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SPLINE IN COMPRESSION METHOD FOR NON-LINEAR TWO POINT BOUNDARY VALUE PROBLEMS ON A GEOMETRIC MESH

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Abstract: In this paper, we propose a new method of order four, based on spline in compression approximation for the numerical solution of non-linear two point boundary value problems on a uniform mesh. The derivation and the convergence of the proposed method are discussed in detail. The method is extended to non-uniform mesh. Numerical results are given to illustrate the usefulness of the proposed method.

Keywords: Spline in compression; Non polynomial spline; Convergence analysis; Root mean square errors; Variable mesh; Burgers' equation.

AMS (MOS) Subject Classification: 65L10; 65L12

1. INTRODUCTION

Consider the non-linear two point boundary value problem

$$-u'' + f(x,u,u') = 0, \qquad 0 < x < 1 \tag{1}$$

subject to the boundary conditions

$$u(0) = A, \quad u(1) = B \tag{2}$$

where A and B are real constants. We assume that for $0 < x < 1, -\infty < u, u' < \infty$.

(i) $f(x,u,u')$ is continuous,

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(ii) $\dfrac{\partial f}{\partial u}$ and $\dfrac{\partial f}{\partial u'}$ exist and continuous,

(iii) $\dfrac{\partial f}{\partial u} > 0$ and $\left|\dfrac{\partial f}{\partial u'}\right| \le b$ for some positive constant *b*.

These conditions assure us that the above boundary value problem has a unique solution (Keller, 1992).

The possibilities of using spline functions for obtaining smooth approximations of the solution of boundary value problems were first briefly discussed by (Ahlberg *et al*, 1967). Following this, (Albasiny *et al*, 1969), (Fyfe, 1969) and (Al-said, 1999; 2001), (Chawla and Subramanian, 1987; 1988), (Kadalbajoo *et al*, 1993), (Khan, 2004), (Kumar, 2007) have used cubic splines for solving two-point boundary value problems. (Jain *et al*, 1981; 1983), (Kadalbajoo *et al*, 2001; 2002), have used lower order spline in compression approximation to obtain the numerical solution of singularly perturbed two point boundary value problems on uniform and non-uniform mesh.

In this paper we follow the approach given by (Jain *et al*, 1981; 1983; 1984) and use the non-polynomial spline in compression approximation to develop a fourth order method for the solution of nonlinear two point boundary value problems. The method involves three evaluations of the function *f*. The resulting spline difference scheme is fourth order accurate. We use the continuity of the first derivative of the spline function and on applying the numerical method we obtain a tri-diagonal system of equations which we solve using the CRAGE method (Evans, 1985; Mohanty *et al*, 2012). The convergence analysis of the method is discussed in details. We have extended the method to non-uniform mesh. In section 2 we discuss the spline in compression approximation. In section 3, we discuss a new method based on spline in compression approximation on uniform mesh. In section 4, we discuss the derivation of the proposed method. In section 5, we discuss the convergence analysis of the proposed method. In section 6, we have extended our method to non-uniform mesh. In section 7, we discuss the numerical illustrations and compare the numerical results obtained from the proposed methods with the corresponding existing methods. Concluding remarks are given in section 8.

2. SPLINE IN COMPRESSION APPROXIMATION

Consider the uniform mesh on [0, 1] such that $0 = x_0 < x_1 < \ldots < x_N < x_{N+1} = 1$.

Let $h = x_l - x_{l-1} > 0, \quad l = 1(1)N+1$ be the mesh size in the x direction so that $h = 1/(N+1)$. Therefore the grid points in the x-direction are given by $x_l = x_0 + lh, \quad l = 1(1)N+1$. Let u_l and U_l be the discrete approximation and the exact value of the solution $u(x)$ at the grid point x_l, respectively.

Let $S(x)$ be the non-polynomial spline function which interpolates $u(x)$ at $x_l, l = 0(1)N+1$, such that $S(x) \in C^2[0,1]$ given by

$$S(x) = a_l + b_l(x - x_l) + c_l \sin w(x - x_l) + d_l \cos w(x - x_l), \quad l = 0(1)N+1 \quad (3)$$

where a_l, b_l, c_l and d_l are constants and w is an arbitrary parameter.

Differentiating (3) we get:

$$S'(x) = b_l + c_l w \cos w(x - x_l) - d_l w \sin w(x - x_l), \quad l = 0(1)N+1 \quad (4)$$

$$S''(x) = -c_l w^2 \sin w(x - x_l) - d_l w^2 \cos w(x - x_l), \quad l = 0(1)N+1 \quad (5)$$

Substituting $S(x_l) = u_l, S(x_{l+1}) = u_{l+1}, S''(x_l) = M_l, S''(x_{l+1}) = M_{l+1}$ in (3), we get

$$a_l = u_l + \frac{M_l}{w^2}, b_l = \frac{u_{l+1} - u_l}{h} + \frac{M_{l+1} - M_l}{w^2 h}, c_l = \frac{(M_l \cos wh - M_{l+1})}{w^2 \sin wh}, d_l = \frac{-M_l}{w^2}.$$

Using the continuity condition of the first derivative, that is, $S'(x_l -) = S'(x_l +)$, we obtain

$$\frac{u_{l+1} - 2u_l + u_{l-1}}{h^2} = \frac{M_l - M_{l-1}}{w^2 h^2} + \frac{M_l - M_{l+1}}{w^2 h^2} + \frac{(M_{l+1} - M_l \cos wh)}{wh \sin wh} + \frac{(M_{l-1} \cos wh - M_l)}{wh \sin wh} + \frac{M_{l-1}}{wh} \sin wh$$

$$= \alpha M_{l+1} + 2\beta M_l + \alpha M_{l-1}, l = 1, 2, \ldots N \quad (6)$$

where

$$\alpha = \frac{1}{w^2 h^2}\left[\frac{wh}{\sin wh} - 1\right], \quad \beta = \frac{1}{w^2 h^2}\left[1 - wh \cot wh\right].$$

Further, we have

$$S'(x_l) = u'_l = \frac{u_{l+1} - u_l}{h} - h(\alpha M_{l+1} + \beta M_l), \quad x_l \le x \le x_{l+1} \quad (7)$$

Also, $S'(x_l) = u'_l = \dfrac{u_l - u_{l-1}}{h} + h(\alpha M_{l-1} + \beta M_l), \quad x_{l-1} \le x \le x_l \quad (8)$

Combining (7) and (8), we get

$$S'(x_l) = u'_l = \frac{u_{l+1} - u_{l-1}}{2h} - \frac{\alpha h}{2}(M_{l+1} - M_{l-1}) \quad (9)$$

Also, from (4) we have:

$$S'(x_{l+1}) = u'_{l+1} = \frac{u_{l+1} - u_l}{h} + h(\alpha M_l + \beta M_{l+1}), \quad x_l \le x \le x_{l+1} \tag{10}$$

$$S'(x_{l-1}) = u'_{l-1} = \frac{u_l - u_{l-1}}{h} - h(\alpha M_l + \beta M_{l-1}), \quad x_{l-1} \le x \le x_l \tag{11}$$

Note that, (6)-(11) are important properties of spline in compression $S(x)$.

3. THE METHOD BASED ON SPLINE IN COMPRESSION APPROXIMATION

For the method based on spline in compression approximation, we follow the technique used by (Jain *et al* 1981; 1983).

We consider the following approximations:

$$\overline{m}_l = \overline{u'}_l = \frac{(u_{l+1} - u_{l-1})}{2h}, \tag{12.1}$$

$$\overline{m}_{l\pm1} = \frac{(\pm 3u_{l\pm1} \mp 4u_l \pm u_{l\mp1})}{2h}, \tag{12.2}$$

$$\overline{f}_l = f(x_l, u_l, \overline{m}_l), \tag{12.3}$$

$$\overline{f}_{l\pm1} = f(x_{l\pm1}, u_{l\pm1}, \overline{m}_{l\pm1}), \tag{12.4}$$

$$\overline{\overline{u'}}_{l+1} = \frac{u_{l+1} - u_l}{h} + h\left(\beta \overline{f}_{l+1} + \alpha \overline{f}_l\right), \tag{12.5}$$

$$\overline{\overline{u'}}_{l-1} = \frac{u_l - u_{l-1}}{h} - h\left(\beta \overline{f}_{l-1} + \alpha \overline{f}_l\right), \tag{12.6}$$

$$\overline{\overline{u'}}_l = \frac{(u_{l+1} - u_{l-1})}{2h} - \frac{\alpha h}{2}(\overline{f}_{l+1} - \overline{f}_{l-1}), \tag{12.7}$$

$$\overline{\overline{f}}_{l\pm1} = f(x_{l\pm1}, u_{l\pm1}, \overline{\overline{u'}}_{l\pm1}), \tag{12.8}$$

$$\overline{\overline{f}}_l = f(x_l, u_l, \overline{\overline{u'}}_l), \tag{12.9}$$

Then the non-polynomial spline method with order of accuracy four for the differential equation (1) may be written as:

$$(u_{l+1} - 2u_l + u_{l-1}) = \frac{h^2}{12}\left[\overline{\overline{f}}_{l+1} + \overline{\overline{f}}_{l-1} + 10\overline{\overline{f}}_l\right], \qquad l = 1(1)N \tag{13}$$

The local truncation error associated with (13) is given by

$$(U_{l+1} - 2U_l + U_{l-1}) = \frac{h^2}{12}\left[\overline{\overline{F}}_{l+1} + \overline{\overline{F}}_{l-1} + 10\overline{\overline{F}}_l\right] + \overline{\overline{T}}_l, \qquad l = 1(1)N \tag{14}$$

where $\overline{\overline{T}}_l = O(h^6)$ and

$$\overline{U'}_l = \frac{(U_{l+1} - U_{l-1})}{2h}, \tag{15.1}$$

$$\overline{U'}_{l\pm1} = \frac{(\pm 3U_{l\pm1} \mp 4U_l \pm U_{l\mp1})}{2h}, \tag{15.2}$$

$$\overline{F}_l = f(x_l, U_l, \overline{U'}_l), \tag{15.3}$$

$$\overline{F}_{l\pm1} = f(x_{l\pm1}, U_{l\pm1}, \overline{U'}_{l\pm1}), \tag{15.4}$$

$$\overline{\overline{U'}}_{l+1} = \frac{U_{l+1} - U_l}{h} + h\left(\beta \overline{F}_{l+1} + \alpha \overline{F}_l\right), \tag{15.5}$$

$$\overline{\overline{U'}}_{l-1} = \frac{U_l - U_{l-1}}{h} - h\left(\beta \overline{F}_{l-1} + \alpha \overline{F}_l\right), \tag{15.6}$$

$$\overline{\overline{U'}}_l = \frac{(U_{l+1} - U_{l-1})}{2h} - \frac{\alpha h}{2}(\overline{F}_{l+1} - \overline{F}_{l-1}), \tag{15.7}$$

$$\overline{\overline{F}}_{l\pm1} = f(x_{l\pm1}, U_{l\pm1}, \overline{\overline{U'}}_{l\pm1}), \tag{15.8}$$

$$\overline{\overline{F}}_l = f(x_l, U_l, \overline{\overline{U'}}_l). \tag{15.9}$$

Note that the boundary conditions are given by $u_0 = A$ and $u_{N+1} = B$. Applying the method (13) to non-linear differential equations and using the boundary conditions, we obtain tri-diagonal system of non-linear difference equations. The resulting tri-diagonal system of non-linear difference equations can be solved using CRAGE iterative method (See Mohanty and Evans (2005); Evans (1999); Mohanty and Talwar (2012)).

4. DERIVATION OF THE METHOD AND LOCAL TRUNCATION ERROR

For the derivation of the finite difference method based on spline in compression approximation for the numerical solution of the differential equation (1), we follow the ideas of (Jain *et al*, 1981; 1983).

At the grid point x_l, we denote: $U_l'' = f(x_l, U_l, U_l') \equiv F_l$ (say)

By Taylor series expansion, we obtain:

$$(U_{l+1} - 2U_l + U_{l-1}) = \frac{h^2}{12}\left[F_{l+1} + F_{l-1} + 10F_l\right] + O(h^6), \qquad l = 1(1)N \tag{16}$$

From equations (15.1)-(15.2) we obtain:

$$\overline{U'}_l = \frac{(U_{l+1} - U_{l-1})}{2h} = U_l' + \frac{h^2}{6}U_l''' + O(h^4) \tag{17}$$

$$\overline{U'}_{l+1} = \frac{(3U_{l+1} - 4U_l + U_{l-1})}{2h} = U_{l+1}' - \frac{h^2}{3}U_l''' + O(h^3) \tag{18}$$

$$\overline{U'}_{l-1} = \frac{\left(-3U_{l-1} + 4U_l - U_{l+1}\right)}{2h} = U'_{l-1} - \frac{h^2}{3} U'''_l - O(h^3) \tag{19}$$

At the grid point x_l, let us denote: $\psi_l = \dfrac{\partial f}{\partial U'_l}$

Using the approximation (17), from (15.3), we obtain

$$\overline{F}_l = f(x_l, U_l, \overline{U'_l}) = F_l + \frac{h^2}{6} U'''_l \psi_l + O(h^3) \tag{20}$$

Similarly,

$$\overline{F}_{l+1} = f(x_{l+1}, U_{l+1}, \overline{U'}_{l+1}) = F_{l+1} - \frac{h^2}{3} U'''_l \Psi_{l+1} + O(h^3) \tag{21}$$

Now, using Taylor Series expansion

$$\psi_{l+1} = \psi_l + h\psi_l{}' + O(h^2) \tag{22}$$

Using (22), from (21), we obtain

$$\overline{F}_{l+1} = F_{l+1} - \frac{h^2}{3} U'''_l \Psi_l + O(h^3) \tag{23}$$

Similarly,

$$\overline{F}_{l-1} = F_{l-1} - \frac{h^2}{3} U'''_l \Psi_l - O(h^3) \tag{24}$$

Using the approximations (20), (23) from (15.5), we obtain

$$\overline{\overline{U'}}_{l+1} = U'_{l+1} + h\left(\alpha + \beta - \frac{1}{2}\right) U''_l + h^2 \left(\beta - \frac{1}{3}\right) U'''_l + O(h^3) \tag{25}$$

Similarly,

$$\overline{\overline{U'}}_{l-1} = U'_{l-1} - h\left(\alpha + \beta - \frac{1}{2}\right) U''_l + h^2 \left(\beta - \frac{1}{3}\right) U'''_l - O(h^3) \tag{26}$$

$$\overline{\overline{U'}}_l = U'_l + \left(\frac{1}{6} - \alpha\right) h^2 U'''_l + O(h^4) \tag{27}$$

From (15.8), using the approximation (25), we obtain

$$\begin{aligned} \overline{\overline{F}}_{l+1} &= f(x_{l+1}, U_{l+1}, \overline{\overline{U'}}_{l+1}) \\ &= F_{l+1} + h\left(\alpha + \beta - \frac{1}{2}\right) U''_l \Psi_l + h^2 \left(\beta - \frac{1}{3}\right) U'''_l \Psi_l + h^2 \left(\alpha + \beta - \frac{1}{2}\right) U''_l \Psi'_l + O(h^3) \end{aligned} \tag{28}$$

Similarly,

$$\bar{\bar{F}}_{l-1} = F_{l-1} - h\left(\alpha+\beta-\frac{1}{2}\right)U_l''\psi_l + h^2\left(\beta-\frac{1}{3}\right)U_l'''\psi_l + h^2\left(\alpha+\beta-\frac{1}{2}\right)U_l''\psi_l' - O(h^3), \quad (29)$$

$$\bar{\bar{F}}_l = F_l + \left(\frac{1}{6}-\alpha\right)h^2 U_l'''\psi_l + O(h^4). \quad (30)$$

Now, using the approximations (28)-(30)from (14), we obtain

$$(U_{l+1}-2U_l+U_{l-1}) = \frac{h^2}{12}\left[F_{l+1}+F_{l-1}+10F_l+2h^2(\beta-5\alpha+\frac{1}{2})U_l'''\psi_l+2h^2(\alpha+\beta-\frac{1}{2})U_l''\psi_l'\right]+\bar{\bar{T}}_l \quad (31)$$

By the help of (16), from (31), we get

$$\bar{\bar{T}}_l = \frac{-h^4}{6}\left[(\beta-5\alpha+\frac{1}{2})U_l'''\psi_l+(\alpha+\beta-\frac{1}{2})U_l''\psi_l'\right]+O(h^6) \quad (32)$$

In order for the proposed method to be of $O(h^4)$, the coefficient of h^4 in (32) must be zero and we obtain $\alpha=\frac{1}{6}, \beta=\frac{1}{3}$, and the local truncation error reduces to $\bar{\bar{T}}_l = O(h^6)$.

5. CONVERGENCE ANALYSIS

We now discuss the convergence of the method (13).

Let $\boldsymbol{U}=[U_1,U_2,...,U_N]^t$ be the exact solution vector and $\boldsymbol{u}=[u_1,u_2,...,u_N]^t$ be the approximate solution to $\boldsymbol{U}$. Let $m_l(U_{l-1},U_l,U_{l+1}) = \frac{h^2}{12}\left[\bar{\bar{F}}_{l+1}+\bar{\bar{F}}_{l-1}+10\bar{\bar{F}}_l\right], l=1(1)N$.

Then (14) can be written in the matrix form as:

$$\boldsymbol{DU}+\boldsymbol{M}(\boldsymbol{U})+\boldsymbol{T}(h)=\boldsymbol{0} \quad (33)$$

where $\boldsymbol{D}=[-1, 2, -1]$ is a $N\times N$ tri-diagonal matrix, $\boldsymbol{M}(\boldsymbol{U})=[m_1-U_0, m_2,...,m_N-U_{N+1}]^t$, and $\bar{\bar{\boldsymbol{T}}}(h)=[\bar{\bar{T}}_1(h),\bar{\bar{T}}_2(h),...,\bar{\bar{T}}_N(h)]^t$.

At each x_k, it is required to solve the method (13), which in matrix form can be written as:

$$\boldsymbol{Du}+\boldsymbol{M}(\boldsymbol{u})=\boldsymbol{0} \quad (34)$$

Let $\varepsilon_k = u_k - U_k, k=1(1)N$, be the discretization error (in the absence of round off error) and $\boldsymbol{\varepsilon}=\begin{bmatrix}\varepsilon_1 & \varepsilon_2 & \dots & \varepsilon_N\end{bmatrix}^t = \boldsymbol{u}-\boldsymbol{U}$ be the error vector.

Therefore, we have

$$\bar{u}'_l - \bar{U}'_l = \frac{(\varepsilon_{l+1} - \varepsilon_{l-1})}{2h},$$

$$\bar{u}'_{l\pm1} - \bar{U}'_{l\pm1} = \frac{(\pm 3\varepsilon_{l\pm1} \mp 4\varepsilon_l \pm \varepsilon_{l\mp1})}{2h}.$$

Let $a_l = \dfrac{\partial f}{\partial u_l}$, $b_l = \dfrac{\partial f}{\partial \bar{u}'_l}$.

Therefore,

$$\bar{f}_{l\pm1} - \bar{F}_{l\pm1} = f(x_{l\pm1}, u_{l\pm1}, \bar{u}'_{l\pm1}) - f(x_{l\pm1}, U_{l\pm1}, \bar{U}'_{l\pm1})$$

$$= a_{l\pm1}\varepsilon_{l\pm1} + (\bar{u}'_{l\pm1} - \bar{U}'_{l\pm1})b_{l\pm1}$$

$$= a_{l\pm1}\varepsilon_{l\pm1} + \frac{(\pm 3\varepsilon_{l\pm1} \mp 4\varepsilon_l \pm \varepsilon_{l\mp1})}{2h} b_{l\pm1} \quad , \quad l = 1(1)N$$

$$\bar{\bar{u}}'_l - \bar{\bar{U}}'_l = \left(\bar{u}'_l - \bar{U}'_l\right) - \frac{\alpha h}{2}\left(\left(\bar{f}_{l+1} - \bar{F}_{l+1}\right) - \left(\bar{f}_{l-1} - \bar{F}_{l-1}\right)\right)$$

$$= \frac{(\varepsilon_{l+1} - \varepsilon_{l-1})}{2h} - \frac{\alpha h}{2}\left[a_{l+1}\varepsilon_{l+1} - a_{l-1}\varepsilon_{l-1}\right] - \alpha\frac{(3\varepsilon_{l+1} - 4\varepsilon_l + \varepsilon_{l-1})}{4} b_{l+1} + \frac{\alpha(-3\varepsilon_{l-1} + 4\varepsilon_l - \varepsilon_{l+1})}{4} b_{l-1}$$

$$\bar{\bar{u}}'_{l+1} - \bar{\bar{U}}'_{l+1} = \frac{(\varepsilon_{l+1} - \varepsilon_l)}{h} + \beta h\left(\bar{f}_{l+1} - \bar{F}_{l+1}\right) + \alpha h\left(\bar{f}_l - \bar{F}_l\right)$$

$$= \frac{(\varepsilon_{l+1} - \varepsilon_l)}{2h} + \beta h\left[a_{l+1}\varepsilon_{l+1} + \frac{(3\varepsilon_{l+1} - 4\varepsilon_l + \varepsilon_{l-1})}{2h} b_{l+1}\right] + \alpha h\left[a_l\varepsilon_l + \frac{(\varepsilon_{l+1} - \varepsilon_{l-1})}{2h} b_l\right]$$

Similarly,

$$\bar{\bar{u}}'_{l-1} - \bar{\bar{U}}'_{l-1} = \frac{(\varepsilon_l - \varepsilon_{l-1})}{h} - \alpha h\left[a_l\varepsilon_l + \frac{(\varepsilon_{l+1} - \varepsilon_{l-1})}{2h} b_l\right] - \beta h\left[a_{l-1}\varepsilon_{l-1} + \frac{(-3\varepsilon_{l-1} + 4\varepsilon_l - \varepsilon_{l+1})}{2h} b_{l-1}\right].$$

Hence

$$\bar{\bar{f}}_l - \bar{\bar{F}}_l = a_l\varepsilon_l + b_l(\bar{\bar{u}}'_l - \bar{\bar{U}}'_l),$$

$$\bar{\bar{f}}_{l+1} - \bar{\bar{F}}_{l+1} = a_{l+1}\varepsilon_{l+1} + b_{l+1}(\bar{\bar{u}}'_{l+1} - \bar{\bar{U}}'_{l+1}),$$

$$\bar{\bar{f}}_{l-1} - \bar{\bar{F}}_{l-1} = a_{l-1}\varepsilon_{l-1} + b_{l-1}(\bar{\bar{u}}'_{l-1} - \bar{\bar{U}}'_{l-1}),$$

for suitable a_l and b_l.

Now $\boldsymbol{M}(\boldsymbol{u}) - \boldsymbol{M}(\boldsymbol{U}) = \dfrac{h^2}{12}\left[\left(\bar{\bar{f}}_{l+1} - \bar{\bar{F}}_{l+1}\right) + \left(\bar{\bar{f}}_{l-1} - \bar{\bar{F}}_{l-1}\right) + 10\left(\bar{\bar{f}}_l - \bar{\bar{F}}_l\right)\right]$, $l = 1(1)N$

$$
\begin{aligned}
=\frac{h^2}{12}\Bigg[& a_{l+1}\varepsilon_{l+1}+\frac{(\varepsilon_{l+1}-\varepsilon_l)}{h}b_{l+1}+\beta h\left[a_{l+1}\varepsilon_{l+1}+\frac{(3\varepsilon_{l+1}-4\varepsilon_l+\varepsilon_{l-1})}{2h}b_{l+1}\right]b_{l+1}+\alpha h\left[a_l\varepsilon_l+\frac{(\varepsilon_{l+1}-\varepsilon_{l-1})}{2h}b_l\right]b_{l+1} \\
& a_{l-1}\varepsilon_{l-1}+\frac{(\varepsilon_l-\varepsilon_{l-1})}{h}b_{l-1}-\alpha h\left[a_l\varepsilon_l+\frac{(\varepsilon_{l+1}-\varepsilon_{l-1})}{2h}b_l\right]b_{l-1}-\beta h\left[a_{l-1}\varepsilon_{l-1}+\frac{(-3\varepsilon_{l-1}+4\varepsilon_l-\varepsilon_{l+1})}{2h}b_{l-1}\right]b_{l-1} \\
& +10a_l\varepsilon_l+10b_l\frac{(\varepsilon_{l+1}-\varepsilon_{l-1})}{2h}-b_l\frac{10\alpha h}{2}\left[a_{l+1}\varepsilon_{l+1}-a_{l-1}\varepsilon_{l-1}\right]-10\alpha b_l\frac{(3\varepsilon_{l+1}-4\varepsilon_l+\varepsilon_{l-1})}{4}b_{l+1} \\
& +10\alpha\frac{(-3\varepsilon_{l-1}+4\varepsilon_l-\varepsilon_{l+1})}{4}b_{l-1}b_l\Bigg],\quad l=1(1)N
\end{aligned}
$$

(35)

Now, $b_{l\pm1}=b_l\pm hb'_l+O(h^2)$ and $a_{l\pm1}=a_l\pm O(h)$

Therefore, $\boldsymbol{M}(\boldsymbol{u})-\boldsymbol{M}(\boldsymbol{U})=\boldsymbol{PE}$
(35)

where $\boldsymbol{P}$ is a tri-diagonal matrix with elements

$$P_{l,l}=\frac{h^2}{12}\left(10a_l+20\alpha b_l^2-2b'_l+2\alpha h^2a_lb'_l-4\beta b_l^2\right)+O(h^4),\quad l=1(1)N$$

$$P_{l,l+1}=\frac{h}{2}b_l+\frac{h^2}{12}\left(a_l+b'_l-10\alpha b_l+2\beta b_l^2\right)+O(h^3),\quad l=2(1)N$$

$$P_{l,l-1}=-\frac{h}{2}b_l+\frac{h^2}{12}\left(a_l+b'_l-10\alpha b_l+2\beta b_l^2\right)+O(h^3),\quad l=1(1)N-1$$

Subtracting (34) from (33), we obtain the error equation

$$(\boldsymbol{D}+\boldsymbol{P})\boldsymbol{E}=\boldsymbol{T}(h) \tag{36}$$

Let $G_*=\min\limits_{0\le x\le1}\frac{\partial f}{\partial u}$, and $G^*=\max\limits_{0\le x\le1}\frac{\partial f}{\partial u}$

then $0\le G_*\le a_{l+i}\le G^*,\quad i=0,\pm1$

Let $|b_{l\pm i}|\le b,\quad i=0,\pm1$ and $|b'_l|\le c$ for some positive constants *b* and *c*.

Therefore, $|P_{l,l+1}|\le\frac{h}{2}b+\frac{h^2}{12}\left(G^*+c+10\alpha b+2\beta b^2\right)+O(h^3)$ (37.1)

Also, $|P_{l,l-1}|\le\frac{h}{2}b+\frac{h^2}{12}\left(G^*+c+10\alpha b+2\beta b^2\right)+O(h^3)$ (37.2)

Now, it is easy to see that for sufficiently small *h*,

$$\left|P_{l,l+1}\right| < 1, \quad l = 1(1)N-1$$
$$\left|P_{l,l-1}\right| < 1, \quad l = 2(1)N$$

Therefore $-1 + P_{l,l+1} < 0, -1 + P_{l,l-1} < 0$.

Hence, $\boldsymbol{D}+\boldsymbol{P}$ is irreducible with non-positive off diagonal entries.

Let S_n be the sum of the elements in the n^{th} row of ***D*+*P***, then

For, l=1

$$S_l = 1 + \frac{h^2}{12}\left(10a_l + 20\alpha b_l - 2b_l' - 4\beta b_l^2\right) + \frac{h^2}{12}\left(a_l + \frac{6}{h}b_l + b_l' - 10\alpha b_l + 2\beta b_l^2\right) + O(h^3), \quad (38.1)$$

For, $l = N$

$$S_l = 1 + \frac{h^2}{12}\left(10a_l + 20\alpha b_l - 2b_l' - 4\beta b_l^2\right) + \frac{h^2}{12}\left(a_l - \frac{6}{h}b_l + b_l' - 10\alpha b_l + 2\beta b_l^2\right) + O(h^3), \quad (38.2)$$

$$\text{For } l = 2(1)N-1, \; S_l = h^2 a_l + O(h^4), \quad (38.3)$$

For sufficiently small h,

$$S_1 > \frac{11h^2}{12}G_* > 0, \quad S_N > \frac{11h^2}{12}G_* > 0 \text{ and } S_l > h^2 G_* > 0, \quad l = 2(1)N-1.$$

Thus, for sufficiently small h, ***D*+*P*** is monotone. Therefore $(\boldsymbol{D}+\boldsymbol{P})^{-1}$ exists and

$\boldsymbol{J} = (\boldsymbol{J}_{i,j})_{NXN} = (\boldsymbol{D}+\boldsymbol{P})^{-1} \geq 0.$

To establish convergence, we proceed as follows to obtain the necessary bounds for $(\boldsymbol{D}+\boldsymbol{P})^{-1}$.

Since, $\sum_{l=1}^{N} J_{i,l}S_l = 1, i = 1(1)N$, hence $J_{i,1}S_1 \leq \sum_{l=1}^{N} J_{i,l}S_l = 1, i = 1(1)N$,

i.e. $J_{i,1} \leq \frac{1}{S_1} \leq \frac{12}{11h^2G_*}, i = 1(1)N$. Similarly, $J_{i,N} \leq \frac{1}{S_N} \leq \frac{12}{11h^2G_*}, i = 1(1)N$.

Also, $\min_{2\leq l\leq N-1} S_l \sum_{l=2}^{N-1} J_{i,l} \leq 1 \Rightarrow \sum_{l=2}^{N-1} J_{i,l} \leq \frac{1}{\min_{2\leq l\leq N-1} S_l} \leq \frac{1}{h^2 G_*}, i = 1(1)N$ ·

Since $(\boldsymbol{D}+\boldsymbol{P})^{-1}$exists, therefore we can write the error equation as

$$\boldsymbol{E}=\left(\boldsymbol{D}+\boldsymbol{P}\right)^{-1}\boldsymbol{T}(h)$$

$$\Rightarrow\left\|\boldsymbol{E}\right\|\leq\left\|\left(\boldsymbol{D}+\boldsymbol{P}\right)^{-1}\right\|.\left\|\boldsymbol{T}(h)\right\|=\left\|\boldsymbol{J}\right\|.\left\|\boldsymbol{T}(h)\right\| \tag{39}$$

Now, $\left\|\boldsymbol{J}\right\|=\max_{1\leq i\leq N}\sum_{l=1}^{N}J_{i,l}\leq\dfrac{35}{11h^2G_*}$

Therefore, $$\left\|\boldsymbol{E}\right\|\leq\frac{35}{11h^2G_*}\left\|\boldsymbol{T}(h)\right\|=O(h^4) \tag{40}$$

This proves the fourth order convergence of the proposed method.

6. EXTENSION TO NON-UNIFORM MESH

In this section we extend the finite difference method based on spline in compression approximation to non-uniform mesh.

To obtain the non-polynomial spline solution of the boundary value problem (1)-(2) on non-uniform mesh, we discretize the solution interval [0,1] with a variable mesh size $h_l=x_l-x_{l-1}$, $l=1(1)N+1$. Let $\sigma_l=h_{l+1}/h_l>0, l=1(1)N$ be the mesh ratio. Grid points are given by $x_i=x_0+\sum_{k=1}^{i}h_k, i=1(1)N+1$. Let $U_l=u(x_l)$ be the exact solution of u at the grid point x_l and is approximated by u_l.

Let $S_l(x)$ be the non-polynomial spline function which interpolates $u(x)$ at $x_l, l=0(1)N+1$, defined on the interval $[x_l,x_{l+1}], l=0(1)N$ such that $S_l(x)\in C^2[0,1]$ given by

$$S_l(x)=a_l+b_l(x-x_l)+c_l\sin w(x-x_l)+d_l\cos w(x-x_l),\quad l=0(1)N+1 \tag{41}$$

where a_l,b_l,c_l and d_l are constants and w is an arbitrary parameter.

At each internal mesh point x_l, we denote

$$M_l=u''(x_l)=f(x_l,u(x_l),u'(x_l)),\quad l=0(1)N+1.$$

Using $S_l(x_l)=u_l, S_l(x_{l+1})=u_{l+1}, S_l''(x_l)=M_l, S_l''(x_{l+1})=M_{l+1}\quad l=0,1,...,N+1$ in (41), we get

$$a_l=u_l+\frac{M_l}{w^2}, b_l=\frac{u_{l+1}-u_l}{h_{l+1}}+\frac{M_{l+1}-M_l}{w^2h_{l+1}}, c_l=\frac{-M_l}{w^2}, d_l=\frac{(M_l\cos wh_{l+1}-M_{l+1})}{w^2\sin wh_{l+1}}$$

Using the continuity condition of the first derivative, that is $S_l'(x_l-)=S_l'(x_l+)$, we obtain:

$$\frac{u_{l+1}-(1+\sigma_l)u_l+\sigma_l u_{l-1}}{h_{l+1}}=\alpha_l h_l M_{l-1}+\left(\beta_l h_l+\beta_{l+1}h_{l+1}\right)M_l+\alpha_{l+1}h_{l+1}M_{l+1},\ l=1,2,...N \tag{42}$$

where

$$\alpha_l=\frac{1}{w^2h_l^2}\left[\frac{wh_l}{\sin wh_l}-1\right],\quad \beta_l=\frac{1}{w^2h_l^2}\left[1-wh_l\cot wh_l\right]$$

Further, we have

$$S_l'(x_{l+1})=u_{l+1}'=\frac{u_{l+1}-u_l}{h_{l+1}}+h_{l+1}(\alpha_{l+1}M_l+\beta_{l+1}M_{l+1}),\quad x_l\le x\le x_{l+1} \tag{43.1}$$

$$S_l'(x_{l-1})=u_{l-1}'=\frac{u_l-u_{l-1}}{h_l}-h_l(\alpha_l M_l+\beta_l M_{l-1}),\quad x_{l-1}\le x\le x_l \tag{43.2}$$

At each grid point x_l, we denote

$$P_l=\sigma_l^2+\sigma_l-1,\quad Q_l=\left(\sigma_l+1\right)\left(\sigma_l^2+3\sigma_l+1\right),\quad R_l=\sigma_l\left(1+\sigma_l-\sigma_l^2\right),\ S_l=\sigma_l\left(1+\sigma_l\right). \tag{44}$$

We consider the following approximations:

Let,

$$\bar{m}_l=\overline{u'}_l=(u_{l+1}-(1-\sigma_l^2)u_l-\sigma_l^2u_{l-1})/(h_lS_l), \tag{45.1}$$

$$\bar{m}_{l+1}=\overline{u'}_{l+1}=\frac{(1+2\sigma_l)u_{l+1}-(1+\sigma_l)^2u_l+\sigma_l^2u_{l-1}}{h_lS_l} \tag{45.2}$$

$$\bar{m}_{l-1}=\overline{u'}_{l-1}=\frac{-u_{l+1}+(1+\sigma_l)^2u_l-\sigma_l(2+\sigma_l)u_{l-1}}{h_lS_l} \tag{45.3}$$

$$\bar{f}_l=f(x_l,u_l,\bar{m}_l), \tag{45.4}$$

$$\bar{f}_{l\pm1}=f(x_{l\pm1},u_{l\pm1},\bar{m}_{l\pm1}), \tag{45.5}$$

$$\hat{m}_l=\hat{u}_l'=\bar{m}_l+\frac{h_{l+1}}{(1+\sigma_l)}(\alpha_l\bar{f}_{l-1}+\left(\beta_l-\beta_{l+1}\right)\bar{f}_l-\alpha_{l+1}\bar{f}_{l+1}), \tag{45.6}$$

$$\hat{u}_{l+1}'=\frac{u_{l+1}-u_l}{\sigma_lh_l}+\sigma_lh_l(\alpha_{l+1}\bar{f}_k+\beta_{l+1}\bar{f}_{k+1}), \tag{45.7}$$

$$\hat{u}_{l-1}'=\frac{u_l-u_{l-1}}{h_l}-h_l(\alpha_l\bar{f}_l+\beta_l\bar{f}_{l-1}), \tag{45.8}$$

$$\hat{f}_{l\pm1}=f(x_{l\pm1},u_{l\pm1},\hat{u}_{l\pm1}'), \tag{45.9}$$

$$\hat{f}_l=f(x_l,u_l,\hat{u}_l'), \tag{45.10}$$

Then the non-polynomial spline in compression method with order of accuracy three for the differential equation (1) may be written as

$$u_{l+1}-(1+\sigma_l)u_l+\sigma_l u_{l-1}=\frac{h_l^2}{12}\left[P_l\hat{f}_{l+1}+Q_l\hat{f}_l+R_l\hat{f}_{l-1}\right],\quad l=1(1)N \tag{46}$$

where $\alpha_l=\frac{1}{6},\beta_l=\frac{1}{3}$ with $u_0=U_0=A$ and $u_{N+1}=U_{N+1}=B.$ The local truncation error associated with (46) is given by $\hat{T}_l=O(h_l^5)$, $\sigma_l\neq 1$. For $\sigma_l=1$, the scheme (46) reduces to (13).

7. NUMERICAL ILLUSTRATIONS

We have solved the following three problems using the method described by the equation (13) and (46). The exact solutions are provided in each case. The boundary conditions and homogeneous functions may be obtained from the exact solutions as a test procedure. On the variable mesh with N internal nodes, with each grid length given by $h_l=x_l-x_{l-1},\ l=1(1)N+1$ and the mesh ratio parameter is given by $\sigma_l=h_{l+1}/h_l,\ l=1(1)N$. For simplicity we take $\sigma_l=\sigma(\text{constant})$. The value of the first mesh spacing on the left is given by

$$h_1=\frac{(1-\sigma)}{\left(1-\sigma^{N+1}\right)},\qquad \sigma\neq 1. \tag{47}$$

Therefore, given the value of N and σ, we can calculate h_1 from the above relation and the remaining mesh points are determined by $h_{l+1}=\sigma h_l, l=1(1)N$.The initial vector $\boldsymbol{u}^{(0)}=\mathbf{0}$ is used for nonlinear problems and the iterations were stopped when the absolute error tolerance $\left|u^{(s+1)}-u^{(s)}\right|\leq 10^{-15}$ was achieved.

We have compared the results with the variable mesh method of $O(h_l^3)$ accuracy based on the finite difference discretization discussed by (Mohanty, 2005).

Example 7.1: $u''=du',\quad 0<x<1.$ (*Convection-diffusion problem*) (48)

The exact solution is $u(x)=\left(1-e^{-d(1-x)}\right)/\left(1-e^{-d}\right)$. The root mean square errors are tabulated in Table 1 for various values of N.

Example 7.2: $\nu u''=\left(u-\frac{1}{2}\right)u',\quad 0<x<1.$ (*Burgers' equation*). (49)

The exact solution is given by $u(x) = \frac{1}{2}\left[1 - \tanh\left(\frac{x}{4v}\right)\right]$, where $v = R^{-1}$. The root mean square errors are tabulated in Table 2 for various values of R and σ.

Example 7.3: $u'' = \lambda u u' + f(x), \quad 0 < x < 1.$ (50)

The exact solution is given by $u(x) = \sin(\pi x)$. The root mean square errors are tabulated in Table 3 for various values of λ and σ.

Table 1: Example 7.1: The root mean square errors

N+1	Proposed Method	Method by(Mohanty, 2005)
	σ =1.0, d =10	
40	0.8707(-06)	0.1134(-04)
80	0.5393(-07)	0.7882(-06)
160	0.3357(-08)	0.5012(-07)
320	0.2091(-09)	0.3116(-08)
640	0.1176(-10)	0.1914(-09)
	σ =1.0, d=10^2	
40	0.2548(-02)	0.5393(-01)
80	0.1770(-03)	0.3375(-02)
160	0.1082(-04)	0.2108(-03)
320	0.6669(-06)	0.1296(-04)
640	0.4148(-07)	0.7784(-06)
1280	0.2587(-08)	0.4823(-07)
	σ =1.0, d=10^3	
160	0.1213(-01)	0.7267(-01)
320	0.1719(-02)	0.4488(-02)
640	0.1349(-03)	0.2936(-03)
1280	0.8396(-05)	0.1844(-04)
2560	0.5156(-06)	0.1123(-05)
	σ =1.0, d=10^4	
1280	0.6239(-02)	0.8866(-02)
2560	0.1097(-02)	0.3655(-02)
5120	0.1011(-03)	0.2981(-03)
10240	0.6526(-05)	0.1832(-04)
20480	0.3996(-06)	0.1124(-05)

Table 2: Example 7.2: The root mean square errors

N+1	Proposed Method	Method by (Mohanty, 2005)
σ =1.0, R =10^1		
80	0.1164(-07)	0.6212(-06)
160	0.7304(-09)	0.3898(-07)
320	0.4586(-10)	0.2516(-08)
640	0.2903(-11)	0.1512(-09)
1280	0.1830(-12)	0.8946(-11)
σ =1.0, R =10^2		
80	0.3927(-04)	0.8412(-03)
160	0.2393(-05)	0.5436(-04)
320	0.1488(-06)	0.3388(-05)
640	0.9297(-08)	0.2102(-06)
1280	0.5815(-09)	0.1322(-07)
σ =1.0, R =10^3		
320	0.5531(-03)	0.1127(-01)
640	0.3117(-04)	0.7923(-03)
1280	0.1862(-05)	0.5221(-04)
2560	0.1152(-06)	0.2912(-05)
5120	0.7184(-08)	0.1818(-06)
σ =1.0, R =10^4		
2560	0.3924(-03)	0.7428(-02)
5120	0.2498(-04)	0.4666(-03)
10240	0.1450(-05)	0.3015(-04)
20480	0.8914(-07)	0.1802(-05)
σ =1.0, R =10^5		
20480	0.2369(-03)	0.6642(-02)
40960	0.2022(-04)	0.5113(-03)
81920	0.1134(-05)	0.3212(-04)
163840	0.6903(-07)	0.1967(-05)
σ =1.4, R =50		
40	0.8769(-03)	0.6214(-02)
80	0.6239(-03)	0.4865(-02)
160	0.4425(-03)	0.3218(-02)
320	0.3134(-03)	0.2115(-02)
640	0.2217(-03)	0.1443(-02)
σ =0.9, R =20		
40	0.1460(-03)	0.1212(-02)
80	0.9774(-04)	0.8943(-03)
160	0.6926(-04)	0.7234(-03)
320	0.4905(-04)	0.5254(-03)
640	0.3417(-04)	0.3678(-03)

Table 3: Example 7.3: The root mean square errors

N+1	Proposed Method	Method by (Mohanty, 2005)
σ =1.0, λ =10		
80	0.1248(-07)	0.2446(-06)
160	0.7856(-09)	0.1768(-07)
320	0.4931(-10)	0.1077(-08)
640	0.3601(-11)	0.6864(-10)
1280	0.2073(-12)	0.4543(-11)
σ =1.0, λ =100		
80	0.1831(-06)	0.2002(-05)
160	0.1152(-07)	0.1244(-06)
320	0.7219(-09)	0.7806(-08)
640	0.4516(-10)	0.4888(-09)
1280	0.2858(-11)	0.2852(-10)
σ =1.1, λ =10		
80	0.1841(-04)	0.2629(-03)
160	0.1305(-04)	0.1971(-03)
320	0.9246(-05)	0.1254(-03)
640	0.6543(-05)	0.9033(-04)
1280	0.4628(-05)	0.6868(-04)
σ =0.9, λ =50		
80	0.1941(-03)	0.3216(-02)
160	0.1377(-03)	0.2006(-02)
320	0.9753(-04)	0.1413(-02)
640	0.6902(-04)	0.9641(-03)
1280	0.4882(-04)	0.7233(-03)

8. CONCLUDING REMARKS

In this paper we have derived a new method of order four, based on spline in compression approximation for the numerical solution of two-point non-linear boundary value problems on uniform mesh. The derivation and the convergence of the proposed method are discussed in detail. The method is further extended to solve two-point nonlinear boundary value problem on non-uniform mesh. To test the efficiency of the proposed method, we have applied it to the Burgers' equation and have received convergent results for high Reynolds number. From Table 2, we can see that for high Reynolds number the method shows fourth order convergence for sufficiently small value of uniform mesh length h.

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