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SOLVING FUZZY BOUNDARY VALUE PROBLEM BY DIFFERENTIAL TRANSFORM METHOD

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ABSTRACT. In this paper, a differential transform method is used to find the solution of second order two-point and third order three-point fuzzy boundary value problems. Numerical experiments demonstrate reliability and efficiency of the proposed method.

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1. Introduction

In recent years, much attention have been given to solve the initial and boundary value problems, which have applications in various branches of pure and applied sciences. The concept of differential transform was first introduced by Zhou [13], to solve linear and nonlinear initial value problems in electric circuit analysis. Abdel-Halim Hassan [1] studied the differential transformation method which is an analytical-numerical method to solve the higher order initial value problems. Arikoglu and Ozkol [3] extended the solution of boundary value problems for integro-differential equations by using differential transform method. Allahviranloo et al. [2] extended the differential transformation method for solving the fuzzy differential equations. They have used the concept of generalised H-differentiability. Mikaeilvand and Khakrangin [8] provided the two-dimensional differential transform method to solve fuzzy partial differential equations. Recently, Salahshour and Allahviranloo [12] studied the solutions of fuzzy Volterra integral equations with separable kernel by using fuzzy differential transform method. In last few years, considerable effort has been made in the development of fuzzy boundary value problems. Bede [4] proved that the fuzzy two-point boundary value problem is not equivalent to the integral equation expressed by Green's function under Hukuhara differentiability in the fuzzy differential equations and using fuzzy Auman-type integral in the integral equation. Also, Bede and Gal [5] introduced the weakly generalized differential of a fuzzy number valued function. Khastan et al. [7] presented a generalized concept of higher-order

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differentiability for fuzzy functions. Satio [11] gave a new representation of fuzzy numbers with bounded supports and proved that fuzzy number means a bounded continuous curve in the two-dimensional metric space. Under this new structure and certain conditions, Prakash et al. [9] presented the solution of third order three-point fuzzy boundary value problem by means of Green's function. However, it should be emphasized that most of the works in this direction are mainly concerned with the fuzzy initial value problems and there has been no attempts made to study the fuzzy boundary value problems by using the differential transform methods. In this paper, we use the differential transform method for solving second order two-point and third order three-point fuzzy boundary value problems and carry out the comparison with the exact and numerical solutions.

2. **Preliminaries**

Let us denote by $\mathbb{R}_F$ the class of fuzzy subsets $u : \mathbb{R} \to [0,1]$, satisfying the following properties:

1. u is normal, that is, there exist $x_0 \in \mathbb{R}$ with $u(x_0) = 1$.
2. u is convex fuzzy set, that is,
$$u(\lambda x + (1-\lambda)y) \quad \geq \quad \min\{u(x), u(y)\},\ \forall x, y \in \mathbb{R},\ \forall \lambda \in [0,1].$$
3. u is upper semi-continuous on $\mathbb{R}$.
4. $\overline{\{x \in \mathbb{R} | u(x) > 0\}}$ is compact, where $\bar{A}$ denotes the closure of A.

Then $\mathbb{R}_F$ is called the space of fuzzy numbers. For $0 < r \leq 1$, set $[u]^r = \{s \in \mathbb{R} | u(s) \geq r\}$ and $[u]^0 = cl\{s \in \mathbb{R} | u(s) > 0\}$. Then the r- level set $[u]^r$ is a non-empty compact interval for all $0 \leq r \leq 1$. The following Theorem gives the parametric form of a fuzzy number.

Theorem 2.1. *The necessary and sufficient conditions for* $(\underline{u}(r), \overline{u}(r))$ *to define the parametric form of a fuzzy number are as follows:*

1. $\underline{u}(r)$ *is a bounded monotonic increasing (non-decreasing) left-continuous function* $\forall r \in (0,1]$ *and right-continuous for* $r = 0$.
2. $\overline{u}(r)$ *is a bounded monotonic decreasing (non-increasing) left-continuous function* $\forall r \in (0,1]$ *and right-continuous for* $r = 0$.
3. $\underline{u}(r) \leq \overline{u}(r), \quad 0 \leq r \leq 1$.

We refer to $\underline{u}$ and $\overline{u}$ as the lower and upper branches on u, respectively. For $u \in \mathbb{R}_F$, we define the length of u as: $len(u) = \overline{u} - \underline{u}$. A crisp number α is simply represented by $\overline{u}(r) = \underline{u}(r) = \alpha \quad (0 \leq r \leq 1)$ is called singleton. For $u, v \in \mathbb{R}_F$ and $\alpha \in \mathbb{R}$, the sum $u + v$ and the scalar multiplication αu are defined by

$$u + v = ((\underline{u+v})(r), (\overline{u+v})(r)) = (\underline{u}(r) + \underline{v}(r), \overline{u}(r) + \overline{v}(r)),$$

$$\alpha u = \begin{cases} (\alpha \underline{u}(r), \alpha \overline{u}(r)), & \alpha \geq 0, \\ (\alpha \overline{u}(r), \alpha \underline{u}(r)), & \alpha < 0. \end{cases}$$

For $u, v \in \mathbb{R}_F$, we say $u = v$ if and only if $\underline{u}(r) = \underline{v}(r)$ and $\overline{u}(r) = \overline{v}(r)$.

The metric structure is given by the Hausdorff distance $D : \mathbb{R}_F \times \mathbb{R}_F \to \mathbb{R}_+ \cup \{0\}$, by

$$D(u, v) = \sup_{r \in [0,1]} max\{|\underline{u}(r) - \underline{v}(r)|, |\overline{u}(r) - \overline{v}(r)|\}.$$

Definition 2.2. Let $x, y \in \mathbb{R}_F$. If there exists $z \in \mathbb{R}_F$ such that $x = y + z$ then z is called the H-difference of x, y and it is denoted $x \ominus y$.

In this paper the sign "$\ominus$" stands always for H-difference and $x \ominus y \neq x + (-1)y$ in general. Usually we denote x+(-1)y by x-y, while $x \ominus y$ stands for the H-difference.

Definition 2.3. Let $F : [a, b] \to \mathbb{R}_F$ for some $a, b \in \mathbb{R}$ and fix $t_0 \in (a, b)$. If there exists an element $F'(t_0) \in \mathbb{R}_F$ such that for all $h > 0$ sufficiently near to 0, $F(t_0 + h) \ominus F(t_0), F(t_0) \ominus F(t_0 - h)$ exist and the limits (in the metric D)

$$\lim_{h \to 0^+} \frac{F(t_0 + h) \ominus F(t_0)}{h} \text{ and } \lim_{h \to 0^+} \frac{F(t_0) \ominus F(t_0 - h)}{h}$$

exist and equal to $F'(t_0)$, then F said to be (1)-differentiable at t_0 and it is denoted by $D_1^1 F(t_0)$. If for all $h > 0$ sufficiently near to 0, $F(t_0) \ominus F(t_0 + h),\ F(t_0 - h) \ominus F(t_0)$ exist and the limits (in the metric D)

$$\lim_{h \to 0^+} \frac{F(t_0) \ominus F(t_0 + h)}{-h} \text{ and } \lim_{h \to 0^+} \frac{F(t_0 - h) \ominus F(t_0)}{-h} = F'(t_0)$$

exist and equal to $F'(t_0)$, then F is said to be (2)-differentiable at t_0 and it is denoted by $D_2^1 F(t_0)$. If t_0 is the end points of I, then we consider the corresponding one-sided derivative.

Theorem 2.4 ([7])**.** *Let $F : [a, b] \to \mathbb{R}_F$ and let $F(t) = (f(t, r),\ g(t, r))$ for each $r \in [0, 1]$.*

1. *If F is (1)-differentiable then $f(t, r)$ and $g(t, r)$ are differentiable functions and $D_1^1 F(t) = (f'(t, r), g'(t, r))$.*
2. *If F is (2)-differentiable then $f(t, r)$ and $g(t, r)$ are differentiable functions and $D_2^1 F(t) = (g'(t, r), f'(t, r))$.*

Definition 2.5. Let $F : [a, b] \to \mathbb{R}_F$ and let $n, m \in \{1, 2\}$. If $D_n^1 F$ exist on a neighborhood of t_0 as a fuzzy number valued function and it is (m)-differentiable at t_0 as a fuzzy number valued function, then F is said to be (n, m)-differentiable at $t_0 \in [a, b]$ and is denoted by $D_{n,m}^2 F(t_0)$.

Theorem 2.6 ([7])**.** *Let $F : [a, b] \to \mathbb{R}_F,\ D_1^1 F : [a, b] \to \mathbb{R}_F$ and $D_2^1 F : [a, b] \to \mathbb{R}_F$ and let $F(t) = (f(t, r), g(t, r))$ for each $r \in [0, 1]$.*

1. *If $D_1^1 F$ is (1)-differentiable, then $f'(t,r)$ and $g'(t,r)$ are differentiable functions and $D_{1,1}^2 F(t) = (f''(t,r), g''(t,r))$.*
2. *If $D_1^1 F$ is (2)-differentiable, then $f'(t,r)$ and $g'(t,r)$ are differentiable functions and $D_{1,2}^2 F(t) = (g''(t,r), f''(t,r))$.*
3. *If $D_2^1 F$ is (1)-differentiable, then $f'(t,r)$ and $g'(t,r)$ are differentiable functions and $D_{2,1}^2 F(t) = (g''(t,r), f''(t,r))$.*
4. *If $D_2^1 F$ is (2)-differentiable, then $f'(t,r)$ and $g'(t,r)$ are differentiable functions and $D_{2,2}^2 F(t) = (f''(t,r), g''(t,r))$.*

Definition 2.7. Let $F : [a,b] \to \mathbb{R}_F$ and let $n, m, l \in \{1,2\}$. If $D_n^1 F$ and $D_{n,m}^2 F$ exist on a neighborhood of t_0 as fuzzy number valued functions and $D_{n,m}^2 F$ is (l)-differentiable at t_0 as a fuzzy number valued function, then F is said to be (n,m,l)-differentiable at $t_0 \in [a,b]$ and it is denoted by $D_{n,m,l}^3 F(t_0)$.

Theorem 2.8 ([7])**.** *Let $F : [a,b] \to \mathbb{R}_F$, $D_n^1 F : [a,b] \to \mathbb{R}_F$, $D_{n,m}^2 F : [a,b] \to \mathbb{R}_F$ for $n, m \in \{1,2\}$ and let $F(t) = (f(t,r), g(t,r))$ for each $r \in [0,1]$.*

1. *If $D_{1,1}^2 F$ is (1)-differentiable, then $f''(t,r)$ and $g''(t,r)$ are differentiable functions and $D_{1,1,1}^3 F(t) = (f'''(t,r), g'''(t,r))$.*
2. *If $D_{1,1}^2 F$ is (2)-differentiable, then $f''(t,r)$ and $g''(t,r)$ are differentiable functions and $D_{1,1,2}^3 F(t) = (g'''(t,r), f'''(t,r))$.*
3. *If $D_{1,2}^2 F$ is (1)-differentiable, then $f''(t,r)$ and $g''(t,r)$ are differentiable functions and $D_{1,2,1}^3 F(t) = (g'''(t,r), f'''(t,r))$.*
4. *If $D_{1,2}^2 F$ is (2)-differentiable, then $f''(t,r)$ and $g''(t,r)$ are differentiable functions and $D_{1,2,2}^3 F(t) = (f'''(t,r), g'''(t,r))$.*
5. *If $D_{2,1}^2 F$ is (1)-differentiable, then $f''(t,r)$ and $g''(t,r)$ are differentiable functions and $D_{2,1,1}^3 F(t) = (g'''(t,r), f'''(t,r))$.*
6. *If $D_{2,1}^2 F$ is (2)-differentiable, then $f''(t,r)$ and $g''(t,r)$ are differentiable functions and $D_{2,1,2}^3 F(t) = (f'''(t,r), g'''(t,r))$.*
7. *If $D_{2,2}^2 F$ is (1)-differentiable, then $f''(t,r)$ and $g''(t,r)$ are differentiable functions and $D_{2,2,1}^3 F(t) = (f'''(t,r), g'''(t,r))$.*
8. *If $D_{2,2}^2 F$ is (2)-differentiable, then $f''(t,r)$ and $g''(t,r)$ are differentiable functions and $D_{2,2,2}^3 F(t) = (g'''(t,r), f'''(t,r))$.*

Remark 2.9. A fuzzy number valued function F on $[a,b]$ is said to be (1)-differentiable (or (2)-differentiable) of order k ($k \in \mathbb{N}$) on $[a,b]$ if $F^{(s)}$ is (1)-differentiable (or (2)-differentiable) for all for $s = 1, \dots, k$. In this paper we only consider this kind of function. Let y be a solution of a fuzzy differential equation of order s. If y is (1) differentiable, then $y(t) = (\underline{y}(t,r), \overline{y}(t,r))$. If y is (2) differentiable, then $y(t) = (\underline{y}(t,r), \overline{y}(t,r))$ if s is even and $y(t) = (\overline{y}(t,r), \underline{y}(t,r))$ if s is odd. In the next section we calculate $\overline{y}(t,r)$ and $\underline{y}(t,r)$ by using differential transform method.

3. The Differential Transform Method

Definition 3.1. If $y:[a,b]\to\mathbb{R}_F$ is differentiable of order k in the domain $[a,b]$, then $\underline{Y}(k,r)$ and $\overline{Y}(k,r)$ are defined by

$$\left.\begin{aligned}\underline{Y}(k,r) &= M(k)\left[\frac{d^k\underline{y}(t,r)}{dt^k}\right]_{t=0}\\ \overline{Y}(k,r) &= M(k)\left[\frac{d^k\overline{y}(t,r)}{dt^k}\right]_{t=0}\end{aligned}\right\} k=0,1,2,\ldots$$

when y is (1)-differentiable and

$$\left.\begin{aligned}\underline{Y}(k,r) &= M(k)\left[\frac{d^k\overline{y}(t,r)}{dt^k}\right]_{t=0}\\ \overline{Y}(k,r) &= M(k)\left[\frac{d^k\underline{y}(t,r)}{dt^k}\right]_{t=0}\end{aligned}\right\} k=1,3,5,\ldots$$

and

$$\left.\begin{aligned}\underline{Y}(k,r) &= M(k)\left[\frac{d^k\underline{y}(t,r)}{dt^k}\right]_{t=0}\\ \overline{Y}(k,r) &= M(k)\left[\frac{d^k\overline{y}(t,r)}{dt^k}\right]_{t=0}\end{aligned}\right\} k=0,2,4,\ldots$$

when y is (2)-differentiable. $\underline{Y}_i(k,r)$ and $\overline{Y}_i(k,r)$ are called the lower and the upper spectrum of $y(t)$ at $t=t_i$ in the domain $[a,b]$ respectively.

If y is (1)-differentiable, then $\underline{y}(t,r)$ and $\overline{y}(t,r)$ can be described as

$$\begin{aligned}\underline{y}(t,r) &= \sum_{k=0}^{\infty}\frac{(t-t_i)^k}{k!}\frac{\underline{Y}(k,r)}{M(k)},\\ \overline{y}(t,r) &= \sum_{k=0}^{\infty}\frac{(t-t_i)^k}{k!}\frac{\overline{Y}(k,r)}{M(k)}.\end{aligned}$$

If y is (2)-differentiable, then $\underline{y}(t,r)$ and $\overline{y}(t,r)$ can be described as

$$\begin{aligned}\underline{y}(t,r) &= \left(\sum_{k=1,odd}^{\infty}\frac{(t-t_i)^k}{k!}\frac{\overline{Y}(k,r)}{M(k)}+\sum_{k=0,even}^{\infty}\frac{(t-t_i)^k}{k!}\frac{\underline{Y}(k,r)}{M(k)}\right),\\ \overline{y}(t,r) &= \left(\sum_{k=1,odd}^{\infty}\frac{(t-t_i)^k}{k!}\frac{\underline{Y}(k,r)}{M(k)}+\sum_{k=0,even}^{\infty}\frac{(t-t_i)^k}{k!}\frac{\overline{Y}(k,r)}{M(k)}\right),\end{aligned}$$

where $M(k)>0$ is called the weighting factor. The above set of equations are known as the inverse transformations of $\underline{Y}(k,r)$ and $\overline{Y}(k,r)$. In this paper, the transformation with $M(k)=\frac{1}{k!}$ is considered. If y is (1)-differentiable, then

$$\begin{aligned}\underline{Y}(k,r) &= \frac{1}{k!}\left[\frac{d^k}{dt^k}\underline{y}(t,r)\right]_{t=0}\\ \overline{Y}(k,r) &= \frac{1}{k!}\left[\frac{d^k}{dt^k}\overline{y}(t,r)\right]_{t=0}\end{aligned}\qquad k=0,1,2,\ldots. \tag{3.1}$$

If y is (2)-differentiable, then

$$\tag{3.2} \left.\begin{aligned} \underline{Y}(k,r) &= \frac{1}{k!}\left[\frac{d^k}{dt^k}\overline{y}(t,r)\right]_{t=0} \\ \overline{Y}(k,r) &= \frac{1}{k!}\left[\frac{d^k}{dt^k}\underline{y}(t,r)\right]_{t=0} \end{aligned}\right\} \quad k=1,3,5,\ldots. \qquad \left.\begin{aligned} \underline{Y}(k,r) &= \frac{1}{k!}\left[\frac{d^k}{dt^k}\underline{y}(t,r)\right]_{t=0} \\ \overline{Y}(k,r) &= \frac{1}{k!}\left[\frac{d^k}{dt^k}\overline{y}(t,r)\right]_{t=0} \end{aligned}\right\} \quad k=0,2,4,\ldots.$$

Using the differential transformation, a differential equation in the domain of interest can be transformed to an algebraic equation in the domain $\{0,1,2,\ldots\}$ and $\underline{y}(t,r)$ and $\overline{y}(t,r)$ can be obtained as the finite-term Taylor series plus a remainder, as

$$\tag{3.3} \begin{aligned} \underline{y}(t,r) &= \sum_{k=0}^{n}(t-t_0)^k\underline{Y}(k,r)+R_{n+1}(t), \\ \overline{y}(t,r) &= \sum_{k=0}^{n}(t-t_0)^k\overline{Y}(k,r)+R_{n+1}(t), \end{aligned}$$

when y is (1)-differentiable and

$$\tag{3.4} \begin{aligned} \underline{y}(t,r) &= \sum_{k=1,odd}^{n}(t-t_0)^k\overline{Y}(k,r)+\sum_{k=0,even}^{n}(t-t_0)^k\underline{Y}(k,r)+R_{n+1}(t), \\ \overline{y}(t,r) &= \sum_{k=1,odd}^{n}(t-t_0)^k\underline{Y}(k,r)+\sum_{k=0,even}^{n}(t-t_0)^k\overline{Y}(k,r)+R_{n+1}(t), \end{aligned}$$

when y is (2)-differentiable. From Definition 3.1, it is easily proven that the transformation function have basic mathematics operation shown in Table 1.

TABLE 1. The fundamental operations of one-dimensional differential transform method

Original function	Transformed function
$c(t)=u(t)\pm v(t)$	$C(k)=U(k)\pm V(k)$
$c(t)=\alpha u(t)$	$C(k)=\alpha U(k)$, where α is a constant
$c(t)=\frac{du(t)}{dt}$	$C(k)=(k+1)U(k+1)$
$c(t)=\frac{d^r u(t)}{dt^r}$	$C(k)=(k+1)(k+2).....(k+r)U(k+r)$
$c(t)=u(t)v(t)$	$C(k)=\sum_{r=0}^{k}U(r)V(k-r)$
$c(t)=t^m$	$C(k)=\delta(k-m)$
$c(t)=e^{\lambda t}$	$C(k)=\frac{\lambda^k}{k!}$
$c(t)=\sin(\omega t+\alpha)$	$C(k)=\frac{\omega^k}{k!}\sin(\frac{\pi k}{2!}+\alpha)$
$c(t)=\cos(\omega t+\alpha)$	$C(k)=\frac{\omega^k}{k!}\cos(\frac{\pi k}{2!}+\alpha)$

4. **Two-point fuzzy boundary value problem**

In this section, we discuss the second order two-point fuzzy boundary value problem of the form,

$$\begin{aligned} y''(t) &= f(t, y(t), y'(t)) \\ y(a) &= A, \quad y(b) = B, \end{aligned} \tag{4.1}$$

where $t \in [a,b]$, $A \in \mathbb{R}_F$, $B \in \mathbb{R}_F$ and $f \in C([a,b] \times \mathbb{R}_F \times \mathbb{R}_F, \mathbb{R}_F)$.

Definition 4.1 ([6])**.** Let $y : [a,b] \to \mathbb{R}_F$ and let $n, m \in \{1,2\}$. We say y is a (n,m) solution for problem (4.1) on $[a,b]$, if $D_n^1 y$ and $D_{n,m}^2 y$ exist on $[a,b]$ as fuzzy number valued functions, $D_{n,m}^2 y(t) = f(t, y(t), D_n^1 y(t))$ for all $t \in [a,b]$, $y(a) = A$ and $y(b) = B$.

Definition 4.2. Let $n, m \in \{1,2\}$ and let I_1 and be an interval such that $I_1 \subset [a,b]$. If $y : I_1 \cup \{a,b\} \to \mathbb{R}_F$, $D_n^1 y$ and $D_{n,m}^2 y$ exist on I_1 as fuzzy number valued functions, $D_{n,m}^2 y(t) = f(t, y(t), D_n^1 y(t))$ for all $t \in I_1$, $y(a) = A$ and $y(b) = B$, then y is said to be a (n,m) solution for the boundary value problem (4.1) on $I_1 \cup \{a,b\}$.

Remark 4.3. I_1 may or may not contains $\{a,b\}$.

The derivatives of type (1) or (2), we may replace the fuzzy boundary value problem by the following equivalent system. For $r \in [0,1]$,

$$\begin{aligned} \underline{y}''(t,r) &= \underline{f}(t, \underline{y}(t,r), \underline{y}'(t,r), \overline{y}(t,r), \overline{y}'(t,r)), \\ \underline{y}(a,r) &= \underline{A}, \quad \underline{y}(b,r) = \overline{A}. \\ \overline{y}''(t,r) &= \overline{f}(t, \underline{y}(t,r), \underline{y}'(t,r), \overline{y}(t,r), \overline{y}'(t,r)), \\ \underline{y}(a,r) &= \underline{B}, \quad \overline{y}(b,r) = \overline{B}. \end{aligned} \tag{4.2}$$

For any fixed $r \in [0,1]$, the system represents an two-point boundary value problem, to which any convergent classical numerical procedure can be applied. We proposed a differential transformation method for solving the problem. Taking the differential transformation of (4.2), the transformed equation describes the relationship between the spectrum of $y(t)$, $y'(t)$ and $y''(t)$ as

$$\begin{aligned} (k+1)(k+2)\underline{Y}(k+2,r) &= \underline{F}(t, \underline{Y}(k,r), \underline{Y}'(k,r), \overline{Y}(k,r), \overline{Y}'(k,r)) \\ (k+1)(k+2)\overline{Y}(k+2,r) &= \overline{F}(t, \underline{Y}(k,r), \underline{Y}'(k,r), \overline{Y}(k,r), \overline{Y}'(k,r)), \end{aligned}$$

and

$$\left.\begin{aligned} (k+1)(k+2)\underline{Y}(k+2,r) &= \overline{F}(t, \underline{Y}(k,r), \underline{Y}'(k,r), \overline{Y}(k,r), \overline{Y}'(k,r)) \\ (k+1)(k+2)\overline{Y}(k+2,r) &= \underline{F}(t, \underline{Y}(k,r), \underline{Y}'(k,r), \overline{Y}(k,r), \overline{Y}'(k,r)), \end{aligned}\right\} \quad k = 1, 3, 5, \ldots.$$

$$\left.\begin{aligned}(k+1)(k+2)\underline{Y}(k+2,r) &= \underline{F}(t,\underline{Y}(k,r),\underline{Y}'(k,r),\overline{Y}(k,r),\overline{Y}'(k,r))\\ (k+1)(k+2)\overline{Y}(k+2,r) &= \overline{F}(t,\underline{Y}(k,r),\underline{Y}'(k,r),\overline{Y}(k,r),\overline{Y}'(k,r)),\end{aligned}\right\}$$
$$k=0,2,4,\ldots.$$

when y is (1) and (2)-differentiable respectively, where $\underline{F}(\cdot)$ and $\overline{F}(\cdot)$ denote the transformed function of $\underline{f}(t,\underline{y}(t,r),\ \underline{y}'(t,r),\ \overline{y}(t,r),\overline{y}'(t,r))$ and $\overline{f}(t,\underline{y}(t,r),\ \underline{y}'(t,r),\ \overline{y}(t,r),\ \overline{y}'(t,r))$ respectively.

5. **Three-point fuzzy boundary value problem**

In this section, we discuss a third order three-point fuzzy boundary value problem of the form

$$\begin{aligned} y'''(t) &= f(t,y(t),y'(t),y''(t))\\ y(a) &= A,\quad y(c)=C,\quad y(b)=B, \end{aligned} \tag{5.1}$$

where $t\in[a,b]$, $a<c<b$, $A\in\mathbb{R}_F$, $B\in\mathbb{R}_F$, $C\in\mathbb{R}_F$ and $f\in C([a,b]\times\mathbb{R}_F\times\mathbb{R}_F\times\mathbb{R}_F,\mathbb{R}_F)$.

Definition 5.1. Let $y:[a,b]\to\mathbb{R}_F$ and let $n,m,l\in\{1,2\}$. We say y is a (n,m,l) solution for problem (5.1) on $[a,b]$, if $D^1_n y, D^2_{n,m}y$ and $D^3_{n,m,l}y$ exist on $[a,b]$, $D^3_{n,m,l}y(t)=f(t,y(t),\ D^1_n y(t),\ D^2_{n,m}y(t))$ for all $t\in[a,b]$, $y(a)=A$, $y(c)=C$ and $y(b)=B$.

Definition 5.2. Let $n,m,l\in\{1,2\}$ and let I_1 and be an interval such that $I_2\subset[a,b]$. If $y:I_2\cup\{a,c,b\}\to\mathbb{R}_F,\ D^1_n y, D^2_{n,m}y$ and $D^3_{n,m,l}y$ exist on I_2 as fuzzy number valued functions, $D^3_{n,m,l}y(t)=f(t,y(t),D^1_n y(t),D^2_{n,m}y(t))$ for all $t\in I_2\cup\{a,c,b\}$, $y(a)=A$, $y(c)=C$ and $y(b)=B$, then y is said to be a (n,m,l) solution for the boundary value problem (5.1) on I_2.

Remark 5.3. I_2 may or may not contains $\{a,c,b\}$.

If the derivatives of type (1), we may replace the fuzzy boundary value problem by the following equivalent system.

$$\begin{aligned}\underline{y}'''(t,r) &= \underline{f}(t,\underline{y}(t,r),\underline{y}'(t,r),\underline{y}''(t,r),\overline{y}(t,r),\overline{y}'(t,r),\overline{y}''(t,r)),\\ &\qquad \underline{y}(a,r)=\underline{A},\quad \underline{y}(b,r)=\underline{B},\quad \underline{y}(c,r)=\underline{C},\\ \overline{y}'''(t,r) &= \overline{f}(t,\underline{y}(t,r),\underline{y}'(t,r),\underline{y}''(t,r),\overline{y}(t,r),\overline{y}'(t,r),\overline{y}''(t,r)),\\ &\qquad \overline{y}(a,r)=\overline{A},\quad \overline{y}(b,r)=\overline{B},\quad \overline{y}(c,r)=\overline{C}\end{aligned}$$

for $r \in [0,1]$. If the derivatives of type (2), then we get

$$\underline{y}'''(t,r) = \overline{f}(t, \underline{y}(t,r), \underline{y}'(t,r), \underline{y}''(t,r), \overline{y}(t,r), \overline{y}'(t,r), \overline{y}''(t,r)),$$
$$\underline{y}(a,r) = \underline{A}, \quad \underline{y}(b,r) = \underline{B}, \quad \underline{y}(c,r) = \underline{C},$$
$$\overline{y}'''(t,r) = \underline{f}(t, \underline{y}(t,r), \underline{y}'(t,r), \underline{y}''(t,r), \overline{y}(t,r), \overline{y}'(t,r), \overline{y}''(t,r)),$$
$$\overline{y}(a,r) = \overline{A}, \quad \overline{y}(b,r) = \overline{B}, \quad \overline{y}(c,r) = \overline{C},$$

for $r \in [0,1]$. Taking the differential transformation of above parametric representation of (5.1), the transformed equation describes the relationship between the spectrum of $y(t)$, $y'(t)$, $y''(t)$ and $y'''(t)$ as

$$\begin{aligned}(k+1)(k+2)(k+3)\underline{Y}(k+3,r) &= \underline{F}(t, \underline{Y}(k,r), \underline{Y}'(k,r), \underline{Y}''(k,r), \overline{Y}(k,r), \overline{Y}'(k,r), \overline{Y}''(k,r))\\ (k+1)(k+2)(k+3)\overline{Y}(k+3,r) &= \overline{F}(t, \underline{Y}(k,r), \underline{Y}'(k,r), \underline{Y}''(k,r), \overline{Y}(k,r), \overline{Y}'(k,r), \overline{Y}''(k,r)),\end{aligned}$$

for $k = 0,1,2,3,\ldots$ when y is (1) differentiable and when y is (2) differentiable, we get

$$\begin{aligned}(k+1)(k+2)(k+3)\underline{Y}(k+3,r) &= \overline{F}(t, \underline{Y}(k,r), \underline{Y}'(k,r), \underline{Y}''(k,r), \overline{Y}(k,r), \overline{Y}'(k,r), \overline{Y}''(k,r))\\ (k+1)(k+2)(k+3)\overline{Y}(k+3,r) &= \underline{F}(t, \underline{Y}(k,r), \underline{Y}'(k,r), \underline{Y}''(k,r), \overline{Y}(k,r), \overline{Y}'(k,r), \overline{Y}''(k,r)),\end{aligned}$$

for $k = 1,3,5,\ldots$ and

$$\begin{aligned}(k+1)(k+2)(k+3)\underline{Y}(k+3,r) &= \underline{F}(t, \underline{Y}(k,r), \underline{Y}'(k,r), \underline{Y}''(k,r), \overline{Y}(k,r), \overline{Y}'(k,r), \overline{Y}''(k,r))\\ (k+1)(k+2)(k+3)\overline{Y}(k+3,r) &= \overline{F}(t, \underline{Y}(k,r), \underline{Y}'(k,r), \underline{Y}''(k,r), \overline{Y}(k,r), \overline{Y}'(k,r), \overline{Y}''(k,r)),\end{aligned}$$

for $k = 0,2,4,\ldots$, where $\underline{F}(.)$ and $\overline{F}(.)$ denote the transformed function of

$$\underline{f}(t, \underline{y}(t,r), \underline{y}'(t,r), \underline{y}''(t,r), \overline{y}(t,r), \overline{y}'(t,r), \overline{y}''(t,r)) \text{ and}$$
$$\overline{f}(t, \underline{y}(t,r), \underline{y}'(t,r), \underline{y}''(t,r), \overline{y}(t,r), \overline{y}'(t,r), \overline{y}''(t,r)) \text{ respectively.}$$

6. **Numerical examples**

Example 6.1. Consider the following second order two-point fuzzy boundary value problem

$$\begin{aligned}y''(t) &= 2(r-1, 1-r),\\ y(0) &= \tfrac{1}{8}(r-1, 1-r), \; y(1) = \tfrac{3}{8}(r-1, 1-r).\end{aligned} \tag{6.1}$$

If y is (1) or (2)-differentiable, then (6.1) can be written as

$$\underline{y}''(t,r) = 2(r-1), \qquad \overline{y}''(t,r) = 2(1-r), \tag{6.2}$$

with boundary conditions

$$\begin{aligned}\underline{y}(0,r) &= \tfrac{1}{8}(r-1), \quad \overline{y}(0,r) = \tfrac{1}{8}(1-r),\\ \underline{y}(1,r) &= \tfrac{3}{8}(r-1), \quad \overline{y}(1,r) = \tfrac{3}{8}(1-r).\end{aligned} \tag{6.3}$$

If y is (1)-differentiable, the differential transformation of (6.2) becomes

$$\left.\begin{aligned}(k+1)(k+2)\underline{Y}(k+2,r) &= 2(r-1)\delta(k-0)\\ (k+1)(k+2)\overline{Y}(k+2,r) &= 2(1-r)\delta(k-0)\end{aligned}\right., \text{ when } k = 0,1,2,\ldots \tag{6.4}$$

From (3.1) and (6.3) we get,

$$\begin{aligned} \underline{Y}(0,r) &= \tfrac{1}{8}(r-1),\ \overline{Y}(0,r) = \tfrac{1}{8}(1-r), \\ \underline{Y}(1,r) &= a_1(r), \qquad \overline{Y}(1,r) = b_1(r), \end{aligned} \tag{6.5}$$

where $a_1(r) = \underline{y}'(0,r)$ and $b_1(r) = \overline{y}'(0,r)$. By recursive method and substituting (6.5) into (6.4), we get

$$\begin{aligned} \underline{Y}(2,r) &= r-1, \overline{Y}(2,r) = 1-r \\ \underline{Y}(k,r) &= 0, \qquad \overline{Y}(k,r) = 0,\ \text{for } k \geq 3. \end{aligned} \tag{6.6}$$

Substituting all $\underline{Y}, \overline{Y}$ in (6.6) into (3.3),

$$\begin{aligned} \underline{y}(t,r) &= \tfrac{1}{8}(r-1) + a_1(r)t + (r-1)t^2, \\ \overline{y}(t,r) &= \tfrac{1}{8}(1-r) + b_1(r)t + (1-r)t^2. \end{aligned} \tag{6.7}$$

$a_1(r)$ and $b_1(r)$ are evaluated from the boundary conditions given in (6.3) at $t = 1$ as follows

$$a_1(r) = -\frac{3}{4}(r-1), \quad b_1(r) = -\frac{3}{4}(1-r).$$

$$y(t,r) = \left(\frac{1}{8}(r-1)(8t^2-6t+1), \frac{1}{8}(1-r)(8t^2-6t+1)\right) \tag{6.8}$$

Theorem 2.1, we see $y(t,r)$ in (6.8) represents a valued fuzzy number when $8t^2-6t+1 \geq 0$. Hence (6.8) represents fuzzy number for $t \in [0, \frac{1}{4}] \cup [\frac{1}{2}, 1]$. The (1)-derivative of (6.8) is given by

$$y'(t) = \left(\frac{r-1}{4}(8t-3), \frac{1-r}{4}(8t-3)\right)$$

and it is a fuzzy number when $t \in [\frac{1}{2}, 1]$. Then it is again (1)-differentiable

$$y''(t) = 2(r-1, 1-r)$$

and it is a fuzzy number when $t \in [\frac{1}{2}, 1]$. Hence y in (6.8) is a (1,1,1)-solution of the fuzzy boundary value problem (6.2)–(6.3) on $\{0\} \cup [\frac{1}{2}, 1]$. Lower and upper branch of (1,1) solution of the fuzzy boundary value problem (6.2)–(6.3) is plotted in Figure 1 for different r.

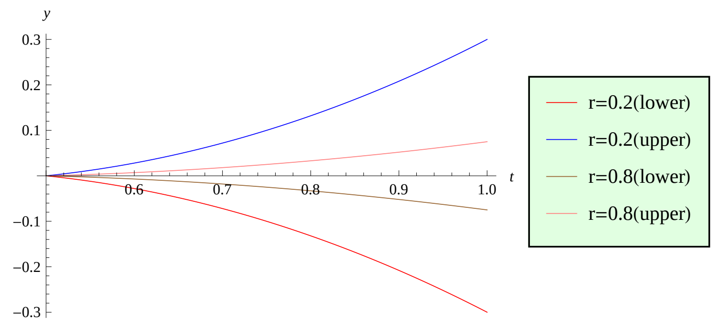

FIGURE 1. (1,1) solution for different r.

If y is (2)-differentiable, the differential transformation of (6.2) becomes

$$\begin{aligned}(k+1)(k+2)\underline{Y}(k+2,r) &= 2(r-1)\delta(k-0),\\ (k+1)(k+2)\overline{Y}(k+2,r) &= 2(1-r)\delta(k-0)\end{aligned}\Bigg\}\ \text{when}\quad k=0,2,\dots$$
$$\begin{aligned}(k+1)(k+2)\underline{Y}(k+2,r) &= 2(1-r)\delta(k-0),\\ (k+1)(k+2)\overline{Y}(k+2,r) &= 2(r-1)\delta(k-0)\end{aligned}\Bigg\}\ \text{when}\quad k=1,3,\dots \tag{6.9}$$

From (3.2) and (6.3) we get,

$$\begin{aligned}&\underline{Y}(0,r)=\tfrac{1}{8}(r-1),\ \overline{Y}(0,r)=\tfrac{1}{8}(1-r),\\ &\underline{Y}(1,r)=a_2(r),\qquad \overline{Y}(1,r)=b_2(r),\end{aligned}\tag{6.10}$$

where $a_2(r)=\overline{y}'(0,r)$ and $b_2(r)=\underline{y}'(0,r)$. By recursive method and substituting (6.10) into (6.9), we get

$$\begin{aligned}\underline{Y}(2,r) &= r-1, \overline{Y}(2,r)=1-r\\ \underline{Y}(k,r) &= 0,\qquad \overline{Y}(k,r)=0,\ \text{for}\ k\geq 3.\end{aligned}\tag{6.11}$$

Substituting all $\underline{Y},\overline{Y}$ into (3.4).

$$\begin{aligned}\underline{y}(t,r) &= \tfrac{1}{8}(r-1)+b_2(r)t+(r-1)t^2,\\ \overline{y}(t,r) &= \tfrac{1}{8}(1-r)+a_2(r)t+(1-r)t^2.\end{aligned}\tag{6.12}$$

The constants $a_2(r)$ and $b_2(r)$ are evaluated from the boundary conditions given in (6.3) at $t=1$ as follows

$$a_2(r)=-\frac{3}{4}(1-r),\quad b_2(r)=-\frac{3}{4}(r-1).$$

We get $y(t,r)$ as in (6.8). We already see that $y(t,r)$ in (6.8) represents a valued fuzzy number when for $t\in[0,\frac{1}{4}]\cup[\frac{1}{2},1]$. The (2)-derivative of (6.8) is given by

$$y'(t)=\left(\frac{1-r}{4}(8t-3),\frac{r-1}{4}(8t-3)\right)$$

and it is a fuzzy number when $t\in[0,\frac{1}{4}]$. Then it is again (2)-differentiable

$$y''(t)=2(r-1,1-r)$$

and it is a fuzzy number when $t \in [0, \frac{1}{4}]$. Hence y in (6.8) is a (2,2,2)-solution of the fuzzy boundary value problem (6.2)–(6.3) on $[0, \frac{1}{4}] \cup \{1\}$. Lower and upper branch of (2,2) solution y is plotted in Figure 2 for different r. The solution of the fuzzy boundary value problem (6.2)–(6.3) for different t is plotted in Figure 3. From this figure we see that y is a fuzzy number valued function.

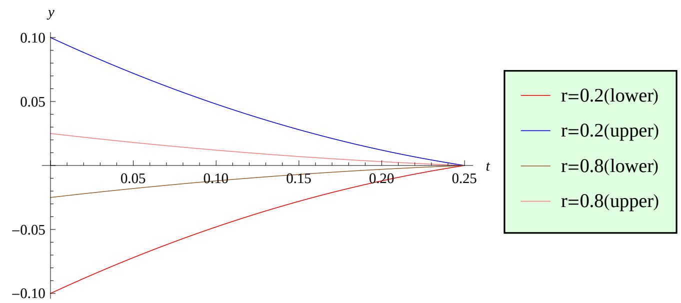

FIGURE 2. (2,2) solution for different r.

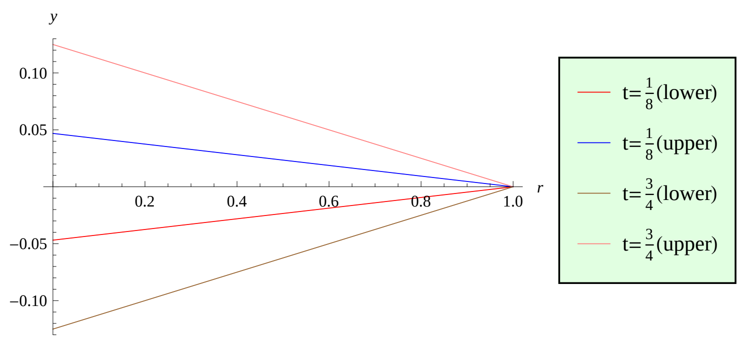

FIGURE 3. $\underline{y}(t, r)$ and $\overline{y}(t, r)$ for different $t \in [0, \frac{1}{4}] \cup [\frac{1}{2}, 1]$.

Example 6.2. Consider the following third order three-point fuzzy boundary value problem

$$\begin{aligned} y'''(t) &= 6(r-1, 1-r), \\ y(0) &= \tilde{0}, \\ y(1) &= \tfrac{1}{8}(r-1, 1-r), \\ y(2) &= \tfrac{17}{4}(r-1, 1-r). \end{aligned} \tag{6.13}$$

If y is (1)-differentiable, then (6.13) can be written as

$$\underline{y}'''(t, r) = 6(r-1), \quad \overline{y}'''(t, r) = 6(1-r), \tag{6.14}$$

with boundary conditions

$$(6.15)\qquad \begin{aligned} \underline{y}(0,r) &= 0, & \overline{y}(0,r) &= 0,\\ \underline{y}(1,r) &= \tfrac{1}{8}(r-1), & \overline{y}(1,r) &= \tfrac{1}{8}(1-r),\\ \underline{y}(2,r) &= \tfrac{17}{4}(r-1), & \overline{y}(2,r) &= \tfrac{17}{4}(1-r). \end{aligned}$$

Taking the differential transformation of (6.14)

$$(6.16)\qquad \begin{aligned} (k+1)(k+2)(k+3)\underline{Y}(k+3,r) &= 6(r-1)\delta(k-0)\\ (k+1)(k+2)(k+3)\overline{Y}(k+3,r) &= 6(1-r)\delta(k-0) \end{aligned} \quad \text{when } k=0,1,\dots$$

From (3.1) and (6.15) we get,

$$(6.17)\qquad \begin{aligned} \underline{Y}(0,r) &= 0, & \overline{Y}(0,r) &= 0,\\ \underline{Y}(1,r) &= c_1(r), & \overline{Y}(1,r) &= c_2(r)\\ \underline{Y}(2,r) &= d_1(r), & \overline{Y}(2,r) &= d_2(r) \end{aligned}$$

where $c_1(r) = \underline{y}'(0,r)$, $c_2(r) = \overline{y}'(0,r)$ $d_1(r) = \frac{\underline{y}''(0,r)}{2}$ and $d_2(r) = \frac{\overline{y}''(0,r)}{2}$. By recursive method and substituting (6.17) into (6.16), we get

$$(6.18)\qquad \begin{aligned} \underline{Y}(3,r) &= r-1, & \overline{Y}(3,r) &= 1-r\\ \underline{Y}(k,r) &= 0, & \overline{Y}(k,r) &= 0, \text{ for } k \geq 4. \end{aligned}$$

Substituting all $\underline{Y}, \overline{Y}$ in (6.18) into (3.3), we get

$$\begin{aligned} \underline{y}(t,r) &= c_1(r)t + d_1(r)t^2 + (r-1)t^3,\\ \overline{y}(t,r) &= c_2(r)t + d_2(r)t^2 + (1-r)t^3. \end{aligned}$$

The constants $c_1(r)$, $d_1(r)$, $c_2(r)$ and $d_2(r)$ are evaluated from the boundary conditions given in (6.15) at $t=1$ and $t=2$ as follows

$$c_1(r) = \frac{1}{8}(r-1),\quad d_1(r) = -(r-1),\quad c_2(r) = \frac{1}{8}(1-r),\quad d_2(r) = -(r-1).$$

We get,

$$(6.19)\qquad y(t) = \left(\frac{(r-1)}{8}(8t^3-8t^2+t), \frac{(1-r)}{8}(8t^3-8t^2+t)\right).$$

By Theorem 2.1, $y(t)$ in (6.19) represents a valued fuzzy number for $t \in [0, \frac{1}{4}(2-\sqrt{2})] \cup [\frac{1}{4}(2+\sqrt{2}), 2]$. The (1)-derivative of (6.19) is given by

$$y'(t) = \left(\frac{r-1}{8}(24t^2-16t+1), \frac{1-r}{8}(24t^2-16t+1)\right)$$

which is a fuzzy number when $t \in [0, \frac{1}{12}(4-\sqrt{10})] \cup [\frac{1}{4}(2+\sqrt{2}), 2]$. Then it is again (1)-differentiable

$$y''(t) = ((r-1)(6t-2), (1-r)(6t-2))$$

which is a fuzzy number when $t \in [\frac{1}{4}(2+\sqrt{2}), 2]$ and the (1)-differentiability of $y''(t)$ is

$$y'''(t) = 6(r-1, 1-r)$$

which is a fuzzy number when $t \in [\frac{1}{4}(2+\sqrt{2}), 2]$. Hence y in (6.19) is a (1,1,1)-solution of the boundary value problem (6.13) on $\{0\} \cup [\frac{1}{4}(2+\sqrt{2}), 2]$.

If y is (2)-differentiable, then (6.13) can be written as

$$\underline{y}'''(t,r) = 6(1-r), \quad \overline{y}'''(t,r) = 6(r-1), \tag{6.20}$$

with the boundary conditions (6.15). Taking the differential transformation of (6.20), we get,

$$\begin{array}{ll} \left.\begin{aligned} (k+1)(k+2)(k+3)\underline{Y}(k+3,r) &= 6(r-1)\delta(k-0) \\ (k+1)(k+2)(k+3)\overline{Y}(k+3,r) &= 6(1-r)\delta(k-0) \end{aligned}\right\} \text{when } k = 1,3,\ldots \\ \left.\begin{aligned} (k+1)(k+2)(k+3)\underline{Y}(k+3,r) &= 6(1-r)\delta(k-0) \\ (k+1)(k+2)(k+3)\overline{Y}(k+3,r) &= 6(r-1)\delta(k-0) \end{aligned}\right\} \text{when } k = 0,2,\ldots \end{array} \tag{6.21}$$

From (3.2) and (6.15) we get,

$$\begin{aligned} \underline{Y}(0,r) &= 0, & \overline{Y}(0,r) &= 0, \\ \underline{Y}(1,r) &= e_1(r), & \overline{Y}(1,r) &= e_2(r), \\ \underline{Y}(2,r) &= f_1(r), & \overline{Y}(2,r) &= f_2(r), \end{aligned} \tag{6.22}$$

where $e_1(r) = \overline{y}'(0,r)$, $e_2(r) = \underline{y}'(0,r)$ $f_1(r) = \frac{\underline{y}''(0,r)}{2}$ and $f_2(r) = \frac{\overline{y}''(0,r)}{2}$. By recursive method and substituting (6.22) into (6.21), we get

$$\begin{aligned} \underline{Y}(3,r) &= (1-r), & \overline{Y}(3,r) &= (r-1), \\ \underline{Y}(k,r) &= 0, & \overline{Y}(k,r) &= 0, \text{ for } k \geq 4. \end{aligned} \tag{6.23}$$

Substituting all $\underline{Y}, \overline{Y}$ in (6.22) and (6.23) into (3.4).

$$\begin{aligned} \underline{y}(t,r) &= e_2(r)t + f_1(r)t^2 + (r-1)t^3, \\ \overline{y}(t,r) &= e_1(r)t + f_2(r)t^2 + (1-r)t^3. \end{aligned}$$

The constants $e_1(r)$, $f_1(r)$, $e_2(r)$ and $f_2(r)$ are evaluated from the boundary conditions given in (6.15) at $t = 1$ and $t = 2$ as follows

$$e_2(r) = \frac{1}{8}(r-1), \; f_1(r) = -(r-1), \; e_1(r) = \frac{1}{8}(r-1), \; f_2(r) = -(r-1).$$

We get,

$$y(t) = \left(\frac{(1-r)}{8}(8t^3 - 8t^2 + t), \frac{(r-1)}{8}(8t^3 - 8t^2 + t) \right). \tag{6.24}$$

Theorem 2.1, we see $y(t,r)$ in (6.24) represents a valued fuzzy number when $8t^3 - 8t^2 + t \leq 0$. Hence (6.24) represents fuzzy number for $t \in [\frac{1}{4}(2-\sqrt{2}), \frac{1}{4}(2+\sqrt{2}), 2]$. The (2)-derivative of (6.24) is given by

$$y'(t) = \left(\frac{r-1}{8}(24t^2 - 16t + 1), \frac{1-r}{8}(24t^2 - 16t + 1) \right)$$

and it is not a fuzzy number. Hence the fuzzy boundary value problem (6.13) has no (2,2,2)-solution. Lower and upper branch of (1,1,1) solution of the fuzzy boundary value problem (6.15) is plotted for different r and for different t is plotted in Figure 4 and Figure 5 respectively. From Figure 5 we see that y is a fuzzy number valued function.

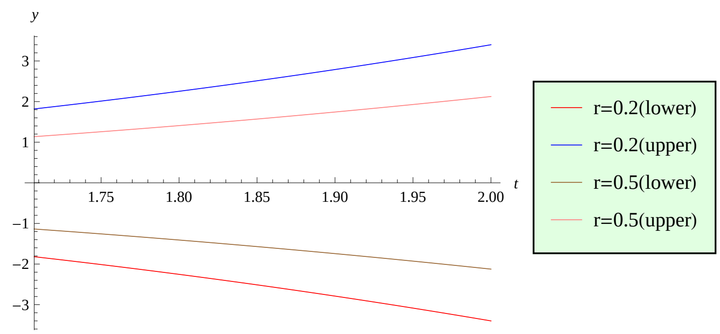

FIGURE 4. (1,1,1) solution for different r.

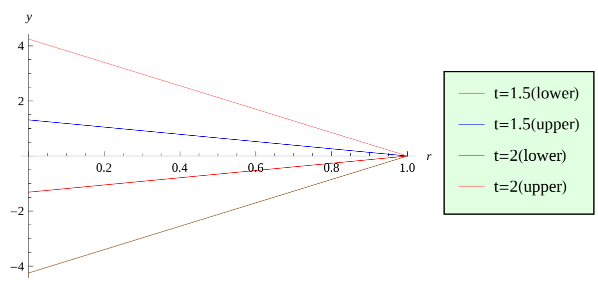

FIGURE 5. $\underline{y}(t,r)$ and $\overline{y}(t,r)$ for different $t \in [0, \frac{1}{4}] \cup [\frac{1}{4}(2+\sqrt{2}), 2]$.

7. Conclusion

In this paper, we have shown that the differential transform method can be successfully applied for the (1,1) and (2,2) solutions of the second order two-point fuzzy boundary value problems and (1,1,1) and (2,2,2) solutions of the third order three-point fuzzy boundary value problems. Construction of numerical methods for finding all kind of solutions of fuzzy boundary value problems will be considered in future.

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