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CONTROLLABILITY OF THE IMPULSIVE FINITE DELAY DIFFERENTIAL EQUATIONS OF FRACTIONAL ORDER WITH NONLOCAL CONDITIONS

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ABSTRACT. In this paper, we study the controllability of the impulsive finite delay differential equations of fractional order with nonlocal conditions in a Banach spaces. The results are obtained by using convex condensing operator and Sadovskii's fixed point theorem via measures of noncompactness and semigroup theory. We do not assume the compactness of the semigroup. An example is presented to illustrate the main result.

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1. Introduction

The fractional differential equations have recently been proved to be valuable tool in modelling of many phenomena in various fields of science and engineering. Indeed, we can find numerous applications in electrochemistry, control, porous media, electromagnetic, etc. (see [1–6]). So the research on fractional differential equations has become an object of extensively study during the past decades.

In 1960, Kalman first introduced the concept of controllability which leads to some very important conclusion regarding the behaviour of linear and nonlinear dynamical systems. The problem of controllability is to show the existence of a control function, which steers the solution of the system from its initial state to final state, where initial and final state may vary over the entire space. In recent years, the various work of controllability of the system represented by differential equations, intgro differential equations, differential inclusions, neutral functional differential equations and impulsive differential inclusions in Banach spaces have been studied by many authors (see [7–14]and references therein). In [15–17], the authors have discussed the controllability of impulsive delay differential equations in abstract spaces. Wang and Zhou [18] have investigated the complete controllability of fractional evolution system without assuming compact semigroup.

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In this paper, we will study the sufficient conditions for the controllability of the impulsive finite delay differential equations of fractional order with nonlocal conditions of the form

$$(1.1)\qquad \begin{cases} {}^cD^\alpha x(t) &= Ax(t) + f(t, x_t, \int_0^t h(t,s,x_s)ds) \\ & \qquad +(Bu)(t), \quad t \in J = [0,b], \quad t \neq t_i, \\ \Delta x|_{t=t_i} &= I_i(x_{t_i}), \quad i = 1,2,\ldots,m, \\ x_0 &= \phi + g(x), \quad t \in [-a,0], \end{cases}$$

where state $x(\cdot)$ takes values in the real Banach space X endowed with norm $\|\cdot\|$; ${}^cD^\alpha$ is the Caputo fractional derivative of order α, $\frac{1}{2} < \alpha < 1$; $A : D(A) \subset X \to X$ is a linear closed densely defined operator; A is an infinitesimal generator of a strongly continuous semigroup $T(t)$ $(t \geq 0)$ on X. The control function $u(\cdot)$ is given in $L^2(J,U)$, a Banach space of admissible control function with U as a Banach space; B is a bounded linear operator from U into X. The non linear operator $h : \Sigma \times D \to X$, $f : J \times D \times X \to X$ are given continuous functions, here $\Sigma = \{(t,s) \mid 0 \leq s \leq t \leq T\}$ and $D = \{\psi : [-a,0] \to X \mid \psi(t)$ is continuous everywhere except for a finite number of points t_i at which $\psi(t_i^+)$ and $\psi(t_i^-)$ exist and $\psi(t_i) = \psi(t_i^-)\}$. $I_i : D \to X$, $i = 1,2,\ldots,m$ are impulsive functions, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = b$, $\Delta\xi(t_i)$ is the jump of a function ξ at t_i defined by $\Delta\xi(t_i) = \xi(t_i^+) - \xi(t_k^-)$.

For any function $x \in PC([-a,b],X)$ and any $t \in J$, x_t denotes the function in D defined by

$$x_t(\theta) = x(t+\theta), \quad \theta \in [-a,0],$$

where $x_t(\cdot)$ represent the time history of the state from the time $t-a$ up to the present time t and $PC([-a,b],X) = \{x : [-a,b] \to X \mid x(t)$ is continuous everywhere except for a finite number of points t_i, at which $x(t_i^+)$ and $x(t_i^-)$ exist and $x(t_i) = x(t_i^-)\}$.

The measure of noncompactness has been used to study differential equation in Banach spaces. Xue [19] studied the existence integral solutions for nonlinear differential equations with nonlocal initial conditions under noncompactness conditions. Zhu et al. [20] have proved the existence results of mild solutions of first order semilinear differential equations with Hausdorff measure of noncompactness by using a fixed point theorem. In these papers the authors has used the Sadovskii fixed point theorem related with condensing operators.

This paper is motivated by recent works [17, 18, 21]. We will study the controllability of the impulsive finite delay differential equation of fractional order with nonlocal conditions by means of fractional calculus, Kurtauskii's measure of noncompactness and Sadovski fixed point theorem. For this we will convert the controllability problem into a fixed point problem with assumption that the controllability operator has an induced inverse on a quotient space. To the best of our knowledge, up to now, no work has been reported on controllability of such type of the problem (1.1).

The rest of paper is organized as follows: In the next section we give some basic definitions and notations. In section 3, we establish the sufficient conditions of controllability of the system (1.1). Finally, in section 4, we present an example to illustrate our results.

2. Preliminaries

In this section, we introduce some basic definition and notation which are used throughout this paper. We denote by X a Banach space with the norm $\|\cdot\|$ and $A : D(A) \to X$ is the infinitesimal generator of a strongly continuous semigroup $\{T(t), t \geq 0\}$. This means that there exists $M \geq 1$ such that $\sup_{t\geq 0} \|T(t)\| \leq M$ (see [22]). Let $L^p([0, b], X)$, $1 \leq p < \infty$ be the space of X-valued Bochner integrable functions on $[0, b]$ with the norm $\|f\|_{L^p} = \left(\int_0^b \|f(t)\|^p\right)^{\frac{1}{p}} dt$. Also note that $PC([-a, b], X)$ is a Banach space with the norm

$$\|x\|_{PC} = \sup\{\|x(t)\| : t \in [-r, b]\}.$$

Definition 2.1 (see [1]). The Riemann-Liouville fractional integral of order $\alpha > 0$ for a function f is given by

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds, \quad t > 0,$$

where Γ is the gamma function, and $f \in L^1([0, T], X)$.

Definition 2.2 (see [1]). The fractional derivative of order $0 \leq n-1 < \alpha < n$ in the Caputo sense is defined as

$${}^cD^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha+1-n}} ds, \quad t > 0,$$

where f is an abstract n-times continuous differentiable function and Γ is a gamma function.

If f is an abstract function with values in a Banach space X, then integral which appear in Definition 2.1 and 2.2 are taken in Bochner's sense.

Definition 2.3 (see [23]). A function $x(\cdot) \in PC([-a, b], X)$ is said to be a mild solution of the system (1.1) if $x(t) = \phi(t) + g(x)(t)$ on $[-a, 0]$, $\Delta x|_{t=t_i} = I_i(x_{t_i})$, $i = 1, 2, \ldots, m$, the restriction of $x(\cdot)$ to the interval J_i $(i = 1, 2, \ldots, m)$ is continuous and the following fractional integral equation is satisfied:

$$\begin{aligned} x(t) = U(t)(\phi(0) + gx(0)) + \int_0^t (t-s)^{\alpha-1} V(t-s) \Bigg[f\left(s, x_s, \int_0^t h(s, \tau, x_\tau)d\tau\right) \\ + Bu(s)\Bigg] ds + \sum_{0<t_i<t} U(t-t_i) I_i(x_{t_i}), \end{aligned} \tag{2.1}$$

where

$$U(t) = \int_0^\infty \phi_\alpha(\theta) T(t^\alpha \theta) d\theta, \quad V(t) = \alpha \int_0^\infty \theta \phi_\alpha(\theta) T(t^\alpha \theta) d\theta, \tag{2.2}$$

and

$$\phi_\alpha(\theta) = \frac{1}{\alpha} \theta^{-1-1/\alpha} \rho_\alpha(\theta^{-1/\alpha}),$$

note that $\phi_\alpha(\theta)$ satisfies the condition of a probability density function defined on $(0, \infty)$, that is $\phi_\alpha(\theta) \geq 0$, $\int_0^\infty \phi_\alpha(\theta) d\theta = 1$ and $\int_0^\infty \theta \phi_\alpha(\theta) = \frac{1}{\Gamma(1+\alpha)}$. Also the term $\rho_\alpha(\theta)$ is defined as

$$\rho_\alpha(\theta) = \frac{1}{\pi} \sum_{n=1}^\infty (-1)^{n-1} \theta^{-n\alpha-1} \frac{\Gamma(n\alpha+1)}{n!} \sin(n\pi\alpha), \quad \theta \in (0, \infty).$$

Lemma 2.4. *The following properties are valid:*

(i) *for fixed $t \geq 0$ and any $x \in X$, we have*

$$\|U(t)x\| \leq M\|x\|, \quad \|V(t)x\| \leq \frac{\alpha M}{\Gamma(1+\alpha)} \|x\| = \frac{M}{\Gamma(\alpha)} \|x\|.$$

(ii) *The operators are $U(t)$ and $V(t)$ are strongly continuous for all $t \geq 0$.*

(iii) *If $S(t)(t \geq 0)$ is a compact semigroup in X, then $U(t)$ and $V(t)$ are norm-continuous in X for $t > 0$.*

(iv) *If $S(t)(t \geq 0)$ is a compact semigroup in X, then $U(t)$ and $V(t)$ are compact operators in X for $t > 0$.*

Now we recall the definition of Kuratowski's measure of noncompactness, which will be used in the next section to study the controllability of the impulsive fractional differential equation.

Definition 2.5 (see [24, 25]). Let X be a Banach space and $\mathcal{B}(X)$ be family of bounded subset of X. Then $\mu : \mathcal{B}(X) \to \mathbb{R}^+$, defined by

$$\mu(S) = \inf\{\delta > 0 : S \text{ admits a finite cover by sets of diameter } \leq \delta\},$$

where $S \in \mathcal{B}(X)$. Clearly $0 \leq \mu(S) < \infty$. $\mu(S)$ is called the Kuratowski measure of noncompactness.

We need to use the following basic properties of the μ measure and Sadovskii's fixed point theorem.

Lemma 2.6 (see [24, 25]). *Let S, S_1 and S_2 be bounded sets of a Banach space X. Then*

(i) $\mu(S) = 0$ *if and only if S is relatively compact set in X.*

(ii) $\mu(S_1) \leq \mu(S_2)$ *if $S_1 \subset S_2$.*

(iii) $\mu(S_1 + S_2) \leq \mu(S_1) + \mu(S_2)$.

(iv) $\mu(\lambda S) \leq |\lambda| \mu(S)$ *for any $\lambda \in \mathbb{R}$.*

Lemma 2.7 (see [24, 25]). *If $W \subset C([a,b],X)$ is bounded and equicontinuous on $[a,b]$, then $\mu(W(t))$ is continuous for $t \in [a,b]$ and*

$$\mu(W) = \sup\{\mu(W(t)), t \in [a,b]\}, \text{ where } W(t) = \{x(t) : x \in W\} \subseteq X.$$

Remark 2.8 (see [24,25]). If B is a bounded set in $C([a,b],X)$, then $B(t)$ is bounded in X, and $\mu(B(t)) \le \mu(B)$.

Lemma 2.9 (see [24, 25]). *Let $B = \{u_n\} \subset C(I,X)$ $(n = 1,2,\ldots)$ be a bounded and countable set. Then $\mu(B(t))$ is Lebesgue integrable on I, and*

$$\mu\left(\left\{\int_I u_n(t)dt \mid n = 1,2,\ldots\right\}\right) \le 2\int_I \mu(B(t))dt, \text{ here } I = [a,b]. \tag{2.3}$$

Lemma 2.10 (see [26]). *(i.) If $W \subset PC([a,b],X)$ is bounded, then $W(t)$ is bounded in X, and $\mu(W(t)) \le \mu(W)$ for any $t \in [a,b]$.*

(ii.) If $W \subset PC([a,b],X)$ is bounded and piecewise equicontinuous on $[a,b]$, then $\mu(W(t))$ is piecewise continuous for $t \in [a,b]$ and

$$\mu(W) = \sup\{\mu(W(t)), t \in [a,b]\}.$$

Lemma 2.11 (see [17]). *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions in $L^1([0,b],\mathbb{R}^+)$. Assume that there exist $\varphi,\eta \in L^1([0,b],\mathbb{R}^+)$ satisfying $\sup_{n\ge1}\|f_n(t)\| \le \varphi(t)$ and $\mu(\{f_n\}_{n=1}^{\infty}) \le \eta(t)$ a.e. $t \in [0,b]$, then for all $t \in [0,b]$, we have*

$$\mu\left(\left\{\int_0^t f_n(s)ds : n \ge 1\right\}\right) \le 2\int_0^t \eta(s)ds.$$

Lemma 2.12 (see [27]). *Let $S \subset X$ be bounded. Then There exists a countable set $S_0 \subset S$ such that $\mu(S) \le 2\mu(S_0)$.*

Lemma 2.13. *Let X and Y be two Banach spaces. A map $F : \Omega \subset X \to Y$ is said to be a condensing map if F is continuous and takes the bounded sets into bounded sets, and $\mu(F(S)) < F(S)$ for all bounded sets $S \subset \Omega$ with $\mu(S) \neq 0$.*

Lemma 2.14 (Sadovskii's fixed point theorem). *Let X be a Banach space and Ω be closed convex bounded subset in X. If $F : \Omega \to \Omega$ is a condensing map. Then F has a fixed point in Ω.*

3. **Main result**

In this section, we prove the result of controllability of problem (1.1). First we take the following assumptions:

(H1) The function $f : J \times D \times X \to X$ satisfies the following:

(i) For $t \in J$, the function $f(t,\cdot,\cdot) : D \times X \to X$ is continuous and for all $(\varphi,x) \in D \times X$, the function $f(\cdot,\varphi,x)$ is strongly measurable.

(ii) For $t \in J$ and $r > 0$, there exist $\frac{a_r(\cdot)}{(t-\cdot)^{1-\alpha}} \in L^1([0,t],\mathbb{R}^+)$ such that

$$\sup_{\|\varphi\|_D \le r} \|f(\cdot,\varphi)\| \le a_r(t) \quad \text{for a.e. } t \in J,$$

and

$$\lim_{r\to\infty} \inf \frac{1}{r}\int_0^t \frac{a_r(s)}{(t-s)^{1-\alpha}} = \sigma < +\infty.$$

(iii) There exists a function $\eta \in L^2([0,b],\mathbb{R}^+)$ such that

$$\mu(f(t,E,S)) \le \eta(t)\left[\sup_{-a\le\theta\le 0}\mu(E(\theta)) + \mu(S)\right],$$

for a.e. $t \in J$ and $E \subset D$, $S \subset X$, where $E(\theta) = \{\phi(\theta) : \phi \in E\}$.

(H2) The function $h : \Sigma \times D \to X$ satisfies the following:

(i) For each $(t,s) \in \Sigma$, the function $h(t,s,\cdot) : D \to X$ is continuous, and for each $x \in D$, the function $h(\cdot,\cdot,x) : \Sigma \to X$ is strongly measurable.

(ii) There exists a function $m \in L^1(\Sigma,\mathbb{R}^+)$ such that

$$\|h(t,s,x_s)\| \le m(t,s)\|x_s\|_D.$$

(iii) There exists a integrable function $\zeta : \Sigma \to [0,\infty)$ such that

$$\mu(h(t,s,H)) \le \zeta(t,s)\sup_{-a\le\theta\le 0}\mu(H(\theta)) \quad \text{a.e. } t \in J$$

and $H \subset D$, where $H(\theta) = \{\phi(\theta) : \phi \in H\}$.

For convenience, we write $L_0 = \max\int_0^t m(t,s)ds$ and $\zeta^* = \max\int_0^t \zeta(t,s)ds$.

(H3) $g : PC([0,b],X) \to X$ is a continuous compact operator such that

$$\lim_{\|y\|_{PC}\to\infty} \frac{\|g(y)\|}{\|y\|_{PC}} = 0.$$

(H4) The linear operator $W : L^2(J,U) \to X$ defined by

$$Wu = \int_0^b (b-s)^{\alpha-1}V(t-s)Bu(s)ds$$

has an inverse operator W^{-1} which takes values in $L^2(J,U)/\ker W$ and there exist two constants $M_2, M_3 > 0$ such that

$$\|B\| \le M_2, \quad \|W^{-1}\| \le M_3$$

and also there is $K_w \in L^2(J,\mathbb{R})$ such that for every bounded set $Q \subset X$,

$$\mu(W^{-1}Q)(t) \le K_w(t)\mu(Q).$$

(H5) The function $I_i : D \to X$, $i = 1,2\ldots,m$, is a continuous operator and there exist nondecreasing functions $L_i : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\|I_i(x)\| \le L_i(\|x\|_D), \quad i = 1,2,\ldots,m, \quad x \in D,$$

$$\lim_{\rho\to\infty}\inf\frac{L_i(\rho)}{\rho} = \lambda_i < \infty, \quad i = 1,2,\ldots m$$

and also there exists constant $k_i \geq 0$ such that

$$\mu(I_i(S)) \leq k_i \sup_{-a\leq\theta\leq 0} \mu(S(\theta)), \quad i = 1, 2, \ldots, m.$$

(H6) Take

$$N = \left[(2\zeta^* + 1)\left(\frac{2M}{\Gamma(\alpha)} + \frac{4M^2M_2}{(\Gamma(\alpha))^2}\sqrt{\frac{b^{2\alpha-1}}{2\alpha-1}}\|K_W\|_{L^2}\right)\|\eta\|_{L^2}\sqrt{\frac{b^{2\alpha-1}}{2\alpha-1}} + \left(M + \frac{2M^2M_2}{\Gamma(\alpha)}\sqrt{\frac{b^{2\alpha-1}}{2\alpha-1}}\|K_W\|_{L^2}\right)\sum_{i=1}^m k_i\right] < \frac{1}{2},$$

and for convenience, we write

$$M^* = M_3\left[\|x_1\| + M\|\hat{\phi}(0)\| + \frac{M}{\Gamma(\alpha)}\int_0^b (b-s)^{\alpha-1}a_{r^*}(s)ds + M\sum_{i=1}^m L_i(r')\right].$$

Theorem 3.1. *Assume that the hypothesis* (H1)–(H6) *hold. Then the impulsive finite delay differential equations of fractional order* (1.1) *is controllable on* J *if the condition*

$$M\left(1 + \frac{MM_2M_3}{\Gamma(\alpha)}\sqrt{\frac{b^{2\alpha-1}}{2\alpha-1}}\right)\left[\frac{\sigma(1+L_0)}{\Gamma(\alpha)} + \sum_{i=1}^m \lambda_i\right] < 1 \tag{3.1}$$

is satisfied.

Proof. Using hypothesis (H4), for every $x \in PC([-a,b],X)$, we define the control

$$u_x(t) = W^{-1}\left[x_1 - U(b)(\phi(0) + gx(0)) - \int_0^b (b-s)^{\alpha-1}V(b-s) \times f\left(s, x_s, \int_0^s h(s,\tau,x_\tau)d\tau\right)ds - \sum_{0<t_i<b} U(b-t_i)I_i(x_{t_i})\right](t).$$

By using this control, we define the operator

$$(Fx)(t) = \begin{cases} \phi(t) + (g(x))(t), & t \in [-a,0], \\ U(t)\left[\phi(0) + (g(x))(0)\right] + \int_0^t (t-s)^{\alpha-1}V(t-s) \\ \times\left[f\left(s, x_s, \int_0^s h(s,\tau,x_\tau)d\tau\right) + Bu_x(s)\right]ds \\ + \sum_{0<t_i<t} U(t-t_i)I_i(x_{t_i}), & t \in J. \end{cases} \tag{3.2}$$

Clearly any fixed point of F is a solution of (1.1) and also note that $x(b) = (Fx)(b) = x_1$. We rewrite the problem (1.1) as follows. For $\phi \in D$, we define $\hat{\phi} \in PC$ by

$$\hat{\phi}(t) = \begin{cases} \phi(t) + (g(x))(t), & t \in [-a,0], \\ U(t)[\phi(0) + (g(x))(0)], & t \in J = [0,b]. \end{cases}$$

So $\hat{\phi} \in PC$. Let $x(t) = y(t) + \hat{\phi}(t)$, $t \in [-a,b]$. Clearly we see that y satisfies $y_0 = 0$ and for $t \in J$,

$$y(t) = \int_0^t (t-s)^{\alpha-1}V(t-s)\left[f\left(s, y_s + \hat{\phi}_s, \int_0^s h(s,\tau,y_\tau + \hat{\phi}_\tau)d\tau\right) + Bu_y(s)\right]ds$$

$$+ \sum_{0<t_i<t} U(t-t_i)I_i(y_{t_i} + \hat{\phi}_{t_i}),$$

where

$$u_y(s) = W^{-1}[x_1 - U(b)(\phi(0) + gx(0)) - \int_0^b (b-s)^{\alpha-1} V(b-s) \times \\ \times f(s, y_s + \hat{\phi}_s, \int_0^s h(s,\tau, y_\tau + \hat{\phi}_\tau)d\tau)ds - \sum_{0<t_i<t} U(t-t_i)I_i(y_{t_i} + \hat{\phi}_{t_i})](s)$$

if and only if x satisfies

$$x(t) = U(t)[\phi(0) + gx(0)] + \int_0^t (t-s)^{\alpha-1} V(t-s)[f(s, x_s, \int_0^s h(s,\tau,x_\tau)d\tau) \\ + Bu_x(s)]ds + \sum_{0<t_i<t} U(t-t_i)I_i(x_{t_i}), \quad t \in J,$$

and $x(t) = \phi(t) + gx(t)$, $t \in [-a, 0]$. Define $PC_0 = \{y \in PC \mid y_0 = 0\}$. Let $Q : PC_0 \to PC_0$ be an operator defined by

(3.3)

$$Qy(t) = \begin{cases} 0, \quad t \in [-a,0], \\ \int_0^t (t-s)^{\alpha-1} V(t-s) \left[f\left(s, y_s + \hat{\phi}_s, \int_0^s h(s,\tau,y_\tau + \hat{\phi}_\tau)d\tau\right) + Bu_y(s)\right] ds \\ \quad + \sum_{0<t_i<t} U(t-t_i)I_i(y_{t_i} + \hat{\phi}_{t_i}), \quad t \in J. \end{cases}$$

Obviously, the operator F has a fixed point if and only if Q has a fixed point. So we are going to prove that Q has a fixed point.

Firstly we show that there exists a number $r \geq 1$ such that $QB_r \subseteq B_r$, where $B_r = \{y \in PC_0 : \|y\|_{PC} \leq r\}$.

If this is not true, then for each positive integer r, there exists $y^r \in B_r$ such that $\|Qy^r\| > r$ for some $t \in J$.

$$\|Qy^r(t)\| \leq \frac{M}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|f\left(s, y_s^r + \hat{\phi}_s, \int_0^s h(s,\tau,y_\tau^r + \hat{\phi}_\tau)d\tau\right) + Bu_{y^r}(s)\|ds \\ + M \sum_{0<t_i<t} \|I_i(y_{t_i}^r + \hat{\phi}_{t_i})\| \\ \leq \frac{M}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|f\left(s, y_s^r + \hat{\phi}_s, \int_0^s h(s,\tau,y_\tau^r + \hat{\phi}_\tau)d\tau\right)\|ds \\ + \frac{MM_2}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|u_{y^r}(s)\|ds + M \sum_{i=1}^m L_i(\|y_{t_i}^r + \hat{\phi}_{t_i}\|_D)\|, \tag{3.4}$$

where

(3.5)

$$\int_0^t (t-s)^{\alpha-1} \|f\left(s, y_s^r + \hat{\phi}_s, \int_0^s h(s,\tau,y_\tau^r + \hat{\phi}_\tau)d\tau\right)\|ds \leq \int_0^t (t-s)^{\alpha-1} a_{r^*}(s)ds,$$

here $r^* = (1+L_0)r'$, $r' = r + \|\hat{\phi}\|_{PC}$, and

$$\int_0^t (t-s)^{\alpha-1}\|u_{y^r}(s)\|ds \leq \sqrt{\frac{b^{2\alpha-1}}{2\alpha-1}}\|u_{y^r}\|_{L^2}, \tag{3.6}$$

here

$$\begin{aligned}
\|u_{y^r}\|_{L^2} &\leq M_3\Bigg[\|x_1\| + M(\|\phi(0)\| + \|gx(0)\|) + \frac{M}{\Gamma(\alpha)}\int_0^b (b-s)^{\alpha-1}a_{r^*}(s)ds \\
&\qquad + M\sum_{i=1}^m L_i(\|x_{t_i}\|_D)\Bigg] \\
&\leq M_3\left[\|x_1\| + M\|\hat{\phi}(0)\| + \frac{M}{\Gamma(\alpha)}\int_0^b (b-s)^{\alpha-1}a_{r^*}(s)ds + M\sum_{i=1}^m L_i(r')\right] \\
&= M^*. \qquad (3.7)
\end{aligned}$$

Thus in view of (3.4) to (3.7), we have that

$$\begin{aligned}
r &< \frac{M}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}a_{r^*}(s)ds + \frac{MM_2}{\Gamma(\alpha)}\sqrt{\frac{b^{2\alpha-1}}{2\alpha-1}}\|u_{y^r}\|_{L^2} + M\sum_{i=1}^m L_i(r') \\
&\leq \frac{M}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}a_{r^*}(s)ds + \frac{MM_2M_3}{\Gamma(\alpha)}\sqrt{\frac{b^{2\alpha-1}}{2\alpha-1}}\Bigg[\|x_1\| + M\|\hat{\phi}(0)\| \\
&\qquad + \frac{M}{\Gamma(\alpha)}\int_0^b (b-s)^{\alpha-1}a_{r^*}(s)ds + M\sum_{i=1}^m L_i(r')\Bigg] + M\sum_{i=1}^m L_i(r'). \qquad (3.8)
\end{aligned}$$

Since $r' = r + \|\hat{\phi}\|_{PC} \to \infty$ and $r^* = (1+L_0)r' \to \infty$ as $r \to \infty$ and by (H1) and (H5), we get

$$\liminf_{r\to\infty} \frac{\int_0^t (t-s)^{\alpha-1}a_{r^*}(s)ds}{r} = \liminf_{r\to\infty} \frac{\int_0^t (t-s)^{\alpha-1}a_{r^*}(s)ds}{r^*}\frac{r^*}{r} = \sigma(1+L_0), \tag{3.9}$$

$$\liminf_{r\to\infty} \frac{\sum_{i=1}^m L_i(r')}{r} = \liminf_{r\to\infty} \frac{\sum_{i=1}^m L_i(r')}{r'}\frac{r'}{r} = \sum_{i=1}^m \lambda_i. \tag{3.10}$$

Dividing both side of (3.8) by r and taking $r \to \infty$, then in view of (3.9) and (3.10), we get

$$1 < M\left(1 + \frac{MM_2M_3}{\Gamma(\alpha)}\sqrt{\frac{b^{2\alpha-1}}{2\alpha-1}}\right)\left[\frac{\sigma(1+L_0)}{\Gamma(\alpha)} + \sum_{i=1}^m \lambda_i\right].$$

This contradicts (3.1). Hence, for some $r > 0$, $QB_r \subseteq B_r$. Now we shall prove that Q is continuous on B_r. Let $\{y^{(n)}\} \subset B_r$ with $y^{(n)} \to y \in B_r$ as $n \to \infty$. Then for any $t \in J = [0, b]$, and by assumptions (H1), (H2) and (H5), we have

(I) $h(t,\tau,y_\tau^{(n)} + \hat{\phi}_\tau) \to h(t,\tau,y_\tau + \hat{\phi}_\tau)$.

(II) $f(t, y_t^{(n)} + \hat{\phi}_t, \int_0^t h(t,\tau,y_\tau^{(n)} + \hat{\phi}_\tau)d\tau) \to f(t, y_t + \hat{\phi}_t, \int_0^t h(t,\tau,y_\tau + \hat{\phi}_\tau)d\tau)$.

(III) $I_i(y_{t_i}^{(n)} + \hat{\phi}_{t_i}) \to I_i(y_{t_i} + \hat{\phi}_{t_i})$, $\quad i = 1, 2, \ldots,$ as $n \to \infty$.

(IV) $\|f(t, y_t^{(n)} + \hat{\phi}_t, \int_0^t h(t,\tau,y_\tau^{(n)} + \hat{\phi}_\tau)d\tau) - f(t, y_t + \hat{\phi}_t, \int_0^t h(t,\tau,y_\tau + \hat{\phi}_\tau)d\tau)\| \leq 2\alpha_{r^*}(t)$.

These together with Lebesgue dominated convergence theorem, we have

$$\begin{aligned}\|Qy^{(n)}(t)-Qy(t)\| \leq & \frac{M}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|f\left(s, y_s^{(n)}+\hat{\phi}_s, \int_0^s h(s,\tau,y_\tau^{(n)}+\hat{\phi}_\tau)d\tau\right)\\ & - f\left(s, y_s+\hat{\phi}_s, \int_0^s h(s,\tau,y_\tau+\hat{\phi}_\tau)d\tau\right)\|ds\\ & + \frac{M}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|Bu_{y^{(n)}}(s)-Bu_y(s)\|ds\\ & + M\sum_{i=1}^m \|I_i(y_{t_i}^{(n)}+\hat{\phi}_{t_i})-I_i(y_{t_i}+\hat{\phi}_{t_i})\|\\ \leq & \frac{M}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|f\left(s, y_s^{(n)}+\hat{\phi}_s, \int_0^s h(s,\tau,y_\tau^{(n)}+\hat{\phi}_\tau)d\tau\right)\\ & - f\left(s, y_s+\hat{\phi}_s, \int_0^s h(s,\tau,y_\tau+\hat{\phi}_\tau)d\tau\right)\|ds\\ & + \frac{MM_2}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|u_{y^{(n)}}-u_y\|_{L^2}ds\\ & + M\sum_{i=1}^m \|I_i(y_{t_i}^{(n)}+\hat{\phi}_{t_i})-I_i(y_{t_i}+\hat{\phi}_{t_i})\|,\end{aligned} \tag{3.11}$$

where

$$\begin{aligned}\|u_{y^{(n)}}-u_y\|_{L^2} \leq M_3 & \left[\frac{M}{\Gamma(\alpha)}\int_0^b (b-s)^{\alpha-1}\|f\left(s, y_s^{(n)}+\hat{\phi}_s, \int_0^s h(s,\tau,y_\tau^{(n)}+\hat{\phi}_\tau)d\tau\right)\right.\\ & - f\left(s, y_s+\hat{\phi}_s, \int_0^s h(s,\tau,y_\tau+\hat{\phi}_\tau)d\tau\right)\|ds\\ & \left. +M\sum_{i=1}^m \|I_i(y_{t_i}^{(n)}+\hat{\phi}_{t_i})-I_i(y_{t_i}+\hat{\phi}_{t_i})\|\right] \to 0 \text{ as } n\to\infty.\end{aligned}$$

So $\|Qy^{(n)}(t)-Qy(t)\| \to 0$ as $n\to\infty$. Hence $\|Qy^{(n)}-Qy\|_{PC} \to 0$ as $n\to\infty$. This mean that Q is continuous on B_r.

Next we shall prove that $Q(B_r)$ is equicontinuous on every J_i, $i=1,2,\dots,m$, i.e. $Q(B_r)$ is piecewise equicontinuous on J. For $t_1,t_2 \in J_i$ with $t_1<t_2$ and $y\in B_r$ and in view of (3.7), we have that

$$\begin{aligned}\|Qy(t_2)-Qy(t_1)\| \leq & \|\int_0^{t_1}(t_2-s)^{\alpha-1}[V(t_1-s)-V(t_2-s)]\\ & \times\left[f\left(s, y_s+\hat{\phi}_s, \int_0^s h(s,\tau,y_\tau+\hat{\phi}_\tau)d\tau\right)+Bu_y(s)\right]ds\|\\ & + \|\int_0^{t_1}\left[(t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}\right]V(t_1-s)\\ & \times\left[f\left(s, y_s+\hat{\phi}_s, \int_0^s h(s,\tau,y_\tau+\hat{\phi}_\tau)d\tau\right)+Bu_y(s)\right]ds\|\end{aligned}$$

$$
\begin{aligned}
&+\Big\|\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}V(t_2-s)\\
&\times\left[f\left(s,y_s+\hat{\phi}_s,\int_0^s h(s,\tau,y_\tau+\hat{\phi}_\tau)d\tau\right)+Bu_y(s)\right]ds\Big\|\\
&+\Big\|\sum_{i=1}^m[U(t_1-t_i)-U(t_2-t_i)]I_i(y_{t_i}+\hat{\phi}_{t_i})\Big\|\\
&\le\int_0^{t_1}(t_2-s)^{\alpha-1}\|V(t_1-s)-V(t_2-s)\|[\alpha_{r^*}(s)+M^*M_2]ds\\
&+\frac{M}{\Gamma(\alpha)}\int_0^{t_1}|(t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}|[\alpha_{r^*}(s)+M^*M_2]ds\\
&+\frac{M}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}[\alpha_{r^*}(s)+M^*M_2]ds\\
&+\sum_{i=1}^m\|U(t_1-t_i)-U(t_2-t_i)\|L_i(r)\\
&=I_1+I_2+I_3+I_4, \qquad (3.12)
\end{aligned}
$$

where

$$
\begin{aligned}
I_1&=\int_0^{t_1}(t_2-s)^{\alpha-1}\|V(t_1-s)-V(t_2-s)\|[\alpha_{r^*}(s)+M^*M_2]ds,\\
I_2&=\frac{M}{\Gamma(\alpha)}\int_0^{t_1}|(t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}|[\alpha_{r^*}(s)+M^*M_2]ds,\\
I_3&=\frac{M}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}[\alpha_{r^*}(s)+M^*M_2]ds,\\
I_4&=\sum_{i=1}^m\|U(t_1-t_i)-U(t_2-t_i)\|L_i(r).
\end{aligned}
$$

For any $\epsilon\in(0,t_1)$, we have

$$
\begin{aligned}
I_1&\le\int_0^{t_1-\epsilon}(t_2-s)^{\alpha-1}\|V(t_1-s)-V(t_2-s)\|[\alpha_{r^*}(s)+M^*M_2]ds\\
&\quad+\int_{t_1-\epsilon}^{t_1}(t_2-s)^{\alpha-1}\|V(t_1-s)-V(t_2-s)\|[\alpha_{r^*}(s)+M^*M_2]ds\\
&\le\int_0^{t_1-\epsilon}(t_2-s)^{\alpha-1}[\alpha_{r^*}(s)+M^*M_2]ds.\sup_{s\in[0,t_1-\epsilon]}\|V(t_1-s)-V(t_2-s)\|\\
&\quad+\frac{2M}{\Gamma(\alpha)}\int_{t_1-\epsilon}^{t_1}(t_2-s)^{\alpha-1}[\alpha_{r^*}(s)+M^*M_2]ds. \qquad (3.13)
\end{aligned}
$$

By Lemma 2.4 we can see that $I_1\to 0$ as $t_2\to t_1$ and $\epsilon\to 0$ independent of $y\in B_r$. From expression of I_2, I_3 and I_4, we can easily see that $I_2\to 0$, $I_3\to 0$ and $I_4\to 0$ $t_2\to t_1$ independent of $y\in B_r$. Thus the $Q(B_r)$ is equicontinuous on J_i $(i=1,2,\dots,m)$.

Let $S \subset B_r$ be any subset. By Lemma 2.12, there exists a countable set $S_1 = \{y^{(n)}\} \subset S$ such that

$$\mu(Q(S)) \leq 2\mu(Q(S_1)). \tag{3.14}$$

Since $Q(S_1) \subset Q(B_r)$ is equicontinuous, then, in view of Lemma 2.10, we have

$$\mu(Q(S_1)) = \sup_{t\in J} \mu(Q(S_1)(t)). \tag{3.15}$$

At last, we are going to prove that Q is condensing mapping from $B_r \to B_r$. Using Lemma 2.11 and (H1)(iii), (H2)(iii), (H4) and (H5) , we have that

$$\begin{aligned}
\mu(Q(S_1)(t)) &\leq \mu\left(\left\{\int_0^t (t-s)^{\alpha-1} V(t-s) f\left(s, y_s^{(n)} + \hat{\phi}_s, \int_0^s h(s,\tau, y_\tau^{(n)} + \hat{\phi}_\tau) d\tau\right) ds\right\}\right) \\
&\quad + \mu\left(\left\{\int_0^t (t-s)^{\alpha-1} V(t-s) B u_{y^{(n)}}(s) ds\right\}\right) \\
&\quad + \mu\left(\left\{\sum_{0<t_i<t} U(t-t_i) I_i(y_{t_i}^{(n)} + \hat{\phi}_{t_i})\right\}\right) \\
&= \chi_1 + \chi_2 + \chi_3, \qquad (3.16)
\end{aligned}$$

where

$$\begin{aligned}
\chi_1 &= \mu\left(\left\{\int_0^t (t-s)^{\alpha-1} V(t-s) f\left(s, y_s^{(n)} + \hat{\phi}_s, \int_0^s h(s,\tau, y_\tau^{(n)} + \hat{\phi}_\tau) d\tau\right) ds\right\}\right), \\
\chi_2 &= \mu\left(\left\{\int_0^t (t-s)^{\alpha-1} V(t-s) B u_{y^{(n)}}(s) ds\right\}\right), \\
\chi_3 &= \mu\left(\left\{\sum_{0<t_i<t} U(t-t_i) I_i(y_{t_i}^{(n)} + \hat{\phi}_{t_i})\right\}\right).
\end{aligned}$$

$$\begin{aligned}
\chi_1 &\leq \frac{2M}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \mu\left(\left\{f\left(s, y_s^{(n)} + \hat{\phi}_s, \int_0^s h(s,\tau, y_\tau^{(n)} + \hat{\phi}_\tau) d\tau\right)\right\}\right) ds \\
&\leq \frac{2M}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \eta(s) \left[\sup_{-a\leq\theta\leq 0} \mu\left(\left\{y^{(n)}(s+\theta) + \hat{\phi}(s+\theta)\right\}\right)\right. \\
&\quad \left. + \mu\left(\left\{\int_0^s h(s,\tau, y_\tau^{(n)} + \hat{\phi}_\tau) d\tau\right\}\right)\right] ds \\
&\leq \frac{2M}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \eta(s) \sup_{0\leq\tau\leq s} \mu\left(\left\{y^{(n)}(\tau)\right\}\right) ds \\
&\quad + \frac{4M}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \eta(s) \left(\int_0^s \zeta(s,\tau) \sup_{0\leq\nu\leq\tau} \mu\left(\left\{y^{(n)}(\nu)\right\}\right) d\tau\right) ds \\
&\leq \frac{2M}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \eta(s) \sup_{0\leq\tau\leq s} \mu\left(\left\{y^{(n)}(\tau)\right\}\right) ds \\
&\quad + \frac{4M}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \eta(s) \zeta^* \sup_{0\leq\nu\leq s} \mu\left(\left\{y^{(n)}(\nu)\right\}\right) ds
\end{aligned}$$

$$\leq \frac{2M}{\Gamma(\alpha)}(2\zeta^*+1)\int_0^t (t-s)^{\alpha-1}\eta(s)\sup_{0\leq\nu\leq s}\mu\left(\left\{y^{(n)}(\nu)\right\}\right)ds$$
$$\leq \frac{2M}{\Gamma(\alpha)}(2\zeta^*+1)\|\eta\|_{L^2}\sqrt{\frac{t^{2\alpha-1}}{2\alpha-1}}\mu\left(\left\{y^{(n)}\right\}\right)$$
$$\leq \frac{2M}{\Gamma(\alpha)}(2\zeta^*+1)\|\eta\|_{L^2}\sqrt{\frac{b^{2\alpha-1}}{2\alpha-1}}\mu\left(\left\{y^{(n)}\right\}\right), \tag{3.17}$$

$$\chi_3 \leq M\sum_{i=1}^m k_i \sup_{-a\leq\theta\leq 0}\mu\left(\left\{y^{(n)}(t_i+\theta)+\hat{\phi}(t_i+\theta)\right\}\right)$$
$$\leq M\sum_{i=1}^m k_i \sup_{0<\tau_i<t_i}\mu\left(\left\{y^{(n)}(\tau_i)\right\}\right), \tag{3.18}$$

$$\mu\left(\left\{u_{y^{(n)}}(s)\right\}\right) \leq K_w(s)\left[\mu\left(\left\{\int_0^b (b-s)^{\alpha-1}V(b-s)\right.\right.\right.$$
$$\left.\left.\times\ f\left(s, y_s^{(n)}+\hat{\phi}_s, \int_0^s h(s,\tau,y_\tau^{(n)}+\hat{\phi}_\tau)d\tau\right)ds\right\}\right)$$
$$\left.+\mu\left(\left\{\sum_{0<t_i<t} U(t-t_i)I_i(y_{t_i}^{(n)}+\hat{\phi}_{t_i})\right\}\right)\right]$$
$$\leq K_w(s)\left[\frac{2M}{\Gamma(\alpha)}(2\zeta^*+1)\|\eta\|_{L^2}\sqrt{\frac{b^{2\alpha-1}}{2\alpha-1}}\mu\left(\left\{y^{(n)}\right\}\right)\right.$$
$$\left.+M\sum_{i=1}^m k_i \sup_{0<\tau_i<t_i}\mu\left(\left\{y^{(n)}(\tau_i)\right\}\right)\right], \tag{3.19}$$

and

$$\chi_2 \leq \frac{2MM_2}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\mu\left(\left\{u_{y^{(n)}}(s)\right\}\right)ds$$
$$\leq \left[\frac{4M^2M_2}{\Gamma(\alpha)^2}(2\zeta^*+1)\|\eta\|_{L^2}\sqrt{\frac{b^{2\alpha-1}}{2\alpha-1}}\mu\left(\left\{y^{(n)}\right\}\right)\right.$$
$$\left.+\frac{2M^2M_2}{\Gamma(\alpha)}\left(\sum_{i=1}^m k_i \sup_{0<\tau_i<t_i}\mu\left(\left\{y^{(n)}(\tau_i)\right\}\right)\right)\right]$$
$$\times\left(\int_0^t (t-s)^{\alpha-1}K_w(s)ds\right). \tag{3.20}$$

From (3.16), (3.17), (3.18), (3.19) and (3.20), we get

$$\mu(Q(S_1)(t)) \leq \frac{2M}{\Gamma(\alpha)}(2\zeta^*+1)\|\eta\|_{L^2}\sqrt{\frac{b^{2\alpha-1}}{2\alpha-1}}\mu\left(\left\{y^{(n)}\right\}\right)$$
$$+\left[\frac{4M^2M_2}{\Gamma(\alpha)^2}(2\zeta^*+1)\|\eta\|_{L^2}\sqrt{\frac{b^{2\alpha-1}}{2\alpha-1}}\mu\left(\left\{y^{(n)}\right\}\right)\right.$$

$$
\begin{aligned}
&+ \frac{2M^2M_2}{\Gamma(\alpha)}\left(\sum_{i=1}^{m} k_i \sup_{0<\tau_i<t_i} \mu\left(\{y^{(n)}(\tau_i)\}\right)\right)\Bigg]\left(\int_0^t (t-s)^{\alpha-1}K_w(s)ds\right)\\
&+ M\sum_{i=1}^{m} k_i \sup_{0<\tau_i<t_i} \mu\left(\{y^{(n)}(\tau_i)\}\right)\\
&\leq (2\zeta^*+1)\left[\frac{2M}{\Gamma(\alpha)} + \frac{4M^2M_2}{\Gamma(\alpha)^2}\|K_w\|_{L^2}\sqrt{\frac{b^{2\alpha-1}}{2\alpha-1}}\right]\|\eta\|_{L^2}\sqrt{\frac{b^{2\alpha-1}}{2\alpha-1}}\mu\left(\{y^{(n)}\}\right)\\
&+ \left[M + \frac{2M^2M_2}{\Gamma(\alpha)^2}\|K_w\|_{L^2}\sqrt{\frac{b^{2\alpha-1}}{2\alpha-1}}\right]\left(\sum_{i=1}^{m} k_i\right)\mu\left(\{y^{(n)}\}\right)\\
&\leq \Bigg[(2\zeta^*+1)\left(\frac{2M}{\Gamma(\alpha)} + \frac{4M^2M_2}{\Gamma(\alpha)^2}\|K_w\|_{L^2}\sqrt{\frac{b^{2\alpha-1}}{2\alpha-1}}\right)\|\eta\|_{L^2}\sqrt{\frac{b^{2\alpha-1}}{2\alpha-1}}\\
&+ \left(M + \frac{2M^2M_2}{\Gamma(\alpha)^2}\|K_w\|_{L^2}\sqrt{\frac{b^{2\alpha-1}}{2\alpha-1}}\right)\left(\sum_{i=1}^{m} k_i\right)\Bigg]\mu(S). \qquad (3.21)
\end{aligned}
$$

This implies $\mu(Q(S_1)) \leq N\mu(S)$ and hence

$$\mu(Q(S)) \leq 2N\mu(S).$$

Thus, In view of (H6), Q is a condensing mapping from B_r to B_r. Hence, using Sadovskii's fixed point theorem (Lemma 2.14), Q has a fixed point on B_r. we can easily see that $x = y + \hat{\phi}$ is a fixed point of F in PC. Hence x becomes a mild solution of the system (1.1) satisfying $x(b) = x_1$, i.e. system (1.1) is nonlocally controllable. The proof is completed. □

4. **An example**

Let $X = L^2([0,\pi],\mathbb{R})$. Consider the following impulsive fractional partial differential equations with finite delay and nonlocal conditions.

$$
\begin{cases}
{}^cD_t^{\alpha} z(t,y) = \frac{\partial}{\partial y} z(t,y) + \rho\omega(t,y) + F\left(t, z_t(\nu,y), \int_0^t h(t,s,z_s(\nu,y))ds\right),\\
z(t,0) = z(t,\pi) = 0, \quad t\in[0,b],\\
\Delta z(t,y)|_{t=t_i} = I_i(z(t_i,y)), \quad i = 1,2,\ldots,m,\\
z(\nu,y) = \phi(\nu,y) + \int_0^b K(s,y)\sin(z(s,y))ds, \quad \nu\in[-a,0],
\end{cases} \qquad (4.1)
$$

where ${}^cD^{\alpha}$ is a Caputo fractional partial derivative of order α, $1/2 < \alpha < 1$, $y \in [0,\pi]$, $\phi \in D = \{\psi : [-a,b]\times[0,\pi] \to \mathbb{R}, \psi(\cdot,y)$ is continuous everywhere except for a countable number of points at which $\psi(t^-,y)$, $\psi(t^+,y)$ exist with $\psi(t^-,y) = \psi(t,y)\}$, $0 = t_0 < t_1 < \ldots < t_{m+1} = b$, $\Delta z(t,y)|_{t=t_i} = z(t_i^+,y) - z(t_i^-,y)$, $z(t_i^+,y) = \lim_{h\to 0^+} z(t_i+h,y)$, $z(t_i^-,y) = \lim_{h\to 0^-} z(t_i+h,y)$ represent the right and left limits of $z(t,y)$ at $t = t_i$ respectively, for $i = 1,2,\ldots,m$ and $z_t(\nu,y) = z(t+\nu,y)$, $t\in[0,b]$, $\nu\in[-a,0]$. The operator $K(s,y)$ is continuous on compact square $[0,b]\times[0,\pi]$.

We define an operator $A : X \to X$ by $Av = v'$ with domain

$$D(A) = \{v \in X : v \text{ is absolutely continuous } v' \in X, v(0) = v(\pi) = 0\}.$$

It is well known that A is infinitesimal generator of a semigroup $\{T(t), t \geq 0\}$ in X and is given by $T(t)v(t) = v(t+s)$, for $v \in X$, $T(t)$ is not a compact semigroup on X. Now we define $x(t)(y) = z(t, y)$, $f(t, x_t, \int_0^t h(t, s, x_s)ds)(y) = F\left(t, z_t(\nu, y), \int_0^t h(t, s, z_s(\nu, y))ds\right)$, $I(x(t_i))(y) = I(z(t_i, y))$ and $x(\nu)(y) = \phi(\nu)(y) + g(x)(y) = \phi(\nu, y) + \int_0^b K(s, y)\sin(z(s, y))ds$, $\quad \nu \in [-a, 0]$ and $g(x)(y) = \int_0^b K(s, y) \times \sin(z(s, y))ds$. The bounded linear control operator $B : X \to X$ is defined by $(Bu)(t)(y) = \rho\omega(t, y)$ for a.e. $y \in [0, \pi]$. Therefore, the above impulsive fractional differential equation (4.1) can be written as the abstract form (1.1).

We can take $f\left(t, x_t, \int_0^t h(t, s, x_s)ds\right) = Ct^{1/3}x_t$. Then f is Lipschitz continuous for the second variable and we can find $\phi_r(t) = Crt^{1/3} \in L^1([0, t], \mathbb{R})$ for each $r > 0$ such that assumption (H1)(ii) is satisfied. Also f satisfies the assumptions H(1) and H(2).

The function $I_i : X \to X$ can be taken as

$$I_i(x)(y) = \int_0^\pi \gamma_i(s, y)\frac{1}{1 + (x(s))^2}ds,$$

where $\gamma_i \in C([0, \pi] \times [0, \pi], \mathbb{R})$, for each $i = 1, 2, \ldots, m$. Then I_i is compact, hence satisfies the assumption (H5).

Since the function $g : PC([0, b], X) \to X$ given by

$$g(x)(y) = \int_0^b K(s, y)\sin(x(s)(y))ds, \text{ where } x(s)(y) = z(s, y),$$

is a continuous and compact. Also g satisfies the assumption (H3). The linear operator $W : L^2(J, U) \to X$ is given by

$$(Wu)(y) = \int_0^b (b - s)^{\alpha - 1}V(t - s)\rho\omega(s, y)ds.$$

Assuming that W defined by above satisfies the assumption (H4). Then the abstract form of problem 4.1 satisfies all the conditions of the Theorem 3.1. Thus the system 4.1 is nonlocally controllable on the interval J.

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