = KNx + Hu \tag{2.1}$$

where $u \in U \subset Z$, called the set of all admissible controls. We shall assume throughout that for each $u \in U$, (2.1) has a unique solution $x \in X$. This x is referred as the response or trajectory of (2.1) corresponding to the control u. For a quick review of the existence and uniqueness of solutions of operator equations of the form (2.1), refer Browder [7].

We shall be interested in finding a control $u \in U$ which minimizes a cost functional J. In most of the practical situations the cost functional J happens to be a function of both the variables u and x and so we take the cost functional J of the form.

$$J(u) = \phi(u, x)$$

If U is just a subset of Z then the minimization problem is referred as a constrained problem, otherwise it is called unconstrained problem. Let x^* be the response corresponding to a control u^* which minimizes J, then the pair (u^*, x^*) is called an optimal pair of (2.1). A pair of coupled equations satisfied by the optimal pair (u^*, x^*) is referred as an 'optimality system' for (2.1) (with respect to the cost functional J), see Seidman and Zhou [16].

In our study, we investigate the following problems.

Problem 2.1. Find a set of sufficient conditions on K, N and H which will guarantee the existence of an optimal pair $(u^*, x^*) \in U \times X$ for the system (2.1).

Problem 2.2. Derive an 'optimality system' for the control system (2.1).

The problem of the type raised above have been investigated by many authors for the systems described by nonlinear differential equations, refer Balachandran and Somasundaram [2], Barbu and Prato [3], Papageorgiou [15] and Seidman and Zhou [16].

Our system (2.1) is in the most general setting and contains the systems considered by the above authors as special cases. We, also derive the result in Datko [9] for linear system as a corollary of our main result in Section 5. For the systems governed by differential equations, we deduce the Hamiltonian system in the Minimum Principle of Pontriagin as a special case of our 'optimality system'. For such systems we relate the 'optimality system' to Riccati Eautions. We use the theory of monotone operators and operators of type (M) for our analysis and our result apply to monotone type nonlinearities rather than Lipschitz type. We note that there is very little discussion in the literature regarding control problems containing monotone nonlinearities.

3. PRELIMINARIES

In this section we give necessary defintions and basic results for for the existence of maximum and minimum for a functional defined on some set. Let X be a real Banach space and X^* be the dual of X. The strong convergence of a sequence $\{x_n\}$ to x_0 in X is denoted by $x_n \to x_0$ and weak convergence by $x_n \rightharpoonup x_0$.

An operator $T : X \to X^*$ is said to be monotone if $\langle Tx_1 - Tx_2, x_1 - x_2\rangle \geq 0$ for all $x_1, x_2 \in X$. T is of type (M) if $x_n \rightharpoonup x_0, Tx_n \rightharpoonup y$ and $\lim_n \langle Tx_n, x_n - x_0\rangle \leq 0$ implies that $y = Tx_0$. T is said to be weakly continuous (completely continuous) if $x_n \rightharpoonup x_0$ in X implies $Tx_n \rightharpoonup Tx_0 (Tx_n \to Tx_0)$ in X^*. T is said to be a bounded operator if it maps bounded subsets of X into a bounded subsets of X^*. T is said to be coercive if $\langle Tx, x\rangle/\|x\| \to \infty$ as $\|x\| \to \infty$.

Let M be the set of all operators $T : X \to X^*$ such that $\langle Tx_1 - Tx_2, x_1 - x_2\rangle \geq \alpha\|x_1 - x_2\|^2$ for all $x_1, x_2 \in X$ and for some constant α.

For $T \in M$, we define $\mu(T)$ as

$$\mu(T) = \inf_{\substack{x_1, x_2 \in X \\ x_1 \neq x_2}} \frac{\langle Tx_1 - Tx_2, x_1 - x_2 \rangle}{\|x_1 - x_2\|^2}$$

Let Lip be the set of all operators $T : X \to X^*$ satisfying the Lipschitz condition. That is, there exists $\alpha > 0$ such that $\|Tx_1 - Tx_2\| \leq \alpha \|x_1 - x_2\|$ for all $x_1, x_2 \in X$. For $T \in$ Lip, we define

$$\|T\|^* = \sup_{\substack{x_1, x_2 \in X \\ x_1 \neq x_2}} \frac{\|Tx_1 - Tx_2\|}{\|x_1 - x_2\|}$$

A functional $f : X \to \mathbb{R}$ is said to be lower semi continuous (weakly lower semicontinous) if $\lim_n f(x_n) \geq f(x_0)$ whenever $x_n \to x_0 (x_n \rightharpoonup x_0)$ and is said to be coercive if $f(x) \to \infty$ as $\|x\| \to \infty$. Let I denote the interval $[t_0, t_1]$, then $f : I \times X \to \mathbb{R}$ is said to be approximately lower semi continuous if for all $\varepsilon > 0$ there exists $I_\varepsilon \subset I$ such that $mes(I \setminus I_\varepsilon) < \varepsilon$ and $F|_{I_\varepsilon \times X}$ is lower semicontinuous. We will use the following known result in our analysis.

Theorem 3.1. *Let $K \subset X$ be a (weakly) compact set and $f : K \to \mathbb{R}$ be (weakly) lower semi continuous then there exists $x_0 \in K$ such that $f(x_0) \leq \inf_{x \in K} f(x)$.*

4. EXISTENCE OF OPTIMAL PAIR FOR THE ABSTRACT SYSTEM

In this section we give the main results concerning Problem 2.1. We first do the analysis for the constrained problem and then extend the analysis to the unconstrained problem.

Let $T : U \to X$ denote the system operator (also known as solution operator) which assigns to each control $u \in U$ a unique solution $x \in X$ satisfying (2.1). To tackle Problem 2.1 we begin by giving different sets of sufficient conditions on K, N and H which will guarantee the weak and complete continuity of T. We assume throughout this section that X is a reflexive Banach space.

Assumptions [A]

A_1 : $K : X^* \to X$ is linear and compact and there exists a constant $d > 0$ such that $\langle Kx, x \rangle \geq d\|Kx\|^2$ for all $x \in X^*$.

A_2 : $N : X \to X^*$ is continuous, bounded and negative monotone.

A_3 : $H : U \to X$ is completely continuous.

Assumptions [B]

$[B_1]$: K is linear and belongs to M.

$[B_2]$: $N \in$ Lip and $\mu(-N) > 0$ with $(\mu(K) + \mu(-N)\|N\|^{*-2}) > 0$

$[B_3]$: H is completely continuous.

Assumptions [C]

$[C_1]$: K and N satisfy either $[A_1]$ and $[A_2]$ or $[B_1]$ and $[B_2]$.

$[C_2]$: H and N are weakly continuous.

Assumptions [D]

D_1 : K is bounded linear operator and there exists a constant $d > 0$ such that $\langle Kx, x\rangle \geq d\|Kx\|^2$ for all $x \in X^*$.

D_2 : N is a continuous bounded operator of type (M)

D_3 : H is completely continuous.

Lemma 4.1. *Suppose that the operator K, N and H satisfy either Assumptions $[A]$ or Assumptions $[B]$ then the system operator T is well defined and is completely continuous.*

Proof. Let Assumptions $[A]$ hold. Then by Theorem 1 of Hess [11] it follows that $[I-KN]^{-1}$ is well defined and so is $T = [I-KN]^{-1}H$. The boundedness and continuity of $[I-KN]^{-1}$ follow from a similar argument given in the proof of Lemma 2.1 of George [14]. This together with complete continuity of H imply that T is completely continuous.

If Assumptions [B] holds, we get the continuity and boundedness of $[I-KN]^{-1}$ by Theorem 2.1 of Dolezal [10] and hence the complete continuity of $[I-KN]^{-1}H$. $\square$

Similarly we have the following result regarding the weak continuity of T.

Lemma 4.2. *Under any one of the set of Assumptions $[C]$ or Assumptions $[D]$ the system operator T is weakly continuous.*

Remark 4.1. If the system operator is guaranteed to be well - defined, a priori, then the linearity assumption on K and the monotonicity assumption on N can be relaxed in the above lemmas (refer Theorem 5 of Brezis and Browder [5] and Theorem 2 of Hess [11]).

Remark 4.2. We note that if K is a bounded linear non-compact operator and $N \in$ Lip with $\|K\|\|N\|^* < 1$ then by using infinite dimensional version of Theorem 3.1 of Joshi and George [13] it follows that the system operator T is well defined and is weakly continuous provided N and H are weakly continuous.We now give our main results on existence of optimal control.

Assumptions [I]

(a) : The operators K, N and H satisfy any one of the sets of Assumptions $[A]$ or Assumptions $[B]$.

(b) : For a fixed $x, u \to \phi(u, x)$ is convex and continuous and $x \to \phi(u, x)$ is continuous and it is uniform for all $u \in U$.

(c) : The control set U is weakly compact.

Theorem 4.1. *Under Assumptions* [I] *the system* (2.1) *has an optimal pair* (u^*, x^*)*.*

Proof. We first show that J is weakly lower semicontinuous. That is, $J(u^*) \leq \lim_n J(u_n)$ whenever $u_n \rightharpoonup u^*$ in U.

Let $g(u, v)$ be real valued function on $U \times X$ defined by

$$g(u, v) = \phi(u, Tv)$$

In view of Assumption [I(b)], $u \to g(u, Tv)$ is continuous and convex for a fixed $v \in X$. Also Assumption [I(a)] implies (by Lemma 4.1) that T is completely continuous. This together with Assumption [I(b)] give that $v \to g(u, v)$ is weakly continuous uniformly for all $u \in U$. As all assumptions of Theorem 1 of Browder [6] are satisfied, it follows that

$$J(u) = \phi(u, Tu) = g(u, v)$$

is weakly lower semicontinuous.

As U is weakly compact and J is weakly lower semicontinuous it follows from Theorem 4.1 that there exists $u^* \in U$ such that

$$\phi(u^*, x^*) = J(u^*) \leq \inf_{u \in U} J(u) = \inf_{u \in U} \phi(u, x)$$

where $x = Tu$ and $x^* = Tu^*$.

This proves that (u^*, x^*) is an optimal pair for the abstract system (2.1). □

When the system operator T is weakly continuous we have the following theorem which follows along the same line as Theorem 4.1 . Here we use Lemma 4.2 instead of Lemma 4.1

Assumptions [II]

(a) : The operators K, N and H satisfy either Assumptions $[C]$ or Assumptions $[D]$.
(b) : For a fixed $x, u \to \phi(u, x)$ is convex and continuous and $x \to \phi(u, x)$ is weakly continuous and is uniform for all $u \in U$.
(c) : The control set U is weakly compact.

Theorem 4.2. *Under Assumptions* [II] *the system* (2.1) *has an optimal pair* (u^*, x^*)*.*

In most of the optimal control problems, the functional $\phi(u, x)$ is of quadratic type with respect to both u and x and in such cases weaker forms of continuity, viz, lower semicontinuity or weak lower semi continuity of ϕ is easier to check. So, in the following theorems we prove the existence of optimal pair (u^*, x^*) with lower semi continuity (weak lower semicontinuity) assumptions on ϕ instead of continuity (weak continuity) assumptions. For such functionals we investigate directly the unconstrained problem, where U is the whole space Z which is assumed to be a reflexive

Banach space.

Assumptions [III]

(a) : The operators K, N and H satisfy either Assumptions $[A]$ or Assumptions $[B]$.

(b) : $\phi : Z \times T(Z) \to \overline{\mathbb{R}}_+$ is a lower semicontinuous function with respect to the weak topology in Z and norm topology in $T(Z)$ and $\|u\| \to \infty$ implies $\phi(u, Tu) \to \infty, u \in U$.

Theorem 4.3. *Under Assumptions* [III]*, the system* (2.1) *has an optimal pair* (u^*, x^*)*.*

Proof. Let $\{u_n\}$ be a minimizing sequence of controls in Z. That is

$$\lim_n \phi(u_n, Tu_n) = \inf_{u \in Z} \phi(u, Tu) = m(say).$$

From Assumption [III(b)], $\{u_n\}$ is bounded. Since Z is reflexive, by extracting a subsequence, we can assume that $u_n \rightharpoonup u^*$ in Z. In view of Assumption [III(a)], Lemma 4.1 implies that the system operator is completely continuous and hence $Tu_n \to Tu^*$ strongly in X. Assumption [III(b)] gives

$$\phi(u^*, Tu^*) \leq \lim_{n \to \infty} \phi(u_n, Tu_n) \text{ whenever } u_n \to u^*.$$

This inturn implies that

$$m = \inf \phi(u, Tu) \geq \phi(u^*, Tu^*)$$

That is, $m = \phi(u^*, Tu^*)$ and (u^*, x^*) is the desired optimal pair, where $x^* = Tu^*$. □

When the system operator T is weakly continuous we have the following result, the proof of which follows by using Lemma 4.2.

Assumptions [IV]

(a) : The operators K, N and H satisfy either Assumptions $[C]$ or Assumptions $[D]$.

(b) : $\phi : Z \times T(Z) \to \overline{\mathbb{R}}_+$ is lower semicontinuous with respect to the weak topologies in Z and $T(Z)$ and further $\|u\| \to \infty$ implies $\phi(u, Tu) \to +\infty, u \in Z$.

Theorem 4.4. *Under Assumptions* [IV] *the nonlinear system* (2.1) *has an optimal pair* (u^*, x^*)*.*

Let the explicit representation for the cost functional be given by

$$\phi(u, x) = \langle u, Ru \rangle + \langle x, Wx \rangle \tag{4.1}$$

where $R : Z \to Z^*$ is a bounded linear symmetric, strictly monotone and coercive operator and $W : X \to X^*$ is a bounded linear symmetric monotone operator.

As a corollary of the above Theorem 4.4 we have the following result.

Corollary 4.1. *If the system operator is weakly continuous then the system* (2.1) *with respect to the cost functional* (4.1) *has a unique optimal pair* (u^*, x^*)*.*

Proof. Set $\phi_1(u) = \langle u, Ru\rangle$ and $\phi_2(x) = \langle x, Wx\rangle$. So $\phi(u,x)$ can be written as

$$\phi(u,x) = \phi_1(u) + \phi_2(x)$$

We first observe that ϕ_1 and ϕ_2 are differentiable functionals with gradients $2R$ and $2W$, respectively. Also, R and W are monotone operators by assumption and hence by Theorem 6.1 and Theorem 8.4 of Vainberg [18] it follows that ϕ_1 and ϕ_2 are weakly lower semicontinuous convex functionals. Moreover, the strict monotonicity of R implies that ϕ_1 is strictly convex and so is ϕ.

Further, since R is coercive, that is $\frac{\langle u,Ru\rangle}{\|u\|} \to \infty$ as $\|u| \ \to \infty$, we have that $\Phi(u,Tu) \to \infty$ as $\|u\| \to \infty$.

Thus $\Phi : Z \times T(Z) \to \overline{R}_+$ is lower semicontinuous in the weak topologies of Z and $T(Z)$ and is coercive. Now applying Theorem 4.4 it follows immediately that there exists an optimal pair (u^*, x^*) for the system (1.1) with respect to the cost functional (4.1). The uniqueness of (u^*, x^*) follows from the fact that the cost functional is strictly convex in Z. □

5. OPTIMALITY SYSTEM FOR THE ABSTRACT PROBLEM

In this section we investigate Problem 2.2. We assume that the state space X and the control space Z are real Hilbert spaces. Also throughout this section the cost functional under consideration is of the form (4.1). That is, $J(u) = \Phi(u,x)$ is of the form

$$J(u) = \langle u, Ru\rangle + \langle x, Wx\rangle \tag{5.1}$$

where, R and W satify the earlier assumptions of Section 4. Note that in view of monotonicity and coercivity assumptions on R, R is invertible as a bounded linear operator (refer Joshi and Bose [12]). This fact will be used in the subsequent analysis.

Recall that the system operator T is of the form

$$Tu = [I - KN]^{-1}Hu, \quad u \in Z \tag{5.2}$$

The following lemma gives the existence of the derivative of the system operator T under certain conditions on K, N and H.

Assumptions [V]

(a) : K, N and H are Frechet differentiable with $K'(x) = K, N'(x) = G(x)$ for all $x \in X$ and $H'(u) = H$ for all $u \in Z$.

(b) : $[I - KG(x)]^{-1}$ exists as a bounded linear operator for all $x = Tu, \quad u \in Z$.

Lemma 5.1. *Under Assumptions* [V], *the system operator* T *is Frechet differentiable with derivative given by*

$$T'(u) = [I - KG(x)]^{-1}H, \quad u \in Z \text{ and } x = Tu$$

Lemma 5.2. *Suppose that the system operator* T *is Frechet differentiable then the cost functional* J *given by* (5.1) *is Frechet differentiable with derivative* $J'(u)$ *given by*

$$\frac{1}{2}J'(u) = \langle T'(u)h, WTu\rangle + \langle h, Ru\rangle, \quad u \in Z$$

Proof. We have $J(u) = \langle u, Ru\rangle + \langle x, Wx\rangle$

As R and W are bounded symmetric linear operators we get

$$\begin{aligned} J(u+h) &- J(u) - 2\langle T'(u)h, WTu\rangle - 2\langle h, Ru\rangle \\ &= \langle T(u+h), WT(u+h)\rangle - \langle Tu, WTu\rangle + \langle u+h, R(u+h)\rangle \\ &- 2\langle h, Ru\rangle - \langle u, Ru\rangle - 2\langle T'(u)h, WTu\rangle \\ &= 2\langle T(u+h) - Tu - T'(u)h, WTu\rangle \\ &+ \langle T(u+h) - Tu, W(T(u+h) - T(u))\rangle + \langle h, Rh\rangle \end{aligned}$$

This implies that

$$\begin{aligned} &\frac{\|J(u+h) - J(u) - 2\langle T'(u)h, WTu\rangle - 2\langle h, Ru\rangle\|}{\|h\|} \\ &\leq \frac{2\|WTu\|\|T(u+h) - T(u+h) - T(u) - T'(u)h\|}{\|h\|} \\ &+ \frac{\|w\|\|T(u+h) - T(u)\|^2}{\|h\|} + \|R\|\|h\| \end{aligned} \tag{5.3}$$

In view of Frechet differentiability of T, the first term on the RHS of (5.3) goes to zero as $\|h\| \to 0$. Also Frechet differentiability of T implies that T is locally Lipschitz (refer Joshi and Bose [12]) and hence the second term also goes to zero as $\|h\| \to 0$. This proves that RHS of (5.3) goes to zero and hence LHS goes to zero as $\|h\| \to 0$.

This gives

$$\frac{1}{2}J'(u)h = \langle T(u)h, WTu\rangle + \langle h, Ru\rangle, \quad u \in Z.$$

□

The following theorem gives optimality system for (2.1). Here the superscript '*' corresponding to a given operator denotes its adjoint.

Theorem 5.1. *Under Assumptions* [V] *the 'optimality system' for* (2.1) *is given by*

$$x^* = KNx^* + Hu^*$$

$$u^* = -R^{-1}H^*[I - KG(x^*)]^{*-1}Wx^*$$

Proof. Existence and uniqueness of the optimal pair (u^*, x^*) is proved in Corollary 5.1. If u^* is an optimal control then it is necessary that $J'(u^*) = 0$. Using Lemma 5.2, we get

$$\langle h, Ru^* \rangle + \langle T'(u^*)h, Wx^* \rangle = 0 \text{ for all } h \in Z \text{ where } x^* = Tu^*.$$

Taking adjoint of the derivative of the system operator, denoted by $[T'(u^*)]^*$, we get

$$\langle h, Ru^* \rangle + \langle h, [T'(u^*)]^* Wx^* \rangle = 0 \text{ for all } h \in Z.$$

This implies that

$$Ru^* = [T'(u^*)]^* Wx^*$$

which gives

$$u^* = -R^{-1}([I - KG(x^*)]^{-1}H)^* Wx^*$$

That is,

$$u^* = -R^{-1}H^*[I - KG(x^*)]^{*-1}Wx^*$$

where x^* satisfies

$$x^* = KNx^* + Hu^*$$

Thus the optimal pair (u^*, x^*) satisfies the coupled operator equations

$$\begin{aligned} x^* &= KNx^* + Hu^* \\ u^* &= -R^{-1}H^*[I - KG(x^*)]^{*-1}Wx^* \end{aligned}$$

As special cases of this result we shall derive optimality system for parabolic equations involving linear and nonlinear operators in Section 6. □

Corollary 5.1. *Suppose that $X = Z$ is a real Hilbert space. Assume that $R = I = W$ and $H = K$ in the above Theorem* 5.1. *Then the unique optimal pair (u^*, x^*) satisfies the following optimality system*

$$\begin{aligned} x^* &= KNx^* + Ku^* \\ u^* &= K^*G^*(x^*)u^* - K^*x^* \end{aligned}$$

Proof. By Theorem 5.1, the optimal pair (u^*, x^*) satisfies the following optimality system

$$\begin{aligned} x^* &= KNx^* + Ku^* \\ u^* &= -K^*[I - KG(x^*)]^{*-1}x^* \end{aligned}$$

That is, $u^* = -K^*[I - G^*(x^*)K^*]^{-1}x^*$. Therefore,

$$\begin{aligned} [I - K^*G^*(x^*)]u^* &= -[I - K^*G^*(x^*)]K^*[I - G^*(x^*)K^*]^{-1}x^* \\ &= -K^*[I - G^*(x^*)K^*][I - G^*(x^*)K^*]^{-1}x^* \end{aligned}$$

This implies

$$-K^*x^* = u^* - K^*G^*(x^*)u^*$$

$$u^* = K^*G^*(x^*)u^* - K^*x^*$$

and hence the proof. □

Corollary 5.2. *For the linear system, that is, when $N = 0$, the optimality system is given by*

$$x^* = Hu^*$$
$$u^* = -R^{-1}H^*Wx^*$$

6. OPTIMALITY RESULTS FOR SYSTEMS GOVERNED BY DIFFERENTIAL EQUATIONS

We now derive sufficient conditions for the existence of optimal control for the class of nonlinear differential equation (1.1) defined in Section I. For that we make the following assumptions on the system components.

Assumptions [VI]

(a) : There exists a positive constant μ such that

$$\langle -A(t)x, x\rangle \geq \mu\|x\|^2 \text{ for all } x \in D, t \in [t_0, t_1]$$

(b) : The nonlinear function $F : [t_0, t_1] \times Y \to Y$ satisfies a growth condition of the form

$$\|F(t, x)\| \leq a(t) + b\|x\| \quad \forall t \in [t_0, t_1],\ \forall x \in Y$$

where $a \in L^2[t_0, t_1]$, $b > 0$. Further F is negative monotone, that is, $\langle F(t, x) - F(t, y), x - y\rangle \leq 0$ for all $x, y \in Y$, $t \in [t_0, t_1]$

(c) : For all $t > s$, $\Phi(t, s)$ is a compact evolution operator and $B(t)$ is a bounded linear operator for all $t \in [t_0, t_1]$.

Lemma 6.1. *Under Assumptions* [VI]*, the system operator T corresponding to the system* (1.1) *is completely continuous.*

Proof. Using the definition of the operators K, N and H (see (1.3)) it follows from infinite version of Joshi and George [13] that K is a bounded linear compact operator. Further, for $x \in X_1$, defining $f(t) = \int_{t_0}^{t} \Phi(t, \tau)x(\tau)d\tau$ we can write

$$\langle Kx, x\rangle_{X_1} = \int_{t_0}^{t_1} \langle f(t), x(t)\rangle_Y dt$$

Clearly

$$f'(t) = \int_{t_0}^{t} A(t)\Phi(t, \tau)x(\tau)d\tau + x(t)$$

By virtue of Assumption (a_0) it follows that $f(t) \in D$ and hence

$$f'(t) = A(t)\int_{t_0}^{t} \Phi(t, \tau)x(\tau)d\tau + x(t) \text{ (refer Curtain [7]).}$$

Therefore

$$\begin{aligned}\langle Kx, x\rangle_{X_1} &= \int_{t_0}^{t_1} \langle f(t), f'(t) - A(t)f(t)\rangle dt \\ &= \int_{t_0}^{t_1} \langle f(t), -A(t)f(t)\rangle dt + \int_{t_0}^{t_1} \langle f(t), f'(t)\rangle dt\end{aligned}$$

Since the second term on RHS is $\frac{1}{2}\int_{t_0}^{t_1} \frac{d}{dt}\|f(t)\|^2 dt$, we have

$$\langle Kx, x\rangle \geq \int_{t_0}^{t_1} \langle f(t), -A(t)f(t)\rangle dt$$

Now Assumption [VI(a)] implies that

$$\begin{aligned}\langle Kx, x\rangle &\geq \mu \int_{t_0}^{t_1} \|f(t)\|^2 dt = \mu \int_{t_0}^{t_1} \|(Kx)(t)\|^2 dt \\ &\geq \mu\|Kx\|^2 \quad \text{for all } x \in X^*\end{aligned}$$

Also, H is a completely continuous operator and -N is monotone. By Lemma 4.1 it follows that the system operator T is completely continuous. □

Remark 6.1. Suppose that the Assumption [VI(c)] in the above theorem is replaced by
(c') : $B(t)$ is a compact bounded linear operator from U to Y for all $t \in [t_0, t_1]$. Then it can be shown by using Lemma 4.2 that the system operator T is weakly continuous.

Remark 6.2. By virtue of Lemma 4.1 and 4.2 we can give different sets of verifiable assumptions on $A(t)$, $B(t)$ and $F(t, x(t))$ which will guarantee the complete and weak continuity of T. For example, if $A(t)$ is a closed linear operator and generator of an almost strong evolution operator, $B(t)$ is a bounded linear operator and $F(t, x(t))$ is Lipschitz then the weak continuity of T can be verified by using Remark 4.2.

We first consider the constrained case. Assume that U_{ad} is a weakly compact subset of U.

Assumption [VII]

(a) : $g : [t_0, t_1] \times U_{ad} \times Y \to \bar{\mathbb{R}}_+$ is approximately lower semi continuous.
(b) : For every $t \in [t_0, t_1]$, $(u, x) \to g(t, u, x)$ is lower semi continuous with respect to the weak topology in U_{ad} and strong topology in Y.
(c) : For all $(t, x) \in [t_0, t_1] \times Y$, $u \to g(t, u, x)$ is convex.

An easy application of Lemma 6.1 and Theorem 3.1 gives us the following result regarding the existence of an optimal pair for the system (1.11).

Theorem 6.1. *Under Assumptions* [VI] *and* [VII] *there exists an optimal pair* $(u^*, x^*) \in U_{ad} \times\ X$ *for the nonlinear evolution equation* (1.1).

Proof. Define $S = \{w \in L^2([t_0, t_1], U) : w(t) \in U_{ad} \text{ a.e.}\}$ Since U_{ad} is weakly compact we have that S is also weakly compact in $L^2([t_0, t_1], U)$.

Let $\{u_n\}$ be a sequence with weak limit u^* in S then by Lemma 6.1 the corresponding response $\{x_n\}$ converges strongly to x^* where x^* is the response of u^* . That is, $x_n(t) \to x^*(t)$ in Y whenever $u_n(t) \rightharpoonup u^*(t)$ in U_{ad}. Following Papageorgiou [14], the Assumptions [VII] implies that

$$\lim_{n\to\infty} \int_{t_0}^{t_1} g(t, u_n(t), x_n(t))dt \geq \int_{t_0}^{t_1} g(t, u^*(t), x^*(t))dt$$

whenever $u_n \rightharpoonup u^*$. That is, $\lim_{n\to\infty} J(u_n) \geq J(u^*)$ proving the weak lower semi continuity of J on the weakly compact set S. Now by Theorem 3.1, there exists an optimal pair (u^*, x^*) for the system (1.1).

For unconstrained problem, we take $g(t, u, x)$ to be of special form

$$g(t, u, x) = \langle u(t), R_0 u(t)\rangle + \langle x(t), W_0 x(t)$$

Then the cost functional $J(u)$ assumes the form

$$J(u) = \Phi(u, x) = \int_{t_0}^{t_1} \langle u(s), R_0(s)\rangle + \langle x(s), W_0\ x(s)\rangle ds \tag{6.1}$$

where $W_0 : Y \to Y$, $R_0 : U \to U$ are bounded linear operators and Y, U are real Hilbert spaces. □

<u>Assumptions [VIII]</u>

(a) : The operators A, B and F satisfy Assumptions [VI].

(b) : W_0 is a bounded linear symmetric monotone operator and R_0 is a bounded linear symmetric strongly monotone operator, that is, there exists a constant $a > 0$ such that

$$\langle u, R_0 u\rangle \ \geq a\|u\|^2 \quad \text{for all } u \in U.$$

Theorem 6.2. *Under Assumptions* [VIII] *the system* (1.1) *has a unique optimal pair* (u^*, x^*) *with respect to the cost functional* (6.1).

Proof. Define $W : L^2([t_0, t_1], Y) \to L^2([t_0, t_1], Y)$ and $R : L^2([t_0, t_1], U) \to L^2([t_0, t_1], U)$ by $(Wx)(t) = W_0 x(t)$ and $(Ru)(t) = R_0 u(t)$. It follows easily that R and W are both bounded linear operators (refer Joshi and Bose [12]). Further,

$$\langle x, Wx\rangle = \int_{t_0}^{t_1} \langle x(s), W_0 x(s)\rangle ds$$

$$\langle u, Ru\rangle = \int_{t_0}^{t_1} \langle u(s), R_0 u(s)\rangle ds$$

Thus (6.1) can be written as

$$J(u) = \langle u, Ru\rangle_Z + \langle x, Wx\rangle_X$$

It is easy to verify that R is a strongly monotone symmetric operator and W is linear, monotone and symmetric. Now the theorem follows by using Theorem 4.4 and Corollary 4.1. □

Remark 6.3. If we assume that the nonlinear function $F(t, x)$ is continuously Frechet differentiable with respect to x for almost all $t \in [t_0, t_1]$ with $G(t, x) = \frac{\partial}{\partial x} F(t, x)$ then it follows that the operator N, as defined by (6.4)(iii) in the space $X = L^2([t_0, t_1], Y)$, is continuously Frechet differentiable (refer [12]) with $[N(x)]u = Gu$ where $G : X \to X$ is defined by $(Gu)(t) = [G(t, x(t))]$.

As a particular case to this we have the following theorem regarding the optimality system for (1.1). We note that the system (1.1) is of the same type as considered by Seidman and Zhou [16]. However, we impose monotonicity assumptions on F in contrast to Lipschitz assumptions imposed by Seidman and Zhou [16]. Also we observe that we do not require Lipschitz continuity on the Frechet derivative of F,whereas Seidman and Zhou require so. Moreover our system (1.1) is non-autonomous.

Theorem 6.3. *Suppose that $Y = U$ is a Hilbert space and $B = R_0 = W_0 = I$ and the operators A and F satisfy Assumptions* [VIII(a)]. *Further, assume that the nonlinear function $F(t, x)$ is continuously Frechet differentiable with respect to x for almost all $t \in [t_0, t_1]$ with $G(t, x) = \frac{\partial}{\partial x} F(t, x)$.*

Suppose that $\int_{t_0}^{t_1} \|G(t, x(t))\|^2 dt < \infty$ for all $x \in Y$. Then the optimality system for (1.1) *with cost functional* (6.1) *is given by*

$$\begin{aligned} \dot{x}^* &= A(t)x^*(t) + F(t, x^*(t)) + u^*(t) \\ \dot{u}^* &= A^*(t)u^*(t) + G^*(t, x^*(t))u^*(t) - x^*(t) \end{aligned}$$

$$x^*(t_0) = x_0, \quad u^*(t_1) = 0$$

Proof. The existence of an optimal pair $(u^*(t), x^*(t))$ follows from Theorem 6.2. Using the definitions of the operators K, N and H and Remark 6.3 it follows that $K'(x) = K$, $N'(x) = G(x)$ and $H'(u) = K$ for all x, $u \in X$. For $v \in X$, consider the operator equation

$$u = KG(x)u + v$$

for a fixed $x \in X$. Now using the definition of operators we can write it as

$$u(t) = \int_{t_0}^{t} \Phi(t, s)G(s, x(s))u(s)ds + v(t) \tag{6.2}$$

Since $\|\Phi(t, s)\| \le M$ and $\int_{t_0}^{t_1} \|G(s, x(s))\|^2 ds < \infty$, we have

$$\int_{t_0}^{t_1} \int_{t_0}^{t_1} \|\Phi(t, s)G(s, x(s))\|^2 ds \; dt < \infty.$$

Hence, for each fixed x and v, (6.2) is a linear Voltera integral equation with L^2 kernel. Thus for each $v \in X$, (6.2) has a unique solution in X. That is $[I - KG(x)]^{-1}$ exists and is linear. Moreover this inverse is bounded. Hence by Theorem 5.1 it follows that the optimal pair $(u^*(t), x^*(t))$ satisfies the equations:

$$\begin{aligned} x^*(t) &= \Phi(t, t_0)x_0 + \int_{t_0}^{t} \Phi(t, \tau)F(\tau, x^*(\tau))d\tau + \int_{t_0}^{t} \Phi(t, \tau)u^*(\tau)d\tau \\ u^*(t) &= \int_{t}^{t_1} \Phi(\tau, t)G^*(\tau, x^*(\tau))u^*(\tau)d\tau - \int_{t}^{t_1} \Phi^*(\tau, t)x^*(\tau)d\tau \end{aligned}$$

where $\Phi(t, \tau)$ and $\Phi^*(\tau, t)$ are the evolution operators generated by $A(t)$ and $A^*(t)$, respectively. Differentiating with respect to t we get

$$\begin{aligned} \dot{x}^*(t) &= A(t)x^*(t) + F(t, x^*(t)) + u^*(t) \\ \dot{u}^*(t) &= A^*(t)u^*(t) + G^*(t, x^*(t))u^*(t) - x^*(t) \end{aligned}$$

$$x^*(t_0) = x_0, \quad u^*(t_1) = 0$$

If $F = 0$, using Remark 6.1 we get the following result due to Datko [Theorem 1., 8] for the autonomous linear system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(t_0) = x_0. \tag{6.3}$$

□

Corollary 6.1. *Suppose that A is a closed linear operator which generates a strongly continuous semigroup $\Phi(t)$ and B is a bounded linear operator. Suppose that W_0 and R_0 satisfy Assumptions* [VIII(b)]. *Then the optimality system for* (6.3) *is given by*

$$\begin{aligned} x^*(t) &= \Phi(t - t_0)x_0 + \int_{t_0}^{t} \Phi(t - \tau)Bu^*(\tau)d\tau \\ u^*(t) &= -R^{-1}B^* \int_{t_0}^{t_1} \Phi^*(\tau - t)Wx^*(\tau)d\tau \end{aligned}$$

We now consider a particular case of the system (1.1) and show that the canonical (Hamiltonian) system in the Minimum Principle of Pontriagin (refer Athans and Falb [1]) can be deduced from the optimality system derived in Theorem 5.1.

Let Y and U be Hilbert spaces and A, B and F given in (1.1) be time indepedent. Let the cost functional be as in (6.1). Assume that $F(x(t))$ is continuously Frechet differentiable with $\frac{\partial}{\partial x}F(x(t)) = G(x(t))$ and $\Phi(t, s) = e^{A(t-s)}$ is the semi-group generated by A. Using the definitions (1.3), we have $H = BK$. Let $p(t)$ denotes the costate of the system. Now the Hamiltonian $H(x, p, u)$ of the system is given by (refer Athans and Falb [1])

$$H(x, p, u) = \frac{1}{2}\langle x, W_0x\rangle + \frac{1}{2}\langle u, R_0u\rangle + \langle Ax + F(x), p\rangle + \langle Bu, p\rangle$$

Suppose that the Assumptions [VIII] are satisfied. Using Theorem 5.1 , the optimality system for the given system can be written as

$$
\begin{aligned}
x^* &= KNx^* + KBu^* \\
u^* &= -R^{-1}B^*p^*
\end{aligned}
\tag{6.4}
$$

where

$$p^* = K^*[I - KG(x^*)]^{*-1}Wx^*$$

where $(Wx)(t) = W_0x(t)$ and $(Ru)(t) = R_0u(t)$. From (6.4) we see that the costate $p^*(t)$ satisfies

$$p^*(t) = K^*G^*(x^*)p + K^*Wx^*$$

Thus the optimal pair $(u^*(t), x^*(t))$ satisfy

$$
\begin{aligned}
x^*(t) &= e^{A(t-t_0)}x_0 + \int_{t_0}^{t} e^{A(t-s)}F(x^*(s))ds + \int_{t_0}^{t} e^{A(t-s)}Bu^*(s)ds \\
u^*(t) &= -R^{-1}B^*p^* \\
p^*(t) &= \int_{t}^{t_1} e^{A^*(s-t)}G^*(x^*(s))p^*(s)ds + \int_{t}^{t_1} e^{A^*(s-t)}W_0x^*(s)ds
\end{aligned}
$$

Differentiating w.r.t. t, the state and costate satisfy

$$\dot{x}^*(t) = Ax^*(t) + Bu^*(t) + F(x^*(t)) \equiv -\frac{\partial}{\partial p}H(x^*(t), p^*(t), u^*(t)) \tag{6.5}$$

$$\dot{p}^*(t) = -A^*p^*(t) - G^*(x^*(t))p^*(t) - W_0x^*(t) \equiv -\frac{\partial}{\partial x}H(x^*(t), p^*(t), u^*(t))$$
$$x^*(t_0) = x_0, \quad p^*(t_1) = 0$$

The pair of equations (6.5) is the canonical system (satisfied by the optimal pair) in the Minimum Principle of Pontriagin. It is also possible derive the optimality system if the canonical system satisfied by the optimal pair is known.

In the following we show how the optimality system is related to the Riccati equations.

Let the state $x^*(t)$ and the costate $p^*(t)$ be related by

$$p^*(t) = R(t)x^*(t) \tag{6.6}$$

where $\{R(t), t \in [t_0, t_1]\}$ is a family of nonlinear operators on Y. Now from (6.4), the optimal control has a feed-back representation

$$u^*(t) = -R^{-1}B^*R(t)x^*(t) \text{ for all } t \in [t_0, t_1] \tag{6.7}$$

Differentiating (6.6) w.r.t t and equating with $p^*(t)$ in (6.5), we obtain the following Riccati type nonlinear equation.

$$
\begin{aligned}
&R_t(t)x^*(t) + R_x * (t, x^*)(Ax^*(t) + F(x^*(t)) - BR^{-1}B^*R(t)x^*(t)) \\
&+A^*R(t)x^*(t) + G^*(x^*(t))R(t)x^*(t) + W_0x^*(t) = 0, \; R(t_1)x^* = 0
\end{aligned}
\tag{6.8}
$$

where R_t and R_x are the partial derivatives of R w.r.t. t and x, respectively. Using the classical variational principle, Barbu and Prato obtained equation of the form (6.8) in [3]. They have given some conditions for the existence and uniqueness of solutions of such equations. We observe that $R(t)x(t) = K^*[I - KG(x(t))]^{*-1}W_0x(t)$ satisfies the above equation. If the solution of (6.8) is known then the optimal control can be calculated from (6.7),that is the optimal control is same as in the optimality system (6.4).

If $F \equiv 0$ (i.e., when the system is linear), the above equation (6.8) reduces to the classical Riccati equation

$$R(t) + R(t)A + A^*R(t) - R(t)BR^{-1}B^*R(t) + W_0 = 0,$$

$R(t_1) = 0$, where $R(t)$ is a linear operator on Y.

To illustrate the applicability of our results for the existence of optimal pair, we consider an example of a control system described by a nonlinear distributed parameter system, where the operator A has a specific representation.

Example 6.1. Let $I = [0 \quad b]$ and W be a bounded open set in $\mathbb{R}^n$ with smooth boundary $\partial W = \Gamma$.

Consider the following distributed parameter system

$$\frac{\partial}{\partial t}x(t,z) = \sum_{k=1}^{n} \frac{\partial}{\partial z_k}\left(p(t,z)\frac{\partial}{\partial z_k}x(t,z)\right) + Bu(t,z) + f(t,z,x(t,z))$$
$$x(t,z) = 0 \ \text{ on } \ I \times \Gamma, \quad x(0,z) = x_0(z) \ \text{ for } \ z \in W \tag{6.9}$$

where $p : I \times W \to \mathbb{R}_+$ is such that it 1s Lipschitz w.r.t. the t variable, C^1-in the z variable and $t \to \|p(t,\cdot)\|_\infty \in L^\infty_+$. B belongs to $\text{Ł}(L^2_m(I \times W),\ H^{-1}(W))$. Assume that $f : I \times \bar{W} \times \mathbb{R} \to \mathbb{R}$ is a nonlinear function such that it is measurable in (t,z) and continuous in x and $|f(t,z,x)| \le \alpha_0(t,z) + b(z)|x|$ a.e. where $\alpha_0(\cdot,\cdot) \in L^2(I,W)$ and $b \in L^\infty(W)$. Let $f(t,z,x)$ be monotone decreasing with respect to x for all $z \in W$ and $t \in I$.

The cost functional to be minimized is given by

$$J(u) = \int_0^b \int_W |x(t,z)|^2 dz\, dt + \int_0^b \int_W \langle R(t,z)u(t,z), u(t,z)\rangle dz\, dt$$

Assume that $R : I \times W \to \mathbb{R}^{m\times m}$ belongs in $L^\infty_{m\times m}$ and $R(t,z)$ is strongly monotone with respect to z.

Let the set of all admissible controls be a closed and bounded subset U_{ad} of $L^2_m(I \times W)$. Clearly U_{ad} is a weakly compact set. Let $Y = H^1_0(W)$, $Y^* = H^{-1}(W) = (H^1_0(W))^*$. Note that the embeddings $H^1_0(W) \subset L^2(W) \subset H^{-1}(W)$ are all continuous and dense. For each $t \in I$, define $A(t)$ from Y to Y^* by $\langle A(t)x, v\rangle = a(t, x, v)$ for all $x, v \in Y$, where $a(t, x, v)$ is given by

$$a(t,x,v) = \sum_{k=1}^{n} \in_W \; p(t,z)\frac{\partial x}{\partial z_k}\frac{\partial v}{\partial z_k}dz$$

The assumptions on $p(\cdot,\cdot)$ imply that $\|A(t)x - A(s)x\| \leq k|t-s|\,\|x\|$ for some constant $k > 0$. By Poincare's inequality we have

$$\langle A(t)x, x\rangle = \int_W \sum_{k=1}^{n} p(t,z)|\frac{\partial x}{\partial z_k}^2 dz \geq \mu\|x\|^2 \;\; \text{for all} \;\; x \in Y, \quad \mu > 0$$

For $x \in Y$, $\|A(t)x\| = \sup\{\langle A(t)x, v\rangle : \|v\| \leq 1\} \leq \|p(t)\|\,\|x\|$ (by Cauchy and Poincare's inequalities). Define $F : I \times L^2(W) \to L^2(W)$ by $F(t, x)(z) = f(t, z, x(z))$ and $R(t) \in L(L^2_m(W))$ by $(R(t)u)(\cdot) = R(t,\cdot)u(\cdot)$. Now denoting $x(t) = x(t,\cdot) \in H^1_0(W) \subseteq L^2(W)$ and $u(t) = u(t,\cdot) \in L^2_m(W)$, our system (6.9) and cost functional J take the form

$$\dot{x}(t) = A(t)x(t) + Bu(t) + F(t, x(t)); \quad x(0) = x_0, u(t) \in U_{\text{ad}} a.e$$

$$J(u) = \int_0^b \|x(t)\|^2_{L^2(W)}dt + \int_0^b \langle R(t)u(t), u(t)\rangle_{L^2(W)}dt$$

Clearly, this is in the form of (6.3). Using Proposition 6.5.1 of Tanabe [17], it follows that the linear system satisfies assumption (a_0). Further the family of linear operators $\{A(t) : t \in I\}$ generates a compact evolution operator $\Phi(t, s)$ for $t > s$. Thus the operators $A(t)$ and $F(t, x)$ satisfy Assumptions [VI] and conditions of Theorem 6.2, and hence there exists a unique optimal pair (u^*, x^*) for the given distributed parameter system (6.9).

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