Neural, Parallel, and Scientific Computations 22 (2014) 89-126

SECOND ORDER STATE AND COVARIANCE ESTIMATION FOR NONLINEAR STOCHASTIC SYSTEMS

OLUSEGUN M. OTUNUGA AND GANGARAM S. LADDE

Department of Mathematics and Statistics, University of South Florida
4202 E. Fowler Avenue, CMC 342, Tampa, FL 33620, USA

ABSTRACT. The Kalman Filter is one of the most commonly used methods in state and parameter estimation problems in linear and nonlinear stochastic systems analysis. This paper deals with further extensions of the existing Extended Kalman Filter (EKF) approach to study state and parameter estimation problems of nonlinear system of stochastic differential equations. The main focus in this paper is to reduce the magnitude of errors in the existing Kalman Filter schemes. This is achieved by approximating the state conditional mean and covariances using a second degree polynomials. In addition, the drift, diffusion and the observations are approximated using a multi-dimensional extension of Sterling's interpolation formula.

AMS (MOS) Subject Classification. 91G60, 91G80, 93E03, 93E10, 93E11.

Keywords: Kalman Filter, Stirling Interpolation, Nonlinear, Algorithm, Taylor's Series

1. Introduction

The Kalman Filter is a powerful and widely used technique in state and parameter estimation problems. It is used for finding minimum mean squared error (MMSE) estimation of linear state dynamic systems and observations [27]. Nonlinear state dynamic and observations are estimated by employing the Extended Kalman Filter (EKF) scheme [27]. Moreover, the EKF scheme deals with state and parameter estimation of linearized version of both nonlinear state dynamic and observations [15]. It is well known [20] that the linearized Taylor scheme does not provide sufficiently accurate representation. Moreover, due to its overly crude approximation, the scheme generates problem in convergence [20].

Several other approaches have been made to find a better filter than the EKF scheme. Unlike the usual EKF approach, Magnus [20], Tor Steinar [30] and Luo [16] propose a new set of estimators which are based on polynomial approximations of the nonlinear transformations using the Stirling's interpolation formula. Under this scheme, derivatives of rate functions are avoided due to interpolation approximation formula. As discussed in [20], the Stirling's interpolation formula accommodates easy implementation of the filters and enables state estimation when the derivatives are

Received March 15, 2014

not smooth. It has been remarked that this approach provides a similar, or superior performance than the existing EKF approach. Simon Julier [25, 26, 27] claims that the EKF filtering strategy is difficult to implement, difficult to tune, and only reliable for systems which are almost linear. This leads to the development of a new linear state and covariance estimator using unscented transformation. The new scheme was claimed to be superior than that of the EKF, and, in fact, the scheme generalizes elegantly to the nonlinear system without the linear step required by the EKF scheme. Higher filters have also been discussed by Jazwinski, [10], Maybeck, [21], and Madsen etal [17].

Our main focus in this paper is to reduce the magnitude of error that occurs during the estimation process of the EKF approach. This error is due to the overly simplified approximation scheme. In the process of the error reduction, we modified the Extended Kalman Filter scheme by incorporating second order polynomial approximation for the expected state variable and covariance. This scheme is applied to study the state and parameter estimation problems of nonlinear system of stochastic differential equation. The drift and diffusion part of the nonlinear differential equations are approximated using the Stirling's interpolation formula [20]. This modified approach estimates the parameters of a system of nonlinear stochastic differential equation with lesser magnitude of error compared to the usual EKF approach [15]. Although the magnitude of error in the state and covariance of the EKF is reduced, it is however important to note that our scheme is computationally too demanding/computer intensive. An algorithm is developed to implement this scheme. The extended Kalman filter approach is compared with the developed modified extended Kalman filtering approach. The scheme is applied to Henry Hub natural gas data and to estimate parameters. The details are exhibited in the graph.

The organization of this paper is as follows:

In Section 2, we present a modified EKF scheme. In section 3, we applied the scheme to estimate the parameters for a stochastic dynamic model for Henry Hub Natural gas.

2. Modified Extended Kalman Filter Approach

In this section, we present a modified Kalman filter parameter estimation scheme for nonlinear stochastic dynamic systems. This is accomplished by approximating the state estimator using a quadratic approximation. The Kalman Filter Approach is modified by employing a second order approximation for state and state variance predictions. To estimate the parameters, we minimized the likelihood function of the prediction error of the measurement process. The approach is described below.

We assume that a dynamic state $x \in \Re^n$ and its observation data $y \in \Re^n$ are described by a general non-linear stochastic dynamic systems.

$$\begin{cases} dx &= \mathbf{f}(x;\boldsymbol{\theta})dt + \mathbf{g}(x;\boldsymbol{\theta})d\mathbf{W}(t), \quad x(t_0) = x_0 \\ y(t) &= \mathbf{h}(x;\boldsymbol{\theta}) + \mathbf{v}(t), \end{cases} \tag{2.1}$$

where x_0 is a stochastic initial condition satisfying $E|\mathbf{x}_0|^2 < \infty$, $\mathbf{f} : \Re^n \times \Re^p \to \Re^n$, $\mathbf{g} : \Re^n \times \Re^p \to \Re^{n\times d}$, $\mathbf{h} : \Re^n \times \Re^p \to \Re^n$ are continuous functions, $\mathbf{W} : \Re \to \Re^d$ is a $d-$ dimensional standard Wiener process on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F})_{t\geq 0}, \mathbb{P})$, the filtration function $(\mathcal{F})_{t\geq 0}$ is right-continuous, and each $\mathcal{F}_t$ with $t \geq 0$ contains all $\mathbb{P}$-null sets in $\mathcal{F}$, x is $\mathcal{F}_t$ adapted process and non-anticipative, and $v : \Re \to \Re^n$ is a n-dimensional zero mean Gaussian white noise process independent of $\mathbf{W}$, $\theta \in \Theta$, the parameter space.

Prior to presenting a procedure for the estimation of parameters, we define the following terminologies and notations used throughout this work.

Define

$$Y_{t_k} = \{y_{t_1}, y_{t_2}, \ldots, y_{t_k}\}, \tag{2.2}$$

as all observations of the data given up to time t_k.

(2.3)

$$\begin{aligned}
\hat{y}(t \mid t_{k-1}) &= \mathbb{E}\left[y(t) \mid Y_{t_{k-1}}\right], \\
\hat{x}(t \mid t_{k-1}) &= \mathbb{E}\left[x(t) \mid Y_{t_{k-1}}\right], \\
P(t \mid t_{k-1}) &= \mathbb{E}\left[(x(t) - \hat{x}(t \mid t_{k-1}))(x(t) - \hat{x}(t \mid t_{k-1}))^T \mid Y_{t_{k-1}}\right], \\
R(t \mid t_{k-1}) &= \mathbb{E}\left[v(t)v^T(t) \mid Y_{t_{k-1}}\right], \\
r_{0,2}(t \mid t_{k-1}) &= \mathbb{E}\left[(y(t) - \hat{y}(t \mid t_{k-1}))(y(t) - \hat{y}(t \mid t_{k-1}))^T \mid Y_{t_{k-1}}\right], \\
r_{1,1}(t \mid t_{k-1}) &= \mathbb{E}\left[(x(t) - \hat{x}(t \mid t_{k-1}))(y(t) - \hat{y}(t \mid t_{k-1}))^T \mid Y_{t_{k-1}}\right], \\
r_{2,2}(t \mid t_{k-1}) &= \mathbb{E}\Big[(x(t) - \hat{x}(t \mid t_{k-1}))\,(x(t) - \hat{x}(t \mid t_{k-1}))^T\,(y(t) - \hat{y}(t \mid t_{k-1}))\times \\
&\qquad (y(t) - \hat{y}(t \mid t_{k-1}))^T \mid Y_{k-1}\Big], \\
r_{1,2}(t \mid t_{k-1}) &= \mathbb{E}\Big[x(t)(y(t) - \hat{y}(t \mid t_{k-1}))^T(\mathbb{Y}(t) - \hat{\mathbb{Y}}(t \mid t_{k-1}))^T \mid Y_{t_{k-1}}\Big], \\
r_{0,3}(t \mid t_{k-1}) &= \mathbb{E}\Big[(y(t) - \hat{y}(t \mid t_{k-1}))(y(t) - \hat{y}(t \mid t_{k-1}))^T(\mathbb{Y}(t) - \hat{\mathbb{Y}}(t \mid t_{k-1}))^T \mid Y_{t_{k-1}}\Big], \\
r_{1,3}(t \mid t_{k-1}) &= \mathbb{E}\Big[(x(t) - \hat{x}(t \mid t_{k-1}))(y(t) - \hat{y}(t \mid t_{k-1}))^T(y(t) - \hat{y}(t \mid t_{k-1}))\times \\
&\qquad (y(t) - \hat{y}(t \mid t_{k-1}))^T \mid Y_{t_{k-1}}\Big] \\
r_{0,4}(t \mid t_{k-1}) &= \mathbb{E}\Big[(y(t) - \hat{y}(t \mid t_{k-1}))(y(t) - \hat{y}(t \mid t_{k-1}))^T(y(t) - \hat{y}(t \mid t_{k-1}))\times \\
&\qquad (y(t) - \hat{y}(t \mid t_{k-1}))^T \mid Y_{t_{k-1}}\Big], \\
M_{0,2}(t \mid t_{k-1}) &= \mathbb{E}\Big[(\mathbb{Y}(t) - \hat{\mathbb{Y}}(t \mid t_{k-1}))(y(t) - \hat{y}(t \mid t_{k-1}))(y(t) - \hat{y}(t \mid t_{k-1}))^T\times
\end{aligned}$$

$$(\mathbb{Y}(t) - \hat{\mathbb{Y}}(t \mid t_{k-1}))^T \mid Y_{t_{k-1}}\Big],$$

$$\sigma_{Y2}(t \mid t_{k-1}) = \mathbb{E}\left[(\mathbb{Y}(t) - \hat{\mathbb{Y}}(t \mid t_{k-1}))(y(t) - \hat{y}(t \mid t_{k-1})) \mid Y_{t_{k-1}}\right],$$

where

(2.4)

$$\mathbb{Y}(t) - \hat{\mathbb{Y}}(t \mid t_{k-1}) = \begin{pmatrix} \mathbf{y}(t) - \hat{\mathbf{y}}(t \mid t_{k-1}) & 0 & \dots & 0 \\ 0 & \mathbf{y}(t) - \hat{\mathbf{y}}(t \mid t_{k-1}) & \dots & 0 \\ \vdots & 0 & \ddots & \vdots \\ 0 & \dots & 0 & \mathbf{y}(t) - \hat{\mathbf{y}}(t \mid t_{k-1}) \end{pmatrix}_{n^2 \times n}$$

Let $\hat{x}(t_k \mid t_{k-1})$ be the a-priori state estimate at step k given the knowledge of process $Y_{t_{k-1}}$, and $\hat{x}(t_k \mid t_k)$ be the posterior state estimate at step k given the knowledge of process Y_{t_k}. The Extended Kalman Filter approach begins with the goal of computing a-posterior state estimate $\hat{x}(t_k \mid t_k)$ as a linearized approximation of the form

$$\begin{cases} \hat{x}(t_k \mid t_k) &= A_0 + A_1(y(t_k) - \hat{y}(t_k \mid t_{k-1})), \\ P(t_k \mid t_k) &= B_0. \end{cases} \tag{2.5}$$

It was shown in Jazwinski [10] that

$$\begin{aligned} A_0(t_k \mid t_{k-1}) &= \hat{x}(t_k \mid t_{k-1}), \\ A_1(t_k \mid t_{k-1}) &= r_{1,1}(t_k \mid t_{k-1}) r_{0,2}^{-1}(t_k \mid t_{k-1}), \\ B_0(t_k \mid t_{k-1}) &= P(t_k \mid t_{k-1}) - A_1(t_k \mid t_{k-1}) r_{0,2}(t_k \mid t_{k-1}) A_1^T(t_k \mid t_{k-1}), \end{aligned} \tag{2.6}$$

where A_1 is the Kalman gain. Instead of approximating the conditional covariance at an observation as a constant, Jazwinski [10] extended it to an approximation of order one.

For the rest of this paper, for the sake of simplicity, we write $\mathbf{f}(x) = \mathbf{f}(x; \boldsymbol{\theta})$, $\mathbf{g}(x) = \mathbf{g}(x; \boldsymbol{\theta})$, and $\mathbf{h}(x) = \mathbf{h}(x; \boldsymbol{\theta})$. In this paper, we extend the approximate equations for the conditional mean and covariance at an observation to that of order two. To do this, we first state the Taylors series expansion of a vector value function $\mathbf{f}$ about the vector $\hat{\mathbf{y}}$,

$$\mathbf{f}(\mathbf{y}) = \mathbf{f}(\hat{\mathbf{y}}) + \frac{\partial \mathbf{f}(\hat{\mathbf{y}})}{\partial \mathbf{y}}(\mathbf{y} - \hat{\mathbf{y}}) + \frac{1}{2}\frac{\partial^2 \mathbf{f}(\hat{\mathbf{y}})}{\partial \mathbf{y}^2} \mathrm{diag}(\mathbf{y} - \hat{\mathbf{y}}, \mathbf{y} - \hat{\mathbf{y}}, \dots, \mathbf{y} - \hat{\mathbf{y}})(\mathbf{y} - \hat{\mathbf{y}}), \tag{2.7}$$

where $\frac{\partial \mathbf{f}(\mathbf{y})}{\partial \mathbf{y}} = \begin{pmatrix} \frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial y_2} & \dots & \frac{\partial f_1}{\partial y_n} \\ \frac{\partial f_2}{\partial y_1} & \frac{\partial f_2}{\partial y_2} & \dots & \frac{\partial f_2}{\partial y_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial y_1} & \frac{\partial f_n}{\partial y_2} & \dots & \frac{\partial f_n}{\partial y_n} \end{pmatrix}_{n \times n}$, $\frac{\partial^2 \mathbf{f}(\mathbf{y})}{\partial \mathbf{y}^2} = \begin{pmatrix} \frac{\partial^2 f_1}{\partial \mathbf{y} \partial y_1} & \frac{\partial^2 f_1}{\partial \mathbf{y} \partial y_2} & \dots & \frac{\partial^2 f_1}{\partial \mathbf{y} \partial y_n} \\ \frac{\partial^2 f_2}{\partial \mathbf{y} \partial y_1} & \frac{\partial^2 f_2}{\partial \mathbf{y} \partial y_2} & \dots & \frac{\partial^2 f_2}{\partial \mathbf{y} \partial y_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f_n}{\partial \mathbf{y} \partial y_1} & \frac{\partial^2 f_n}{\partial \mathbf{y} \partial y_2} & \dots & \frac{\partial^2 f_n}{\partial \mathbf{y} \partial y_n} \end{pmatrix}_{n \times n^2}$,

$\mathrm{diag}(\mathbf{y} - \hat{\mathbf{y}}, \dots, \mathbf{y} - \hat{\mathbf{y}}) = \mathbb{Y}(t) - \hat{\mathbb{Y}}(t \mid t_{k-1})$.

We note that $\frac{\partial^2 \mathbf{f}(\mathbf{y})}{\partial \mathbf{y}^2}$ and $\mathbb{Y}(t) - \hat{\mathbb{Y}}(t \mid t_{k-1})$ are $n \times n$ block matrices whose entries are n-dimensional row vectors and column vectors, respectively. Moreover, $\frac{\partial^2 \mathbf{f}(\mathbf{y})}{\partial \mathbf{y}^2}$ is referred to as Hessian matrix, and $\mathbb{Y}(t) - \hat{\mathbb{Y}}(t \mid t_{k-1})$ is a diagonal matrix defined in (2.4).

Following these definitions and notations, we define the a-posterior state estimate $\hat{x}(t_k \mid t_k)$ and a-posterior covariance estimate $P(t_k \mid t_k)$ as a quadratic approximation of the form

$$
\begin{aligned}
\hat{x}(t_k \mid t_k) &= A_0 + A_1(y(t_k) - \hat{y}(t_k \mid t_{k-1})) \\
&\quad + A_2(\mathbb{Y}(t) - \hat{\mathbb{Y}}(t \mid t_{k-1}))(y(t_k) - \hat{y}(t_k \mid t_{k-1})) \\
P(t_k \mid t_k) &= B_0 + B_1(y(t_k) - \hat{y}(t_k \mid t_{k-1}))(y(t_k) - \hat{y}(t_k \mid t_{k-1}))^T,
\end{aligned} \tag{2.8}
$$

where A_0 is an $n \times 1$ matrix (column vector), A_1 is an $n \times n$ matrix, A_2 is an $n \times n$ block matrix whose entries are $1 \times n$ matrix (row vector), B_0 and B_1 are square $n \times n$ matrices.

In order to develop an algorithm for $\hat{x}(t_k \mid t_k)$ and $P(t_k \mid t_k)$, we need to solve for the A_i and B_i for $i = 0, 1$, A_2. For this purpose, we need to evaluate each quantity in equation (2.3). We use the multi-dimensional extension of Stirling's interpolation formula discussed in Magnus [20] and Luo [16] to approximate the state drift, diffusion and the observation functions in equation (2.1) up to the second order.

Using the second-order polynomials, we define the multidimensional interpolation formula as

$$
\begin{aligned}
\mathbf{f}(x) &= \mathbf{f}(\hat{x}) + \tilde{D}_{\Delta x}\mathbf{f}(\hat{x}) + \tfrac{1}{2}\tilde{D}^2_{\Delta x}\mathbf{f}(\hat{x}), \\
\mathbf{g}(x) &= \mathbf{g}(\hat{x}) + \tilde{D}_{\Delta x}\mathbf{g}(\hat{x}) + \tfrac{1}{2}\tilde{D}^2_{\Delta x}\mathbf{g}(\hat{x}), \\
\mathbf{h}(x) &= \mathbf{h}(\hat{x}) + \tilde{D}_{\Delta x}\mathbf{h}(\hat{x}) + \tfrac{1}{2}\tilde{D}^2_{\Delta x}\mathbf{h}(\hat{x}),
\end{aligned} \tag{2.9}
$$

where the operator $\tilde{D}_{\Delta x}$, and $\tilde{D}^2_{\Delta x}$ are described in [20] and are defined by

$$
\begin{aligned}
\tilde{D}_{\Delta x} &= \frac{1}{h}\left(\sum_{p=1}^{n} \Delta x_p \mu_p \delta_p\right), \\
\tilde{D}^2_{\Delta x} &= \frac{1}{h^2}\left(\sum_{p=1}^{n} \Delta x_p^2 \delta_p^2 + \sum_{p=1}^{n}\sum_{\substack{q=1 \\ q \neq p}}^{n} \Delta x_p \Delta x_q (\mu_p \delta_p)(\mu_q \delta_q)\right),
\end{aligned} \tag{2.10}
$$

where Δx, δ_p and μ_p are defined by

$$
\begin{aligned}
\Delta x &= x - \hat{x}, \\
\delta_p \mathbf{f}(\hat{x}) &= \mathbf{f}\left(\hat{x} + \tfrac{h}{2}e_p\right) - \mathbf{f}\left(\hat{x} - \tfrac{h}{2}e_p\right), \\
\mu_p \mathbf{f}(\hat{x}) &= \tfrac{1}{2}\left[\mathbf{f}\left(\hat{x} + \tfrac{h}{2}e_p\right) + \mathbf{f}\left(\hat{x} - \tfrac{h}{2}e_p\right)\right],
\end{aligned} \tag{2.11}
$$

and $h > 0$ is the step size, e_p is the pth unit vector.

Using the Cholesky transformation, we transform x to a variable z which is mutually uncorrelated. Following [20], we write

$$\begin{aligned} z &= S_x^{-1}x, \\ \tilde{\mathbf{f}}(z) &= \mathbf{f}(S_x z) = \mathbf{f}(x). \end{aligned} \tag{2.12}$$

From (2.12), (2.9) reduces to

$$\begin{aligned} \tilde{\mathbf{f}}(z) &= \tilde{\mathbf{f}}(\hat{z}) + \tilde{D}_{\Delta z}\tilde{\mathbf{f}}(\hat{z}) + \tfrac{1}{2}\tilde{D}^2_{\Delta z}\tilde{\mathbf{f}}(\hat{z}), \\ \tilde{\mathbf{g}}(z) &= \tilde{\mathbf{g}}(\hat{z}) + \tilde{D}_{\Delta z}\tilde{\mathbf{g}}(\hat{z}) + \tfrac{1}{2}\tilde{D}^2_{\Delta z}\tilde{\mathbf{g}}(\hat{z}), \\ \tilde{\mathbf{h}}(z) &= \tilde{\mathbf{h}}(\hat{z}) + \tilde{D}_{\Delta z}\tilde{\mathbf{h}}(\hat{z}) + \tfrac{1}{2}\tilde{D}^2_{\Delta z}\tilde{\mathbf{h}}(\hat{z}). \end{aligned} \tag{2.13}$$

Let σ_i represent the ith moment of an arbitrary element in Δz. We shall use the interpolation approximations (2.13) to evaluate the expressions in (2.3). For this purpose, we prove the following Lemma.

Assumption $\mathcal{B}$:

As discussed in Magnus [20], we assume Δz to be iid Gaussian. Hence,

$$\sigma_{2i-1} = 0, \quad i \in \mathbb{N}. \tag{2.14}$$

Lemma 2.1. *Under the Assumption $\mathcal{B}$, we have*

$$\begin{aligned}
r_{0,2}(t_k \mid t_{k-1}) &= \frac{\sigma_2}{h^2}\sum_{p=1}^{n}\mu_p\delta_p\tilde{\boldsymbol{h}}(\hat{z})\mu_p\delta_p\tilde{\boldsymbol{h}}(\hat{z})^T + \frac{\sigma_4-\sigma_2^2}{4h^4}\sum_{p=1}^{n}\delta_p^2\tilde{\boldsymbol{h}}(\hat{z})\delta_p^2\tilde{\boldsymbol{h}}(\hat{z})^T \\
&\quad + \frac{\sigma_2^2}{4h^4}\sum_{\substack{p,q=1\\ q\neq p}}^{n}\mu_p\delta_p\mu_q\delta_q\tilde{\boldsymbol{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\boldsymbol{h}}^T + R \\
r_{1,1}(t_k \mid t_{k-1}) &= \frac{\sigma_2}{h}\sum_{p=1}^{n}S_x\left(\mu_p\delta_p\tilde{\boldsymbol{h}}(\hat{z})\right)^T \\
r_{1,2}(t_k \mid t_{k-1}) &= S_x\left(D(t_k \mid t_{k-1})_{\{\tilde{\boldsymbol{h}}_i,\tilde{\boldsymbol{h}}^T\}}\right)_{1\le i\le n} + \hat{x}(t_k \mid t_{k-1})r_{0,2}(t_k \mid t_{k-1})_{\{\tilde{\boldsymbol{h}}_i,\tilde{\boldsymbol{h}}^T\}} \\
r_{1,3}(t_k \mid t_{k-1}) &= S_xJ - 2S_xE(t_k \mid t_{k-1}) - 2r_{1,1}C(t_k \mid t_{k-1})C^T - r_{1,1}C^TC(t_k \mid t_{k-1}) \\
&\quad - S_xD(t_k \mid t_{k-1})_{\{\tilde{\boldsymbol{h}}^T,\tilde{\boldsymbol{h}}\}}C^T, \\
r_{2,2}(t_k \mid t_{k-1}) &= S_x\left(Q_{i,j}\right)_{\substack{1\le i\le n\\ 1\le j\le n}} \\
r_{0,3}(t_k \mid t_{k-1}) &= \left(L_{i,j}\right)_{\substack{1\le i\le n\\ 1\le j\le n}} \\
r_{0,4}(t_k \mid t_{k-1}) &= \mathbb{E}\left[\mathcal{A}\mathcal{A}^T\mathcal{A}\mathcal{A}^T \mid Y_{t_{k-1}}\right] - \mathbb{E}\left[\mathcal{A}\mathcal{A}^T\mathcal{A}C^T \mid Y_{t_{k-1}}\right] - \mathbb{E}\left[\mathcal{A}\mathcal{A}^TC\mathcal{A}^T \mid Y_{t_{k-1}}\right] \\
&\quad + \mathbb{E}\left[\mathcal{A}\mathcal{A}^TCC^T \mid Y_{t_{k-1}}\right] - \mathbb{E}\left[\mathcal{A}C^T\mathcal{A}\mathcal{A}^T \mid Y_{t_{k-1}}\right] + \mathbb{E}\left[\mathcal{A}C^T\mathcal{A}C^T \mid Y_{t_{k-1}}\right] \\
&\quad + \mathbb{E}\left[\mathcal{A}C^TC\mathcal{A}^T\right] - \mathbb{E}\left[\mathcal{A}C^TCC^T \mid Y_{t_{k-1}}\right] \\
\sigma_{Y^2}(t_k \mid t_{k-1}) &= \left(F_i\right)_{1\le i\le n},
\end{aligned}$$

where

$$\mathcal{A} = \tilde{D}_{\Delta z}\tilde{\boldsymbol{h}}(\hat{z}) + \frac{1}{2}\tilde{D}^2_{\Delta z}\tilde{\boldsymbol{h}}(\hat{z}) + v$$

$$
\begin{aligned}
C &= \frac{\sigma_2}{2h^2}\sum_{p=1}^{n}\delta_p^2\tilde{\boldsymbol{h}}(\hat{z}),\\
F_i &= \frac{\sigma_2}{h^2}\sum_{p=1}^{n}\left(\mu_p\delta_p\tilde{\boldsymbol{h}}(\hat{z})\right)\mu_p\delta_p\tilde{\boldsymbol{h}}_i(\hat{z}) + \frac{\sigma_4-\sigma_2^2}{4h^4}\sum_{p=1}^{n}\left(\delta_p^2\tilde{\boldsymbol{h}}(\hat{z})\right)\delta_p^2\tilde{\boldsymbol{h}}_i(\hat{z})\\
&\quad + \frac{\sigma_2^2}{4h^4}\sum_{p=1}^{n}\sum_{\substack{q=1\\ q\neq p}}^{n}\left(\mu_p\delta_p\mu_q\delta_q\tilde{\boldsymbol{h}}(\hat{z})\right)\mu_p\delta_p\mu_q\delta_q\tilde{\boldsymbol{h}}_i(\hat{z}) + e_iR_{i,i}\\
r_{0,2}(t_k\mid t_{k-1})_{\{\tilde{\boldsymbol{h}}_i,\tilde{\boldsymbol{h}}^T\}} &= \Bigg(\frac{\sigma_2}{h^2}\sum_{p=1}^{n}\mu_p\delta_p\tilde{\boldsymbol{h}}_i(\hat{z})\mu_p\delta_p\tilde{\boldsymbol{h}}^T(\hat{z}) + \frac{\sigma_4-\sigma_2^2}{4h^4}\sum_{p=1}^{n}\delta_p^2\tilde{\boldsymbol{h}}_i(\hat{z})\delta_p^2\tilde{\boldsymbol{h}}^T(\hat{z})\\
&\quad + \frac{\sigma_2^2}{4h^4}\sum_{\substack{p,q=1\\ q\neq p}}^{n}\mu_p\delta_p\mu_q\delta_q\tilde{\boldsymbol{h}}_i\mu_p\delta_p\mu_q\delta_q\tilde{\boldsymbol{h}}^T + R_{i,i}e^T\Bigg)_{1\le i\le n},
\end{aligned}
$$

$\tilde{\boldsymbol{h}} \equiv \tilde{\boldsymbol{h}}(\hat{z}) = \tilde{\boldsymbol{h}}(\hat{z}(t_k \mid t_{k-1}))$, $r_{0,2}(t_k \mid t_{k-1}) = r_{0,2}(t_k \mid t_{k-1})_{\{\tilde{\boldsymbol{h}},\tilde{\boldsymbol{h}}^T\}}$, and detailed expressions for $r_{0,2}(t_k \mid t_{k-1})_{\{\tilde{\boldsymbol{h}},\tilde{\boldsymbol{h}}^T\}}$, $J(t_k \mid t_{k-1})$, $E(t_k \mid t_{k-1})$, $D(t_k \mid t_{k-1})_{\{\tilde{\boldsymbol{h}},\tilde{\boldsymbol{h}}^T\}}$, $D(t_k \mid t_{k-1})_{\{\tilde{\boldsymbol{h}}^T,\tilde{\boldsymbol{h}}\}}$, $D(t_k \mid t_{k-1})_{\{\tilde{\boldsymbol{h}}_i,\tilde{\boldsymbol{h}}^T\}}$, $Q_{i,j}$, $\mathbb{E}\left[\mathcal{A}\mathcal{A}^T\mathcal{A}\mathcal{A}^T \mid Y_{t_{k-1}}\right]$, $\mathbb{E}\left[\mathcal{A}\mathcal{A}^T\mathcal{A}C^T \mid Y_{t_{k-1}}\right]$, $\mathbb{E}\left[\mathcal{A}\mathcal{A}^TC\mathcal{A}^T \mid Y_{t_{k-1}}\right]$, $\mathbb{E}\left[\mathcal{A}\mathcal{A}^TCC^T \mid Y_{t_{k-1}}\right]$, $\mathbb{E}\left[\mathcal{A}C^T\mathcal{A}\mathcal{A}^T \mid Y_{t_{k-1}}\right]$, $\mathbb{E}\left[\mathcal{A}C^T\mathcal{A}C^T \mid Y_{t_{k-1}}\right]$, $\mathbb{E}\left[\mathcal{A}C^TC\mathcal{A}^T \mid Y_{t_{k-1}}\right]$, $\mathbb{E}\left[\mathcal{A}C^TCC^T \mid Y_{t_{k-1}}\right]$, and $L_{i,j}$ are given in Appendix A.

Proof. The proof is given in Appendix B. □

We can now use these values to solve for A_i, B_i, $i = 0, 1$, and A_2. The first step in the algorithm is to solve for A_i, B_i, $i = 0, 1$ and A_2 in (2.8). For this purpose, we use the following Lemma by following the description of the moment propagation procedure across the observations described in Jazwinski [10].

Lemma 2.2. *Under the assumptions in Lemma* 2.1*, we have*

$$
\begin{aligned}
&A_0(t_k \mid t_{k-1}) = r_{1,0}(t_k \mid t_{k-1}) - A_2(t_k \mid t_{k-1})\sigma_{Y2}(t_k \mid t_{k-1}),\\
&A_1(t_k \mid t_{k-1}) = \left[r_{1,1}(t_k \mid t_{k-1}) - A_2(t_k \mid t_{k-1})r_{0,3}(t_k \mid t_{k-1})^T\right] r_{0,2}(t_k \mid t_{k-1})^{-1},\\
&A_2(t_k \mid t_{k-1}) = T_1(t_k \mid t_{k-1})T_2^{-1}(t_k \mid t_{k-1}),
\end{aligned}
\tag{2.15}
$$

where

$$
\begin{aligned}
T_1(t_k \mid t_{k-1}) &= r_{1,2}(t_k \mid t_{k-1}) - r_{1,0}(t_k \mid t_{k-1})\sigma_{Y2}^T(t_k \mid t_{k-1})\\
&\quad - r_{1,1}(t_k \mid t_{k-1})r_{0,2}^{-1}(t_k \mid t_{k-1})r_{0,3}(t_k \mid t_{k-1})\\
T_2(t_k \mid t_{k-1}) &= M_{0,2}(t_k \mid t_{k-1}) - \sigma_{Y2}(t_k \mid t_{k-1})\sigma_{Y2}^T(t_k \mid t_{k-1})\\
&\quad - r_{0,3}^T(t_k \mid t_{k-1})r_{0,2}^{-1}(t_k \mid t_{k-1})r_{0,3}(t_k \mid t_{k-1}).
\end{aligned}
$$

Proof. Proof is in Appendix C. □

Remark 2.3. If $A_2 = 0$, (2.2) reduces to

$$\begin{aligned} A_0 &= \hat{x}(t_k \mid t_{k-1}) \\ A_1 &= r_{1,1}(t_k \mid t_{k-1}) r_{0,2}(t_k \mid t_{k-1})^{-1}. \end{aligned}$$

Now, we present a Lemma for finding B_0 and B_1.

Lemma 2.4. *Under the assumptions in Lemma* 2.1*, we have*

$$\begin{aligned} B_1 = {} & \left(N_2(t_k \mid t_{k-1}) r_{0,2}^{-1}(t_k \mid t_{k-1}) - N_1(t_k \mid t_{k-1})\right) \left[r_{0,4}(t_k \mid t_{k-1}) r_{0,2}^{-1}(t_k \mid t_{k-1}) \right. \\ & \left. - r_{0,2}(t_k \mid t_{k-1})\right]^{-1} \\ B_0 = {} & N_1(t_k \mid t_{k-1}) - B_1(t_k \mid t_{k-1}) r_{0,2}(t_k \mid t_{k-1}), \end{aligned} \tag{2.16}$$

where

$$\begin{aligned} N_1 &= \mathbb{E}\left[(x(t_k) - \hat{x}(t_k \mid t_k))(x(t_k) - \hat{x}(t_k \mid t_k))^T \mid Y_{t_{k-1}}\right] \\ &= P(t_k \mid t_{k-1}) - r_{1,1}(t_k \mid t_{k-1}) A_1^T - r_{1,2}(t_k \mid t_{k-1}) A_2^T - A_1 r_{1,1}(t_k \mid t_{k-1})^T \\ &\quad - A_2 r_{1,2}(t_k \mid t_{k-1}))^T + (\hat{x}(t_k \mid t_{k-1}) - A_0)(\hat{x}(t_k \mid t_{k-1}) - A_0)^T \\ &\quad - (\hat{x}(t_k \mid t_{k-1}) - A_0) r_{0,2}(t_k \mid t_{k-1})_{\{\tilde{\boldsymbol{h}}_i, \tilde{\boldsymbol{h}}^T\}} A_2 \\ &\quad + A_1 r_{0,2}(t_k \mid t_{k-1}) A_1^T + A_1 r_{0,2}(t_k \mid t_{k-1})_{\{\tilde{\boldsymbol{h}}_i, \tilde{\boldsymbol{h}}^T\}} A_2^T \\ &\quad - A_2 r_{0,2}(t_k \mid t_{k-1})_{\{\tilde{\boldsymbol{h}}_i, \tilde{\boldsymbol{h}}^T\}} (\hat{x}(t_k \mid t_{k-1}) - A_0) \\ &\quad + A_2 r_{0,3}(t_k \mid t_{k-1}) A_1 + A_2 M_{0,2}(t_k \mid t_{k-1}) A_2^T \end{aligned}$$

$$\begin{aligned} N_2 &= \mathbb{E}\left[(x(t_k) - \hat{x}(t_k \mid t_k))(x(t_k) - \hat{x}(t_k \mid t_k))^T (y(t_k) - \hat{y}(t_k \mid t_k)) \right. \\ &\quad \left. \times (y(t_k) - \hat{y}(t_k \mid t_k))^T \mid Y_{t_{k-1}}\right] \\ &= \mathbb{E}\left(\left[(x(t_k) - A_0)(x(t_k) - A_0)^T - (x(t_k) - A_0)(y(t_k) - \hat{y}(t_k \mid t_{k-1}))^T A_1^T \right.\right. \\ &\quad - (x(t_k) - A_0)(y(t_k) - \hat{y}(t_k \mid t_{k-1}))^T (\mathbb{Y}(t_k) - \hat{\mathbb{Y}}(t_k \mid t_{k-1}))^T A_2 \\ &\quad - A_1 (y(t_k) - \hat{y}(t_k \mid t_{k-1}))(x(t_k) - A_0)^T \\ &\quad \left. + A_1 (y(t_k) - \hat{y}(t_k \mid t_{k-1}))(y(t_k) - \hat{y}(t_k \mid t_{k-1}))^T A_1^T\right] \\ &\quad \left. \times (y(t_k) - \hat{y}(t_k \mid t_{k-1}))(y(t_k) - \hat{y}(t_k \mid t_{k-1}))^T \mid Y_{t_{k-1}}\right) \\ &= r_{2,2} + (\hat{x}(t_k \mid t_{k-1}) - A_0)(\hat{x}(t_k \mid t_{k-1}) - A_0)^T r_{0,2} \\ &\quad - \mathbb{E}\left[(x(t_k) - \hat{x}(t_k \mid t_{k-1}))(y(t_k) - \hat{y}(t_k \mid t_{k-1}))^T A_1^T (y(t_k) - \hat{y}(t_k \mid t_{k-1})) \right. \\ &\quad \left. \times (y(t_k) - \hat{y}(t_k \mid t_{k-1}))^T \mid Y_{k-1}\right] \\ &\quad - \mathbb{E}\left[(x(t_k \mid t_{k-1}) - A_0)(y(t_k) - \hat{y}(t_k \mid t_{k-1}))^T A_1^T (y(t_k) - \hat{y}(t_k \mid t_{k-1})) \right. \\ &\quad \left. \times (y(t_k) - \hat{y}(t_k \mid t_{k-1}))^T \mid Y_{t_{k-1}}\right] \\ &\quad - \mathbb{E}\left[A_1 (y(t_k) - \hat{y}(t_k \mid t_{k-1}))(x(t_k) - \hat{x}(t_k \mid t_{k-1}))^T (y(t_k) - \hat{y}(t_k \mid t_{k-1})) \right. \\ &\quad \left. \times (y(t_k) - \hat{y}(t_k \mid t_{k-1}))^T \mid Y_{t_{k-1}}\right] \end{aligned}$$

$$-\mathbb{E}\left[A_1(y(t_k)-\hat{y}(t_k \mid t_{k-1}))(x(t_k \mid t_{k-1})-A_0)^T(y(t_k)-\hat{y}(t_k \mid t_{k-1}))\right.$$
$$\left.\times\,(y(t_k)-\hat{y}(t_k \mid t_{k-1}))^T \mid Y_{t_{k-1}}\right]$$
$$+\mathbb{E}\left[A_1(y(t_k)-\hat{y}(t_k \mid t_{k-1}))(y(t_k)-\hat{y}(t_k \mid t_{k-1}))^T A_1^T(y(t_k)-\hat{y}(t_k \mid t_{k-1}))\right.$$
$$\left.\times\,(y(t_k)-\hat{y}(t_k \mid t_{k-1}))^T \mid Y_{t_{k-1}}\right],$$

and $\hat{x}(t_k \mid t_k)$ *is given in equation* (2.8).

Proof. The proof is shown in Appendix D. □

Remark 2.5. If $B_1 = 0$,and $A_2 = 0$, then from (2.16) and Remark 2.3, we have

$$\begin{aligned} A_0 &= r_{1,0}(t_k \mid t_{k-1}) = \hat{x}(t_k \mid t_{k-1}), \\ A_1 &= r_{1,1}(t_k \mid t_{k-1})r_{0,2}(t_k \mid t_{k-1})^{-1} \\ B_0 &= N_1 = P_{t_k|t_{k-1}} - A_1 r_{0,2}(t_k \mid t_{k-1})A_1^T. \end{aligned}$$

Thus, the presented state and covariance algorithm includes the EKF scheme [22] as a special case.

2.1. **Posterior Prediction of State and Covariance of Nonlinear System.** A final step in the recursive algorithm is to predict the state $\hat{x}(t_{j+1} \mid t_j)$ and state variance $P(t_{j+1} \mid t_j)$ at the time of the following measurement. Using equations (2.1), the definition of $P(t \mid t_{k-1})$ in (2.3), and (2.9),we have

(2.17)

$$\hat{x}(t_{k+1} \mid t_k) = \hat{x}(t_k \mid t_k) + \left[\tilde{\mathbf{f}}(\hat{z}(t_k \mid t_k)) + \frac{\sigma_2}{2h^2}\sum_{p=1}^{n}\delta_p^2\tilde{\mathbf{f}}(\hat{z}(t_k \mid t_k))\right]\Delta t,$$

$$\begin{aligned} P(t_{k+1} \mid t_k) = P(t_k \mid t_k) + [&\frac{\sigma_2}{h}\sum_{p=1}^{n}\mu_p\delta_p\tilde{\mathbf{f}}(\hat{z}(t_k \mid t_k))e_p^T S_x^T + \frac{\sigma_2}{h}\sum_{p=1}^{n}S_x e_p \mu_p\delta_p\tilde{\mathbf{f}}^T(\hat{z}(t_k \mid t_k)) \\ &+ \frac{\sigma_2}{h^2}\sum_{p=1}^{n}\mu_p\delta_p\tilde{\mathbf{f}}(\hat{z}(t_k \mid t_k))\mu_p\delta_p\tilde{\mathbf{f}}^T(\hat{z}(t_k \mid t_k)) \\ &+ \frac{\sigma_4-\sigma_2^2}{4h^2}\sum_{p=1}^{n}\delta_p^2\tilde{\mathbf{f}}(\hat{z}(t_k \mid t_k))\delta_p^2\tilde{\mathbf{f}}^T(\hat{z}(t_k \mid t_k)) \\ &+ \frac{\sigma_2}{2h^2}\tilde{\mathbf{g}}(\hat{z}(t_k \mid t_k))\sum_{p=1}^{n}\delta_p^2\tilde{\mathbf{g}}^T(\hat{z}(t_k \mid t_k)) \\ &+ \frac{\sigma_2^2}{4h^2}\sum_{\substack{p,q=1\\ p\neq q}}^{n}\Big[\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{f}}(\hat{z}(t_k \mid t_k))\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{f}}^T(\hat{z}(t_k \mid t_k)) \\ &+\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{f}}(\hat{z}(t_k \mid t_k))\mu_q\delta_q\mu_p\delta_p\tilde{\mathbf{f}}^T(\hat{z}(t_k \mid t_k))\Big] \\ &+ \frac{\sigma_2}{h^2}\sum_{p=1}^{n}\mu_p\delta_p\tilde{\mathbf{g}}(\hat{z}(t_k \mid t_k))\mu_p\delta_p\tilde{\mathbf{g}}^T(\hat{z}(t_k \mid t_k)) + \tilde{\mathbf{g}}(\hat{z}(t_k \mid t_k))\tilde{\mathbf{g}}^T(\hat{z}(t_k \mid t_k))\ldots \end{aligned}$$

$$+\frac{\sigma_2}{2h^2}\sum_{p=1}^{n}\delta_p^2\tilde{\mathbf{g}}(\hat{z}(t_k \mid t_k))\tilde{\mathbf{g}}^T(\hat{z}(t_k \mid t_k))$$

$$+\frac{\sigma_4}{4h^2}\sum_{p=1}^{n}\delta_p^2\tilde{\mathbf{g}}(\hat{z}(t_k \mid t_k))\delta_p^2\tilde{\mathbf{g}}^T(\hat{z}(t_k \mid t_k))$$

$$+\frac{\sigma_2^2}{4h^2}\sum_{\substack{p,q=1\\p\neq q}}^{n}\big[\delta_p^2\tilde{\mathbf{g}}(\hat{z}(t_k \mid t_k))\delta_q^2\tilde{\mathbf{g}}^T(\hat{z}(t_k \mid t_k))$$

$$+\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{g}}(\hat{z}(t_k \mid t_k))\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{g}}^T(\hat{z}(t_k \mid t_k))\big]\,\Delta t$$

$$+\frac{\sigma_2^2}{4h^2}\sum_{\substack{p,q=1\\p\neq q}}^{n}\big[\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{g}}(\hat{z}(t_k \mid t_k))\mu_q\delta_q\mu_p\delta_p\tilde{\mathbf{g}}^T(\hat{z}(t_k \mid t_k))\big]]\Delta t$$

The one step prediction error

$$\Delta y(k) = y_k - \hat{y}(t_k \mid t_{k-1}), \tag{2.18}$$

is assumed to be normal with mean 0 and variance $r_{0,2}$. Hence, for N independent random observations, the Maximum Likelihood approach is equivalent to maximizing

$$L(\boldsymbol{\Theta}) = -\frac{1}{2}\sum_{k=1}^{N}\left[\frac{1}{2}\Delta y^T(k)r_{0,2}^{-1}(t_k \mid t_{k-1})\Delta y(k) + \log \mid r_{0,2}(t_k \mid t_{k-1}) \mid\right], \tag{2.19}$$

where $\boldsymbol{\Theta}$ is the parameter space.

Remark 2.6. The presented predicted algorithm extends the algorithm generated by the EKF approach in a systematic way. We further remark that the second order estimation for nonlinear stochastic systems can be extended to higher order estimation. The scheme is highly complex mathematical expressions. Further detailed examination (applicability/computational, feasibility, etc.) is under investigation.

3. Some Results: Natural Gas

In this section, we give the parameter estimates for the stochastic differential equation (2.1). We consider the nonlinear stochastic differential equation that was developed for describing continuous time stochastic dynamic model of energy commodities log-spot price processes [23],

$$\begin{aligned} dx_1 &= \mu(x_1+\kappa_0)(\kappa_2-x_1)dt+\delta(\kappa_2-x_1)dW_1(t), \quad x_1(t_0)=x_{10},\\ dx_2 &= \gamma(x_2+\kappa_1)(x_1-x_2)dt+\sigma(x_2+\kappa_1)dW_2(t), \quad x_2(t_0)=x_{02}.\\ y(t) &= x(t)+v(t). \end{aligned} \tag{3.1}$$

It follows from (3.1) that

$$\mathbf{f}(x;\boldsymbol{\theta}) = \begin{pmatrix}\mu(x_1+\kappa_0)(\kappa_2-x_1)\\ \gamma(x_2+\kappa_1)(x_1-x_2)\end{pmatrix}, \mathbf{g}(x;\boldsymbol{\theta}) = \begin{pmatrix}\delta(\kappa_2-x_1) & 0\\ 0 & \sigma(x_2+\kappa_1)\end{pmatrix},$$

and $x = \{x_1, x_2\}^T$, where $\mu, \gamma, \kappa_0, \kappa_1, \kappa_2, \delta, \sigma \in \Re^+$, v is a white noise, and $\mathbf{W} = \{W_1, W_2\}^T$, W_1 and W_2 are independent Wiener processes. This model governs the price for energy commodity at time t. $x_2(t)$ is the nonseasonal log of spot price at a time t and $x_1(t)$ describes a mean process of non-seasonal log spot price at time t. The model (3.1) follows the principle of demand and supply processes which suggest that the price of a energy commodity will remain within a given finite lower and upper bounds $\kappa_1 > 0$ and $\kappa_2 > 0$, respectively. In this case, κ_2 characterizes the fixed cost, $(x_1(t)+\kappa_0)(\kappa_2 - x_1)$ characterizes the market potential for $x_1(t)$ per unit of time at a time t. We note that the first component of (3.1) has a unique non-zero equilibrium κ_2. Moreover, we observe that whenever the price x_1 lies above κ_2, there is a tendency for the price to fall and whenever the price is below κ_2, the price rises back. Hence, κ_2 is the equilibrium of the first component of (3.1). Furthermore, μ and γ are the rate of mean reversion for x_1 and x_2 respectively, δ and σ are the volatility for x_1 and x_2 respectively.

We apply this model to the Henry-Hub natural gas data set [5]. We use the Henry-Hub natural gas spot price data set [5] for the observation data for x_2. We generate observation data for x_1 from the forward price $F(t,T)$ at time t of an energy goods with maturity at time T. We define the forward price as

$$F(t,T) = \mathbb{E}_{\mathbb{P}}\left(x_2(T)\right). \tag{3.2}$$

By definition, $x_1(t)$ is the expected log-spot price, which in this case is the observation data $F(t,T)$. We use Henry-Hub natural gas observed future price at a time t with delivery time T.

The existence and uniqueness of the solution of (3.1) is given in [23].

Remark 3.1. Using the techniques decribed in [13, 23], it follows that the solution to the nonlinear system of stochastic differential equation (3.1) is given by:

$$\begin{cases} x_1(t) = \left[\frac{\phi_1(t,t_0)}{x_{10}-\kappa_2} + \mu\int_{t_0}^t \phi_1(t,s)ds\right]^{-1} + \kappa_2 \\ x_2(t) = \left[\frac{\phi_2(t,t_0)}{\vartheta_{02}+\kappa_1} + \gamma\int_{t_0}^t \phi_2(t,s)ds\right]^{-1} - \kappa_1. \end{cases} \tag{3.3}$$

where

$$\begin{cases} \phi_1(t,t_0) &= \exp\left[\left(\mu(\kappa_2+\kappa_0)+\frac{1}{2}\delta^2\right)(t-t_0) + \delta\left(W_1(t)-W_1(t_0)\right)\right] \\ \phi_2(t,t_0) &= \exp\left[\int_{t_0}^t \left(-\gamma(x_1(s)+\kappa_1)+\frac{1}{2}\sigma^2\right)ds - \sigma(W_2(t)-W_2(t_0))\right], \end{cases} \tag{3.4}$$

and c_i, $i = 1, 2$ are constants.

3.1. **Algorithm.** We describe the algorithm used in the computation of the estimates for nonlinear log-spot price stochastic differential equation (3.1).

Algorithm 1 Estimating parameters

Given initial parameters and initial predictions $\hat{x}(t_1 \mid t_0)$ and $P(t_1 \mid t_0)$,
for $k = 1$ to n **do**,
  for $j = 0$ to 2 **do**,
    for $m = 1$ to 6 **do**,
      Compute $\hat{y}(t_k \mid t_{k-1})$ and $r_{j,m}(t_k \mid t_{k-1})$
      Compute $\hat{x}(t_k \mid t_k)$ and $P(t_k \mid t_k)$ using equation (2.8),
      Compute $\hat{x}(t_{k+1} \mid t_k)$ and $P(t_{k+1} \mid t_k)$ using equation (2.17),
      Compute e_k using (2.18),
    end for
  end for
end for
Return e_k.
Compute L($\mathbf{\Theta}$) using equation (2.19),
$\check{\theta} = \arg\min L(\mathbf{\Theta})$
Return L.

Remark 3.2. We further remark that all codes are written in Matlab. To compute the maximization $\arg\min L(\mathbf{\Theta})$ in the algorithm, we use the Nelder-Mead Simplex Method developed in Matlab. Maximizing equation (2.19) is equivalent to minimizing

$$L(\mathbf{\Theta}) = \frac{1}{2}\sum_{k=1}^{N}\left[\frac{1}{2}\Delta y^T(k) r_{0,2}^{-1}(t_k \mid t_{k-1})\Delta y(k) + \log \mid r_{0,2}(t_k \mid t_{k-1}) \mid\right]. \tag{3.5}$$

In this section, we present the calibration results of our model. The initial state of the model is $\hat{x}_1(t_1 \mid t_0) = 1.23$, $\hat{x}_2(t_1 \mid t_0) = 1.456$, $P(t_1 \mid t_0) = \begin{pmatrix} 0.1182 & 0 \\ 0 & 0.22 \end{pmatrix}$.

Table 1 shows the parameter estimates of Henry Hub daily natural gas.

Table 1. Estimated Parameters of (3.1) for Henry Hub daily natural gas spot prices (20 run average)

μ	γ	κ_0	κ_1	κ_2	δ	σ
1.6	1.78	.69	.56	1.5	0.65	0.47

Table 1 shows the estimates of the parameters of (3.1). These parameters are derived by taking the average of the estimated parameters of (3.1) for 20 runs.

Furthermore, we show some of the estimates of the simulations for the modified EKF scheme compared with the usual EKF scheme.

TABLE 2. Estimates for the simulations using the modified extended kalman filter (MEKF) and the usual EKF .

Data		Modified EKF	EKF	Modified EKF Error	EKF Error
t (days)	Real data	Simulated data	Simulated data	*Real − Simulated*	*Real − Simulated*
20	2.6990	2.6739	2.6651	0.0251	0.0339
21	2.7590	2.6649	2.7232	0.0941	0.0358
22	2.6590	2.6523	2.6975	0.0067	-0.0385
23	2.7420	2.7544	2.6265	-0.0124	0.1155
24	2.5620	2.5809	2.5338	-0.0189	0.0282
25	2.4950	2.4836	2.4779	0.0114	0.0171
26	2.540	2.5379	2.6115	0.0021	-0.0715
27	2.5920	2.5948	2.6037	-0.0028	-0.0117
28	2.5700	2.5955	2.5451	-0.0255	0.0249
29	2.5410	2.6038	2.5701	-0.0628	-0.0291
30	2.6180	2.5844	2.6920	0.0336	-0.0740
31	2.5640	2.6071	2.6563	-0.0431	-0.0923
32	2.6670	2.6819	2.6118	-0.0149	0.0552
33	2.6330	2.6251	2.6613	0.0079	-0.0283
34	2.5150	2.5311	2.6376	-0.0161	-0.1226
35	2.5300	2.5104	2.6580	0.0196	-0.1280
...	...	...	...	...	...
...	...	...	...	...	...
...	...	...	...	...	...
255	9.8190	7.9849	4.6175	1.8341	5.2015
256	9.1280	8.1730	4.6536	0.9550	4.4744
257	8.7080	8.4447	4.7375	0.2633	3.9705
258	8.4720	8.6903	4.6959	-0.2183	3.7761
259	8.1030	9.3793	4.8368	-1.2763	3.2662
260	6.9090	6.5429	4.7359	0.3661	2.1731
261	7.1360	6.8047	4.8381	0.3313	2.2979
262	7.4590	6.6625	4.5907	0.7965	2.8683
263	7.4570	6.6334	4.5749	0.8236	2.8821
264	6.9460	6.5458	4.5208	0.4002	2.4252
265	7.1150	5.8724	4.6052	1.2426	2.5098
266	7.2700	5.7080	5.0362	1.5620	2.2338
267	7.2560	5.4032	4.9095	1.8528	2.3465
268	6.2930	5.3971	4.7343	0.8959	1.5587
269	6.2930	5.5606	5.0757	0.7324	1.2173
270	6.2930	5.6742	5.4394	0.6188	0.8536
...	...	...	...	...	...
...	...	...	...	...	...
...	...	...	...	...	...

Data		Modified EKF	EKF	Modified EKF Error	EKF Error
t (days)	Real data	Simulated data	Simulated data	$Real - Simulated$	$Real - Simulated$
...	...	...	...	...	...
...	...	...	...	...	...
...	...	...	...	...	...
315	5.4220	5.5496	5.6280	-0.1276	-0.2060
316	5.3880	5.2974	5.3527	0.0906	0.0353
317	5.4770	5.5581	5.1236	-0.0811	0.3534
318	5.5590	5.6172	5.6023	-0.0582	-0.0433
319	5.3850	5.5434	4.8462	-0.1584	0.5388
320	5.3810	5.5147	4.6067	-0.1337	0.7743
321	5.5160	5.3233	6.1508	0.1927	-0.6348
322	5.2480	5.5923	4.6280	-0.3443	0.6200
323	5.1480	5.1522	4.4990	-0.0042	0.6490
324	5.1010	5.2694	4.4632	-0.1684	0.6378
325	5.1280	5.0945	4.3613	0.0335	0.7667
326	5.1250	4.9006	4.3357	0.2244	0.7893
327	5.0780	4.6212	4.4024	0.4568	0.6756
328	4.9810	4.4826	4.4873	0.4984	0.4937
329	4.8910	4.3396	4.4108	0.5514	0.4802
330	4.8670	4.5609	4.5258	0.3061	0.3412
...	...	...	...	...	...
...	...	...	...	...	...
...	...	...	...	...	...
1040	5.4430	5.5370	5.3943	-0.0940	0.0487
1041	5.3930	5.3264	5.0916	0.0666	0.3014
1042	5.4380	5.1881	5.7584	0.2499	-0.3204
1043	5.3970	5.3663	5.7657	0.0307	-0.3687
1044	5.6430	5.5917	6.0480	0.0513	-0.4050
1045	5.5960	5.6060	5.8662	-0.0100	-0.2702
1046	5.7180	5.3299	5.6337	0.3881	0.0843
1047	5.6880	5.2528	5.1967	0.4352	0.4913
1048	5.7220	5.2462	5.0721	0.4758	0.6499
1049	5.6310	5.3174	4.8923	0.3136	0.7387
1050	5.5820	5.3509	5.0484	0.2311	0.5336
1051	5.5460	5.6582	4.9531	-0.1122	0.5929
1052	5.5300	5.8134	6.1617	-0.2834	-0.6317
1053	5.4290	5.7149	5.8370	-0.2859	-0.4080
1054	5.3360	5.5240	5.6794	-0.1880	-0.3434
1055	5.3950	5.4978	4.7768	-0.1028	0.6182

Table 2 shows the real data sets, estimated simulation results for the Modified EKF scheme and the ordinary EKF scheme. Column 4 and 5 contains the estimation errors of both methods. The estimated error is calculated by subtracting the simulated estimates from the real data set. We further remark that all codes are written in Matlab. The optimization procedure is performed using the Nelder-Mead Simplex Method developed in Matlab.

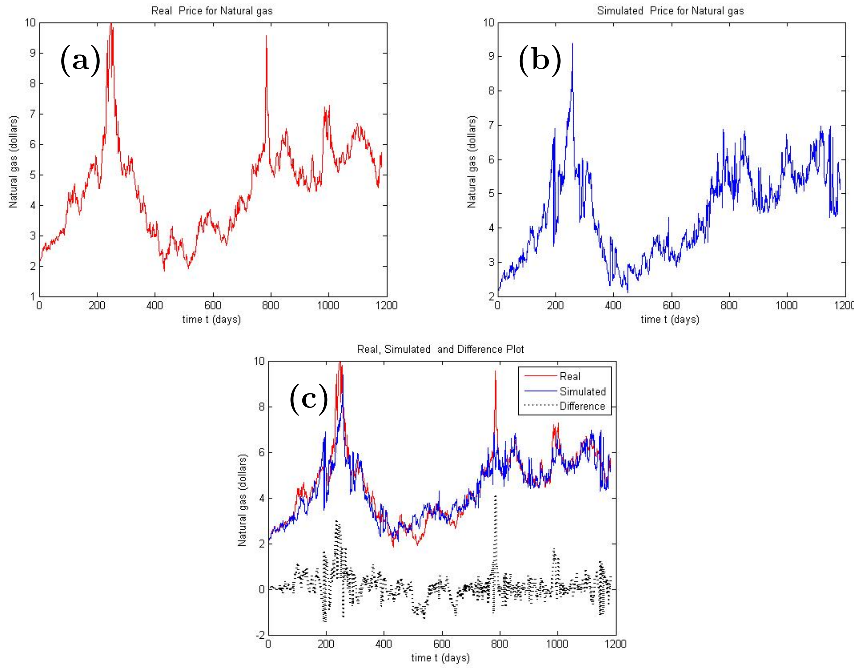

FIGURE 1. Real and Simulated price for Natural gas data set using Modified EKF scheme

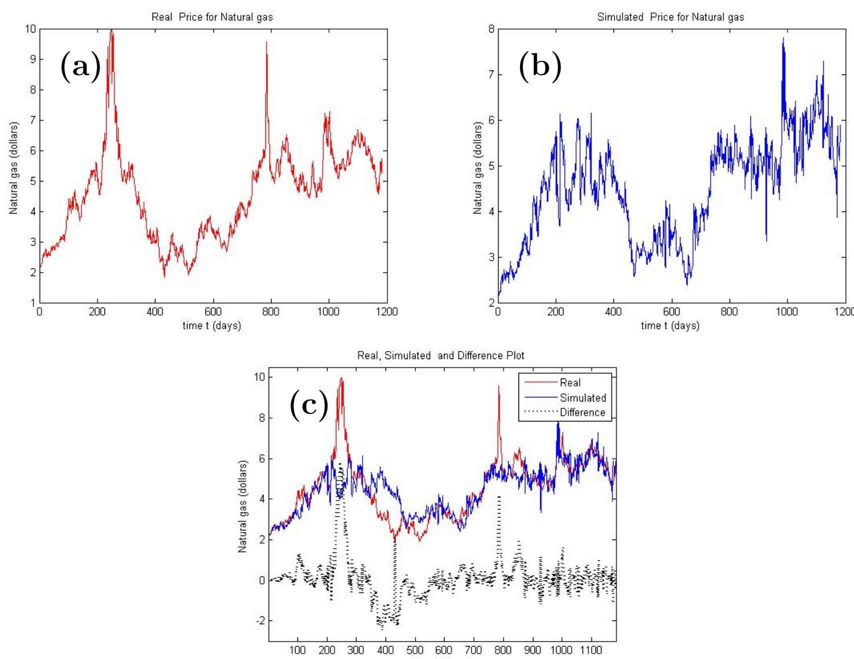

FIGURE 2. Real and Simulated price for Natural gas data set using EKF scheme

Figure 1 (a) shows the graph of the real natural gas data set, Figure 1 (b) shows the simulated price using Modified EKF scheme, and Figure 1 **(c)** shows the combination of the real, simulated and difference of the real and simulated price of the natural gas data set using the modified extended Kalman filter second order estimation scheme.

Figure 2**(a)** shows the graph of the real natural gas data set, Figure 2 (b) shows the simulated price the usual ordinary EKF scheme, and Figure 2 **(c)** shows the combination of the real, simulated and difference of the real and simulated price of the natural gas data set using the usual ordinary EKF scheme.

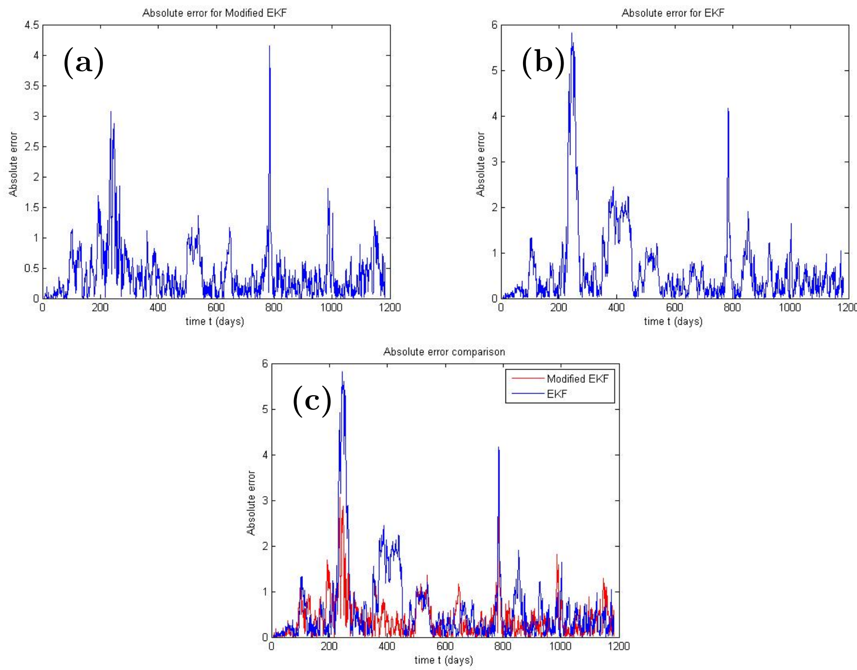

Figure 3. Absolute error for natural gas estimate

Figure 3 **(a)** shows the absolute error of the simulations of natural gas data set using the modified extended Kalman filter scheme, Figure 3 (b) shows the absolute error of the simulations of natural gas data set using the usual extended Kalman filter scheme, and Figure 3 **(c)** shows the comparison of the absolute error for the modified and usual EKF scheme.

Remark 3.3. We note from Figures 1 and 2, it is clear that the presented scheme is superior than the EKF approach. This shows that the modified extended Kalman filter does in fact reduce the magnitude of error tremendously. Furthermore, the modified extended Kalman filter scheme was able to capture the upward price spike in the neighborhood of time $t = 250$ days better than the EKF scheme. Both scheme were not able to campture the upward spike around the time $t = 800$ days. This might be as a result of the kind of model we are using to describe the dynamics of the natural gas data set. As mentioned earlier, the nonlinear stochastic differential equation (3.1) was developed to describe continuous time stochastic dynamic for energy commodities log-spot price processes [23]. We also mention that in [23], the volatility σ is not treated as a constant. Rather, the volatility process $\sigma(t, x_t)$ follows a continuous time-delay Garch model. $\sigma(t, x_t)$ is a functional defined on $[0, T] \times \mathcal{C}[[-\tau, 0], \Re]$ into $\Re$, x_t is a segment of continuous function x defined by $x_t(\theta) = x(t+\theta)$, $\theta \in [-\tau, 0]$ for $t \geq 0$.The upward spike in price at these region was due to the decline in production of natural gas and the increase in demand for electricity generation. We will like to also mention one disadvantage with this scheme. It is computational intensive.

ACKNOWLEDGMENTS

This research is supported by the Mathematical Sciences Division, the U.S. Army Office, under Grant Number W911NF-12-1-0090.

REFERENCES

[1] J. Appell, H.T. Nguyen and P. Zabreiko, Multivalued superposition operators in ideal spaces of vector functions III, *Indag. Math.*, 3:1–9, 1992.

[2] J. Aubin and A. Cellina, *Differential Inclusions*, Springer Verlag, New York, 1984.

[3] G. Birkhoff, *Lattice Theory*, *Amer. Math. Soc. Coll. Publ.* Vol 25, New York, 1967.

[4] H. Cox, *On Estimation of State Variables and Parameters*, Joint AIAA-IMS-SIAM-ONR Symp. Control and System Optimization, Monterey, California, 1964.

[5] Daily Henry Hub Natural gas data set for the period 01/04/2000-09/30/2004, *U. S. Energy Information Administration Website, URL http: //www.eia.gov/.*

[6] F. Daum , *Nonlinear Filters: Beyond the Kalman Filter*, *IEEE A and E Systems Magazine*, Vol 20, No. 8, pg 57–69, 2005.

[7] R. F. Engle, *Autoregressive Conditional Heteroskedasticity with Estimates of the Variance of United Kingdom Inflation* , *Econometrica* Vol. 50 No. 4, pp. 987-1007, 1982

[8] A. Gelb, J. F. Kasper, R. A. Nash, C. F. Price and A. A. Sutherland, *Applied optimal estimation*, Cambridge, MA MIT Press, 1974.

[9] Hermann Singer, *Parameter Estimation of Nonlinear stochastic Differential Equations: Simulated Maximum Likelihood versus Extended Kalman Filter and Ito-Taylor expansion*, *Journal of Computational and Graphical Statistics*, Vol. 11, No. 4, 972-995, 2002.

[10] A. H. Jazwinski, *Stochastic Processes and Filtering Theory*, *Mathematics in Science and Engineering*, Academy Press, New York, Volume 64, 1970.

[11] A. H. Jazwinski, *Stochastic Processes and Filtering Theory*, Dover Publications, Inc, 2007

[12] Anil G. Ladde, and G. S. Ladde, *An Introduction to Differential Equations: Deterministic Modelling and Analysis, Volume 1 World Scientific Publishing Company*, Singapore, 2012.

[13] Anil G. Ladde, and G. S. Ladde,*An Introduction to Differential Equations: Stochastic Modelling and Analysis*, Volume 2 World *Scientific Publishing Company*, Singapore, 2013.

[14] J. C. Lagarias, J. A. Reeds, M. H. Wright, and P. E. Wright, *Convergence Properties of the Nelder-Mead Simplex Method in Low Dimensions*, *SIAM Journal of Optimization*, Vol. 9, 1998 Number 1, pp. 112–147.

[15] F. L. Lewis, *Optimal Estimation*, New York, 1986

[16] X. Luo, I. Hoteit, I.M. Moroz, , *On an Nonlinear Kalman Filter with Simplified Divided Difference Approximation*, *Elsevier Physica D*, 241, pg 671–680, 2012.

[17] H. Madsen , J. N. Nielsen , J. L. Kierkegaard, M. Vestergaard , *Estimating Continuous-time Stochastic Volatility Models Using a Second Order Filter.*

[18] J. N. Nielsen , and M. Vestergaard , *Estimation in Continuous-time Stochastic Volatility Models Using a Second Order Filter*, *International Journal of Theoretical and Applied Finance*, Vol 3, Issue 2, pp 279–308, 2000.

[19] Magnus Nørgaard, *Kalmtool for Use with MATLAB: State Estimation for Nonlinear Systems*, Technical Report IMM-REP-200-6, 2000.

[20] Magnus Nørgaard, Niels K. Poulsen, and Ole Ravn, *New developments in state estimation for nonlinear systems*, *Automatica* Vol. 36 1627-1638, 2000.

[21] P. S. Maybeck, *Stochastic Models, Estimating and Control*, *Academic Press*, London, 1982.

[22] Mohinder S. Grewal, Angus P. Andrews, *Kalman Filtering, Theory and Practice using Mathlab*, John Wiley and Sons, 2008.

[23] Olusegun M. Otunuga, and G. Ladde *Non-Linear Stochastic Modeling of Energy Commodity's Spot Price Processes with Delay in Volatility*, Ph.D Dissertation, 2014.

[24] Simo Ali-Löytty, Niilo Sirola and Robert Piché, *Consistency of Three Kalman Filter Extensions in Hybrid Navigation*, *Proceedings of the European Navigation Conference GNSS*, 2005.

[25] Simon J. Julier, Jeffrey K. Uhlmann and Hugh F. Durrant-Whyte, *A new approach for Filtering Nonlinear systems*, *Proceedings of the America Control Conference Seatle*, Washington, IEEE Press, pg 1628–1631, 1995.

[26] Simon J. Julier, Jeffrey K. *A new Extension of the Kalman Filter to Nonlinear Systems*, The Robotic Research Group, Department of Engineering Science, The University of Oxford, Oxford, 1997.

[27] Simon J. Julier, Jeffrey K. Uhlmann and Hugh F. Durrant, *A new Method for the Nonlinear Transformation of Means and Covariances in Filters and Estimators*, IEEE Transactions on Automatic Control, Vol. 45, No. 3, pg 477-482, 2000.

[28] Sitz A., Schwarz U., Kurths J., Voss H. U., *Estimation of Parameters and Unobserved Components for Nonlinear Systems from Noisy Time Series*, Physical Review E 66, 016210, pg 1–9, 2002.

[29] Søren Hansen, Enis Bayramoglu, Jens Christian Andersen, Ole Ravn, Nils Andersen and Niels Kjølstad Poulsen, (2011), *Orchard navigation using derivative free Kalman filtering*, *American Control Conference*, pg 4679–4684.

[30] Tor Steinar Schei, *A Finite Difference Method for Linearization in Nonlinear Estimation Algorithms*, *Automatica*, Vol. 33, No. 11, pg 2053-2058, 1997.

[31] G. E. Uhlenbeck and L. S. Ornstein , *On the Theory of the Brownian Motion*, *Physical Review*, Vol 36, pg 823–841, 1930

Appendix A. Expressions in Lemma 2.1

$$
\begin{aligned}
D(t_k \mid t_{k-1})_{\{\tilde{\mathbf{h}},\tilde{\mathbf{h}}^T\}} &= \frac{\sigma_4-\sigma_2^2}{2h^3}\sum_{p=1}^{n}\left[e_p\mu_p\delta_p\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T + e_p\delta_p^2\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T\right] \\
&+\frac{\sigma_2^2}{2h^3}\sum_{\substack{p,q=1\\p\neq q}}^{n}\left[e_q\mu_p\delta_p\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T + e_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\right] \\
&+\frac{\sigma_2^2}{4h^4}\sum_{\substack{p,q=1\\p\neq q}}^{n}\left[e_q\delta_q^2\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T + e_p\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\delta_q^2\tilde{\mathbf{h}}^T\right] \\
&+\frac{\sigma_2^2}{2h^3}\sum_{\substack{p,q=1\\p\neq q}}^{n}\left[e_q\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T + e_p\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T\right] \\
E(t_k \mid t_{k-1}) &= \frac{\sigma_4-\sigma_2^2}{2h^3}\sum_{p=1}^{n}\left[e_pC^T\mu_p\delta_p\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T + e_pC^T\delta_p^2\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T\right] \\
&+\frac{\sigma_2^2}{2h^3}\sum_{\substack{p,q=1\\p\neq q}}^{n}\left[e_qC^T\mu_p\delta_p\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T + e_pC^T\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\right] \\
&+\frac{\sigma_2^2}{4h^4}\sum_{\substack{p,q=1\\p\neq q}}^{n}\left[e_qC^T\delta_q^2\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T + e_pC^T\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\delta_q^2\tilde{\mathbf{h}}^T\right] \\
&+\frac{\sigma_2^2}{2h^3}\sum_{\substack{p,q=1\\p\neq q}}^{n}e_qC^T\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T \\
&+\frac{\sigma_2^2}{2h^3}\sum_{\substack{p,q=1\\p\neq q}}^{n}e_pC^T\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T
\end{aligned}
$$

$$
\begin{aligned}
J(t_k \mid t_{k-1}) &= \frac{\sigma_4}{h^3}\sum_{p=1}^{n}e_p\mu_p\delta_p\tilde{\mathbf{h}}^T\mu_p\delta_p\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T \\
&+ \frac{\sigma_2^2}{h^3}\sum_{\substack{p,q=1\\p\neq q}}^{n}\Big[e_q\mu_p\delta_p\tilde{\mathbf{h}}^T\mu_p\delta_p\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T + e_q\mu_p\delta_p\tilde{\mathbf{h}}^T\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T \\
&+e_p\mu_p\delta_p\tilde{\mathbf{h}}^T\mu_q\delta_q\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T\Big] \\
&+ \frac{\sigma_2^3}{4h^5}\sum_{\substack{p,q,r=1\\p\neq q\neq r}}^{n}\Big[e_r\mu_p\delta_p\tilde{\mathbf{h}}^T\delta_q^2\tilde{\mathbf{h}}\mu_p\delta_p\mu_r\delta_r\tilde{\mathbf{h}}^T + e_r\mu_p\delta_p\tilde{\mathbf{h}}^T\delta_q^2\tilde{\mathbf{h}}\mu_r\delta_r\mu_p\delta_p\tilde{\mathbf{h}}^T \\
&+e_q\mu_p\delta_p\tilde{\mathbf{h}}^T\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\delta_r^2\tilde{\mathbf{h}}^T\Big]
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad \frac{\sigma_2\sigma_4}{4h^5}\sum_{\substack{p,q\\p\neq q}}^{n}\Big[e_q\mu_p\delta_p\tilde{\mathbf{h}}^T\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T + e_p\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\delta_q^2\tilde{\mathbf{h}}^T\\
&\quad +e_q\delta_p^2\tilde{\mathbf{h}}^T\mu_p\delta_p\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\Big]\\
&+ \quad \frac{\sigma_2^3}{4h^5}\sum_{\substack{p,q,r=1\\p\neq q\neq r}}^{n}\Big[e_r\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\mu_r\delta_r\tilde{\mathbf{h}}^T + e_r\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_r\delta_r\mu_p\delta_p\tilde{\mathbf{h}}^T\\
&\quad +e_p\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\delta_r^2\tilde{\mathbf{h}}^T + e_r\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_r\delta_r\tilde{\mathbf{h}}^T\Big]\\
&+ \quad \frac{\sigma_2\sigma_4}{4h^5}\sum_{\substack{p,q\\p\neq q}}^{n}\Big[e_p\delta_p^2\tilde{\mathbf{h}}^T\mu_p\delta_p\tilde{\mathbf{h}}\mu_q\delta_q\mu_p\delta_p\tilde{\mathbf{h}}^T + e_q\delta_p^2\tilde{\mathbf{h}}^T\mu_q\delta_q\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T\\
&\quad +e_q\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_p\delta_p\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T\Big]\\
&+ \quad \frac{\sigma_2^3}{4h^5}\sum_{\substack{p,q,r=1\\p\neq q\neq r}}^{n}\Big[e_q\delta_p^2\tilde{\mathbf{h}}^T\mu_q\delta_q\tilde{\mathbf{h}}\delta_r^2\tilde{\mathbf{h}}^T + e_r\delta_p^2\tilde{\mathbf{h}}^T\mu_q\delta_q\tilde{\mathbf{h}}\mu_r\delta_r\mu_q\delta_q\tilde{\mathbf{h}}^T\\
&\quad +e_r\delta_p^2\tilde{\mathbf{h}}^T\mu_q\delta_q\tilde{\mathbf{h}}\mu_q\delta_q\mu_r\delta_r\tilde{\mathbf{h}}^T + e_r\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_p\delta_p\tilde{\mathbf{h}}\delta_r\mu_r\mu_q\delta_q\tilde{\mathbf{h}}^T\Big]\\
&+ \quad \frac{\sigma_2^3}{4h^5}\sum_{\substack{p,q,r=1\\p\neq q\neq r}}^{n}\Big[e_q\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_p\delta_p\tilde{\mathbf{h}}\delta_r^2\tilde{\mathbf{h}}^T + e_r\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_p\delta_p\tilde{\mathbf{h}}\mu_q\delta_q\delta_r\mu_r\tilde{\mathbf{h}}^T\Big]\\
&+ \quad \frac{\sigma_2\sigma_4}{4h^5}\sum_{\substack{p,q\\p\neq q}}^{n}\Big[e_p\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_q\delta_q\tilde{\mathbf{h}}\delta_q^2\tilde{\mathbf{h}}^T + e_r\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_q\delta_q\tilde{\mathbf{h}}\mu_r\delta_r\mu_p\delta_p\tilde{\mathbf{h}}^T\\
&\quad +e_q\delta_p^2\tilde{\mathbf{h}}^T\delta_p^2\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T\Big]\\
&+ \quad \frac{\sigma_2^3}{4h^5}\sum_{\substack{p,q,r=1\\p\neq q\neq r}}^{n}e_r\Big[\mu_r\delta_r\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_q\delta_q\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T + \mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_q\delta_q\tilde{\mathbf{h}}\delta_q^2\tilde{\mathbf{h}}^T + \delta_p^2\tilde{\mathbf{h}}^T\delta_q^2\tilde{\mathbf{h}}\mu_r\delta_r\tilde{\mathbf{h}}^T\Big]\\
&+ \quad \frac{\sigma_2\sigma_4}{4h^5}\sum_{\substack{p,q\\p\neq q}}^{n}\Big[e_q\delta_p^2\tilde{\mathbf{h}}^T\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T + e_p\delta_p^2\tilde{\mathbf{h}}^T\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T\\
&\quad +e_q\delta_q^2\tilde{\mathbf{h}}^T\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T + e_q\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\delta_p^2\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T\Big]\\
&+ \quad \frac{\sigma_2\sigma_4}{4h^5}\sum_{\substack{p,q\\p\neq q}}^{n}\Big[e_p\delta_q^2\tilde{\mathbf{h}}^T\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T + e_q\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\delta_p^2\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T\Big]\\
&+ \quad \frac{\sigma_2\sigma_4}{4h^5}\sum_{\substack{p,q\\p\neq q}}^{n}\Big[e_q\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\delta_q^2\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T + e_p\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\delta_q^2\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T\Big]
\end{aligned}
$$

$$+ \quad \frac{\sigma_2^3}{4h^5} \sum_{\substack{p,q,r=1\\ p\neq q\neq r}}^{n} e_r \left[\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\delta_r^2\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T + \mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\delta_r^2\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T\right]$$

$$\begin{aligned}
L_{i,j} \;=\; & \frac{\sigma_4-\sigma_2^2}{2h^4}\sum_{p=1}^{n}\left[\mu_p\delta_p\tilde{\mathbf{h}}_i\mu_p\delta_p\tilde{\mathbf{h}}_j\delta_p^2\tilde{\mathbf{h}}^T + \mu_p\delta_p\tilde{\mathbf{h}}_i\delta_p^2\tilde{\mathbf{h}}_j\mu_p\delta_p\tilde{\mathbf{h}}^T + \delta_p^2\tilde{\mathbf{h}}_i\mu_p\delta_p\tilde{\mathbf{h}}_j\mu_p\delta_p\tilde{\mathbf{h}}^T\right] \\
& + \frac{\sigma_2^2}{2h^4}\sum_{\substack{p,q=1\\ p\neq q}}^{n}\mu_p\delta_p\tilde{\mathbf{h}}_i\mu_q\delta_q\tilde{\mathbf{h}}_j\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T + \frac{\sigma_2}{2h^2}\sum_{p=1}^{n}\delta_p^2\tilde{\mathbf{h}}^T\delta_{i,j}R_{i,j} \\
+ \;& \frac{\sigma_2^2}{2h^4}\sum_{\substack{p,q=1\\ p\neq q}}^{n}\left[\mu_p\delta_p\tilde{\mathbf{h}}_i\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}_j\mu_q\delta_q\tilde{\mathbf{h}}^T + \mu_q\delta_q\tilde{\mathbf{h}}_i\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}_j\mu_p\delta_p\tilde{\mathbf{h}}^T\right] \\
+ \;& \frac{\sigma_2}{2h^2}\sum_{p=1}^{n}\delta_p^2\tilde{\mathbf{h}}_i e_{j^T}R_{j,j} + \frac{\sigma_2^2}{2h^4}\sum_{\substack{p,q=1\\ p\neq q}}^{n}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}_i\mu_p\delta_p\tilde{\mathbf{h}}_j\mu_q\delta_q\tilde{\mathbf{h}}^T \\
+ \;& \frac{\sigma_2^2}{2h^4}\sum_{\substack{p,q=1\\ p\neq q}}^{n}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}_i\mu_q\delta_q\tilde{\mathbf{h}}_j\mu_p\delta_p\tilde{\mathbf{h}}^T + \frac{\sigma_6+2\sigma_2^3-3\sigma_2\sigma_4}{8h^6}\sum_{p=1}^{n}(\delta_p^2\tilde{\mathbf{h}}_i\delta_p^2\tilde{\mathbf{h}}_j\delta_p^2\tilde{\mathbf{h}}^T) \\
+ \;& \frac{\sigma_4\sigma_2-\sigma_2^3}{8h^6}\sum_{\substack{p,q,r=1\\ p\neq q\neq r}}^{n}\delta_p^2\tilde{\mathbf{h}}_i\delta_q^2\tilde{\mathbf{h}}_j\delta_r^2\tilde{\mathbf{h}}^T + \frac{\sigma_4\sigma_2}{8h^6}\sum_{\substack{p,q=1\\ p\neq q}}^{n}\delta_q^2\tilde{\mathbf{h}}_i\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}_j\mu_q\delta_q\mu_p\delta_p\tilde{\mathbf{h}}^T \\
& + \frac{\sigma_4\sigma_2-\sigma_2^3}{8h^6}\sum_{\substack{p,q=1\\ p\neq q}}^{n}\left[\delta_p^2\tilde{\mathbf{h}}_i\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}_j\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T + \delta_q^2\tilde{\mathbf{h}}_i\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}_j\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\right] \\
& + \frac{\sigma_4\sigma_2-\sigma_2^3}{8h^6}\sum_{\substack{p,q=1\\ p\neq q}}^{n}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}_i\delta_p^2\tilde{\mathbf{h}}_j\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T + \frac{\sigma_2}{2h^2}\sum_{p=1}^{n}\delta_p^2\tilde{\mathbf{h}}_j e_i R_{i,i} \\
+ \;& \frac{\sigma_4\sigma_2}{8h^6}\sum_{\substack{p,q=1\\ p\neq q}}^{n}\left[\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}_i\delta_p^2\tilde{\mathbf{h}}_j\mu_q\delta_q\mu_p\delta_p\tilde{\mathbf{h}}^T + \mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}_i\delta_q^2\tilde{\mathbf{h}}_j\mu_q\delta_q\mu_p\delta_p\tilde{\mathbf{h}}^T\right] \\
& + \frac{\sigma_4\sigma_2-\sigma_2^3}{8h^6}\sum_{\substack{p,q=1\\ p\neq q}}^{n}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}_i\delta_q^2\tilde{\mathbf{h}}_j\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T \\
& + \frac{\sigma_4\sigma_2-\sigma_2^3}{8h^6}\sum_{\substack{p,q=1\\ p\neq q}}^{n}\left(\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}_i\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}_j\delta_p^2\tilde{\mathbf{h}}^T + \mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}_i\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}_j\delta_q^2\tilde{\mathbf{h}}^T\right) \\
+ \;& \frac{\sigma_2^3}{8h^6}\sum_{\substack{p,q,r=1\\ p\neq q\neq r}}^{n}\Big(\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}_i\mu_q\delta_q\mu_r\delta_r\tilde{\mathbf{h}}_j\mu_p\delta_p\mu_r\delta_r\tilde{\mathbf{h}}^T + \mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}_i\delta_r^2\tilde{\mathbf{h}}_j\mu_q\delta_q\mu_p\delta_p\tilde{\mathbf{h}}^T \\
& + \mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}_i\mu_q\delta_q\mu_r\delta_r\tilde{\mathbf{h}}_j\mu_r\delta_r\mu_p\delta_p\tilde{\mathbf{h}}^T\Big) + \frac{\sigma_2}{4h^2}\sum_{p=1}^{n}\delta_p^2 h_i e_j^T R_{j,j} + \delta_p^2 h_j e_i^T R_{i,i}
\end{aligned}$$

$$+ \quad \frac{\sigma_2^3}{8h^6} \sum_{\substack{p,q,r=1 \\ p\neq q\neq r}}^{n} \Big(\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}_i\mu_p\delta_p\mu_r\delta_r\tilde{\mathbf{h}}_j\mu_q\delta_q\mu_r\delta_r\tilde{\mathbf{h}}^T$$

$$+\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}_i\mu_p\delta_p\mu_r\delta_r\tilde{\mathbf{h}}_j\mu_r\delta_r\mu_q\delta_q\tilde{\mathbf{h}}^T \Big) - \frac{\sigma_2}{2h^2}\sum_{p=1}^{n}\delta_p^2\tilde{\mathbf{h}}_i\left(e_j^T R_{j,j}\right)$$

$$+ \quad \frac{\sigma_2^2}{8h^6} \sum_{\substack{p,q=1 \\ p\neq q}}^{n} \mu_p\delta_p\tilde{\mathbf{h}}_i\mu_q\delta_q\tilde{\mathbf{h}}_j\mu_q\delta_q\mu_p\delta_p\tilde{\mathbf{h}}^T$$

$$+\frac{\sigma_2^3}{8h^6} \sum_{\substack{p,q,r=1 \\ p\neq q\neq r}}^{n} \left[\delta_p^2\tilde{\mathbf{h}}_i\mu_q\delta_q\mu_r\delta_r\tilde{\mathbf{h}}_j\mu_r\delta_r\mu_q\delta_q\tilde{\mathbf{h}}^T + \mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}_i\mu_q\delta_q\mu_p\delta_p\tilde{\mathbf{h}}_j\delta_r^2\tilde{\mathbf{h}}^T\right]$$

$$+ \quad \frac{\sigma_4\sigma_2}{8h^6} \sum_{\substack{p,q,r=1 \\ p\neq q\neq r}}^{n} \Big[\delta_p^2\tilde{\mathbf{h}}_i\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}_j\mu_q\delta_q\mu_p\delta_p\tilde{\mathbf{h}}^T + \mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}_i\mu_q\delta_q\mu_p\delta_p\tilde{\mathbf{h}}_j\delta_p^2\tilde{\mathbf{h}}^T$$

$$+\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}_i\mu_q\delta_q\mu_p\delta_p\tilde{\mathbf{h}}_j\delta_q^2\tilde{\mathbf{h}}^T\Big]$$

$$\mathbb{E}\left[\mathcal{A}\mathcal{A}^T\mathcal{A}\mathcal{A}^T \mid Y_{t_{k-1}}\right] = \frac{1}{4h^6}\Bigg(\sum_{p=1}^{n}\sigma_6\delta_p^2\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T\delta_p^2\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T$$

$$+\sum_{\substack{p,q,r=1 \\ p\neq q\neq r}}^{n}\sigma_2^3\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_r\delta_r\tilde{\mathbf{h}}\mu_r\delta_r\tilde{\mathbf{h}}^T$$

$$+ \quad \sum_{\substack{p,q=1 \\ p\neq q}}^{n}\sigma_2\sigma_4\Big[\delta_q^2\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T\delta_p^2\tilde{\mathbf{h}}(t_k,\hat{z}(t_k))\mu_q\delta_q\tilde{\mathbf{h}}^T + \delta_q^2\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T$$

$$+\delta_p^2\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T\delta_q^2\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T + \mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T\delta_q^2\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T\Big]$$

$$+ \quad \sum_{\substack{p,q=1 \\ p\neq q}}^{n}\sigma_2\sigma_4\left[\delta_p^2\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T\delta_p^2\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T + \delta_p^2\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T\right]$$

$$+ \quad \sum_{\substack{p,q,r=1 \\ p\neq q\neq r}}^{n}\sigma_2^3\Big[\delta_p^2\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_r\delta_r\mu_q\delta_q\tilde{\mathbf{h}}\mu_r\delta_r\tilde{\mathbf{h}}^T + \delta_p^2\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_q\delta_q\mu_r\delta_r\tilde{\mathbf{h}}\mu_r\delta_r\tilde{\mathbf{h}}^T$$

$$+\mu_r\delta_r\mu_q\delta_q\tilde{\mathbf{h}}\mu_r\delta_r\tilde{\mathbf{h}}^T\delta_p^2\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T\Big]$$

$$+ \quad \sum_{\substack{p,q=1 \\ p\neq q}}^{n}\sigma_2\sigma_4\Big[\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T\delta_p^2\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T + \mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T\delta_p^2\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T$$

$$+\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T\Big]$$

$$+ \sum_{\substack{p,q,r=1 \\ p\neq q\neq r}}^{n} \sigma_2^3 \Big[\delta_p^2\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T\delta_r^2\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T + \mu_q\delta_q\mu_r\delta_r\tilde{\mathbf{h}}\mu_r\delta_r\tilde{\mathbf{h}}^T\delta_p^2\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T$$

$$+\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T\mu_q\delta_q\mu_r\delta_r\tilde{\mathbf{h}}\mu_r\delta_r\tilde{\mathbf{h}}^T\Big]$$

$$+ \sum_{\substack{p,q=1 \\ p\neq q}}^{n} \sigma_2\sigma_4 \Big[\mu_q\delta_q\mu_p\delta_p\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T$$

$$+\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T + \mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T\delta_p^2\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T\Big]$$

$$+ \sum_{\substack{p,q=1 \\ p\neq q}}^{n} \sigma_2\sigma_4 \Big[\delta_p^2\tilde{\mathbf{h}}(t_k,\hat{z}(t_k))\mu_p\delta_p\tilde{\mathbf{h}}^T\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T + \delta_p^2\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T$$

$$+\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T\mu_q\delta_q\mu_p\delta_p\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T\Big]$$

$$+ \sum_{\substack{p,q,r=1 \\ p\neq q\neq r}}^{n} \sigma_2^3 \Big[\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T\mu_r\delta_r\mu_q\delta_q\tilde{\mathbf{h}}\mu_r\delta_r\tilde{\mathbf{h}}^T$$

$$+\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_p\delta_p\mu_r\delta_r\tilde{\mathbf{h}}\mu_r\delta_r\tilde{\mathbf{h}}^T + \mu_q\delta_q\mu_r\delta_r\tilde{\mathbf{h}}\mu_r\delta_r\tilde{\mathbf{h}}^T\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T\Big]$$

$$+ \sum_{\substack{p,q,r=1 \\ p\neq q\neq r}}^{n} \sigma_2^3\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_r\delta_r\tilde{\mathbf{h}}^T\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_r\delta_r\tilde{\mathbf{h}}^T + \sum_{p=1}^{n}\sigma_6\delta_p^2\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T\mu_p\delta_p\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T$$

$$+ \sum_{\substack{p,q=1 \\ p\neq q}}^{n} \sigma_2\sigma_4 \Big[\delta_q^2\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T\mu_q\delta_q\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T + \delta_q^2\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T$$

$$+\delta_p^2\tilde{\mathbf{h}}\delta_q^2\tilde{\mathbf{h}}^T\mu_q\delta_q\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T\Big]$$

$$+ \sum_{\substack{p,q=1 \\ p\neq q}}^{n} \sigma_2\sigma_4 \Big[\delta_p^2\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T\mu_q\delta_q\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T + \delta_p^2\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T$$

$$+\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\delta_q^2\tilde{\mathbf{h}}^T\mu_p\delta_p\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T\Big]$$

$$+ \sum_{\substack{p,q,r=1 \\ p\neq q\neq r}}^{n} \sigma_2^3 \Big[\delta_p^2\tilde{\mathbf{h}}\mu_r\delta_r\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_q\delta_q\tilde{\mathbf{h}}\mu_r\delta_r\tilde{\mathbf{h}}^T + \delta_p^2\tilde{\mathbf{h}}\mu_q\delta_q\mu_r\delta_r\tilde{\mathbf{h}}^T\mu_q\delta_q\tilde{\mathbf{h}}\mu_r\delta_r\tilde{\mathbf{h}}^T$$

$$+\mu_r\delta_r\mu_q\delta_q\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T\mu_r\delta_r\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T\Big]$$

$$+ \sum_{\substack{p,q=1 \\ p\neq q}}^{n} \sigma_2\sigma_4 \Big[\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T\mu_p\delta_p\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T + \mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T\mu_q\delta_q\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T$$

$$+\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_p\delta_p\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T\Big]$$

$$+ \sum_{\substack{p,q,r=1\\p\neq q\neq r}}^{n} \sigma_2^3 \Big[\delta_p^2\tilde{\mathbf{h}}\delta_r^2\tilde{\mathbf{h}}^T\mu_q\delta_q\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T + \mu_q\delta_q\mu_r\delta_r\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T\mu_r\delta_r\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T$$

$$+\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_q\delta_q\mu_r\delta_r\tilde{\mathbf{h}}^T\mu_p\delta_p\tilde{\mathbf{h}}\mu_r\delta_r\tilde{\mathbf{h}}^T\Big]$$

$$+ \sum_{\substack{p,q=1\\p\neq q}}^{n} \sigma_2\sigma_4 \Big[\mu_q\delta_q\mu_p\delta_p\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_p\delta_p\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T$$

$$+\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_q\delta_q\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T + \mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T\Big]$$

$$+ \sum_{\substack{p,q=1\\p\neq q}}^{n} \sigma_2\sigma_4 \Big[\delta_p^2\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_p\delta_p\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T + \delta_p^2\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_q\delta_q\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T$$

$$+\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_q\delta_q\mu_p\delta_p\tilde{\mathbf{h}}^T\mu_p\delta_p\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T\Big]$$

$$+ \sum_{\substack{p,q,r=1\\p\neq q\neq r}}^{n} \sigma_2^3 \Big[\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_r\delta_r\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_p\delta_p\tilde{\mathbf{h}}\mu_r\delta_r\tilde{\mathbf{h}}^T$$

$$+\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\mu_r\delta_r\tilde{\mathbf{h}}^T\mu_q\delta_q\tilde{\mathbf{h}}\mu_r\delta_r\tilde{\mathbf{h}}^T + \mu_q\delta_q\mu_r\delta_r\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_r\delta_r\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T\Big]$$

$$+ \sum_{p=1}^{n}\sigma_6\mu_p\delta_p\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T\delta_p^2\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T + \sum_{\substack{p,q,r=1\\p\neq q\neq r}}^{n} \sigma_2^3\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_r\delta_r\tilde{\mathbf{h}}^T\mu_r\delta_r\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T$$

$$+ \sum_{\substack{p,q=1\\p\neq q}}^{n} \sigma_2\sigma_4 \Big[\mu_q\delta_q\tilde{\mathbf{h}}\delta_q^2\tilde{\mathbf{h}}^T\delta_p^2\tilde{\mathbf{h}}(t_k,\hat{z}(t_k))\mu_q\delta_q\tilde{\mathbf{h}}^T + \mu_q\delta_q\tilde{\mathbf{h}}\delta_q^2\tilde{\mathbf{h}}^T\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T$$

$$+\mu_q\delta_q\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T\delta_q^2\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T\Big]$$

$$+ \sum_{\substack{p,q=1\\p\neq q}}^{n} \sigma_2\sigma_4 \Big[\mu_q\delta_q\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T\delta_p^2\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T + \mu_q\delta_q\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T$$

$$+\mu_p\delta_p\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\delta_q^2\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T\Big]$$

$$+ \sum_{\substack{p,q,r=1\\p\neq q\neq r}}^{n} \sigma_2^3 \Big[\mu_q\delta_q\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T\mu_r\delta_r\mu_q\delta_q\tilde{\mathbf{h}}\mu_r\delta_r\tilde{\mathbf{h}}^T + \mu_q\delta_q\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T\mu_q\delta_q\mu_r\delta_r\tilde{\mathbf{h}}\mu_r\delta_r\tilde{\mathbf{h}}^T$$

$$+\mu_r\delta_r\tilde{\mathbf{h}}\mu_r\delta_r\mu_q\delta_q\tilde{\mathbf{h}}^T\delta_p^2\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T\Big]$$

$$+ \sum_{\substack{p,q=1\\p\neq q}}^{n} \sigma_2\sigma_4 \Big[\mu_p\delta_p\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\delta_p^2\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T + \mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\delta_p^2\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T$$

$$+\mu_p\delta_p\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T\Big]$$

$$+ \sum_{\substack{p,q,r=1\\p\neq q\neq r}}^{n} \sigma_2^3 \Big[\mu_q\delta_q\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T\delta_r^2\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T + \mu_r\delta_r\tilde{\mathbf{h}}\mu_q\delta_q\mu_r\delta_r\tilde{\mathbf{h}}^T\delta_p^2\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T$$

$$+\mu_p\delta_p\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_q\delta_q\mu_r\delta_r\tilde{\mathbf{h}}\mu_r\delta_r\tilde{\mathbf{h}}^T\Big]$$

$$+ \sum_{\substack{p,q=1\\p\neq q}}^{n} \sigma_2\sigma_4 \Big[\mu_p\delta_p\tilde{\mathbf{h}}\mu_q\delta_q\mu_p\delta_p\tilde{\mathbf{h}}^T\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T$$

$$+\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T + \mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\delta_p^2\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T\Big]$$

$$+ \sum_{\substack{p,q=1\\p\neq q}}^{n} \sigma_2\sigma_4 \Big[\mu_p\delta_p\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T + \mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T\mu_p\delta_p\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T$$

$$+\mu_q\delta_q\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T + \mu_p\delta_p\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_q\delta_q\mu_p\delta_p\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T\Big]$$

$$+ \sum_{\substack{p,q,r=1\\p\neq q\neq r}}^{n} \sigma_2^3 \Big[\mu_p\delta_p\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_r\delta_r\mu_q\delta_q\tilde{\mathbf{h}}\mu_r\delta_r\tilde{\mathbf{h}}^T$$

$$+\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_p\delta_p\mu_r\delta_r\tilde{\mathbf{h}}\mu_r\delta_r\tilde{\mathbf{h}}^T + \mu_r\delta_r\tilde{\mathbf{h}}\mu_q\delta_q\mu_r\delta_r\tilde{\mathbf{h}}^T\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T\Big]$$

$$+ \sum_{\substack{p,q,r=1\\p\neq q\neq r}}^{n} \sigma_2^3\mu_r\delta_r\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_r\delta_r\tilde{\mathbf{h}}^T + \sum_{p=1}^{n}\sigma_6\delta_p^2\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T\mu_p\delta_p\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T$$

$$+ \sum_{\substack{p,q=1\\p\neq q}}^{n} \sigma_2\sigma_4 \Big[\delta_q^2\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_q\delta_q\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T + \delta_q^2\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_p\delta_p\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\Big]$$

$$+ \sum_{\substack{p,q=1\\p\neq q}}^{n} \sigma_2\sigma_4 \Big[\delta_p^2\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_q\delta_q\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T + \delta_p^2\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_p\delta_p\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T$$

$$+\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T\mu_q\delta_q\tilde{\mathbf{h}}\delta_q^2\tilde{\mathbf{h}}^T + \delta_p^2\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_q\delta_q\tilde{\mathbf{h}}\delta_q^2\tilde{\mathbf{h}}^T\Big]$$

$$+ \sum_{\substack{p,q,r=1\\p\neq q\neq r}}^{n} \sigma_2^3 \Big[\delta_p^2\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_r\delta_r\tilde{\mathbf{h}}\mu_r\delta_r\mu_q\delta_q\tilde{\mathbf{h}}^T + \delta_p^2\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_r\delta_r\tilde{\mathbf{h}}\mu_q\delta_q\mu_r\delta_r\tilde{\mathbf{h}}^T$$

$$+\mu_r\delta_r\mu_q\delta_q\tilde{\mathbf{h}}\mu_r\delta_r\tilde{\mathbf{h}}^T\mu_q\delta_q\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T\Big]$$

$$+ \sum_{\substack{p,q=1\\p\neq q}}^{n} \sigma_2\sigma_4 \Big[\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T\mu_q\delta_q\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T + \mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_q\delta_q\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T\Big]$$

$$+ \sum_{\substack{p,q,r=1\\p\neq q\neq r}}^{n} \sigma_2^3 \Big[\delta_p^2\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_q\delta_q\tilde{\mathbf{h}}\delta_r^2\tilde{\mathbf{h}}^T + \mu_q\delta_q\mu_r\delta_r\tilde{\mathbf{h}}\mu_r\delta_r\tilde{\mathbf{h}}^T\mu_q\delta_q\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T$$

$$+\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T\mu_r\delta_r\tilde{\mathbf{h}}\mu_q\delta_q\mu_r\delta_r\tilde{\mathbf{h}}^T\Big]$$

$$+\sum_{\substack{p,q=1\\p\neq q}}^{n}\sigma_2\sigma_4\Big[\mu_q\delta_q\mu_p\delta_p\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T\mu_p\delta_p\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T$$

$$+\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T+\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_p\delta_p\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T\Big]$$

$$+\sum_{\substack{p,q=1\\p\neq q}}^{n}\sigma_2\sigma_4\Big[\delta_p^2\tilde{\mathbf{h}}(t_k,\hat{z}(t_k))\mu_p\delta_p\tilde{\mathbf{h}}^T\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T+\delta_p^2\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T$$

$$+\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T\mu_p\delta_p\tilde{\mathbf{h}}\mu_q\delta_q\mu_p\delta_p\tilde{\mathbf{h}}^T\Big]$$

$$+\sum_{\substack{p,q,r=1\\p\neq q\neq r}}^{n}\sigma_2^3\Big[\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T\mu_r\delta_r\tilde{\mathbf{h}}\mu_r\delta_r\mu_q\delta_q\tilde{\mathbf{h}}^T$$

$$+\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_r\delta_r\tilde{\mathbf{h}}\mu_p\delta_p\mu_r\delta_r\tilde{\mathbf{h}}^T+\mu_q\delta_q\mu_r\delta_r\tilde{\mathbf{h}}\mu_r\delta_r\tilde{\mathbf{h}}^T\mu_p\delta_p\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\Big]$$

$$+\sum_{p=1}^{n}\sigma_6\mu_p\delta_p\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T\mu_p\delta_p\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T+\sum_{\substack{p,q,r=1\\p\neq q\neq r}}^{n}\sigma_2^3\mu_r\delta_r\tilde{\mathbf{h}}\mu_r\delta_r\tilde{\mathbf{h}}^T\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T$$

$$+\sum_{\substack{p,q=1\\p\neq q}}^{n}\sigma_2\sigma_4\Big[\mu_q\delta_q\tilde{\mathbf{h}}\delta_q^2\tilde{\mathbf{h}}^T\mu_q\delta_q\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T+\mu_q\delta_q\tilde{\mathbf{h}}\delta_q^2\tilde{\mathbf{h}}^T\mu_p\delta_p\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T$$

$$+\mu_q\delta_q\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T\mu_q\delta_q\tilde{\mathbf{h}}\delta_q^2\tilde{\mathbf{h}}^T\Big]$$

$$+\sum_{\substack{p,q=1\\p\neq q}}^{n}\sigma_2\sigma_4\Big[\mu_q\delta_q\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T\mu_q\delta_q\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T+\mu_q\delta_q\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T\mu_p\delta_p\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T$$

$$+\mu_p\delta_p\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_q\delta_q\tilde{\mathbf{h}}\delta_q^2\tilde{\mathbf{h}}^T\Big]$$

$$+\sum_{\substack{p,q,r=1\\p\neq q\neq r}}^{n}\sigma_2^3\Big[\mu_q\delta_q\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T\mu_r\delta_r\tilde{\mathbf{h}}\mu_r\delta_r\mu_q\delta_q\tilde{\mathbf{h}}^T+\mu_q\delta_q\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T\mu_r\delta_r\tilde{\mathbf{h}}\mu_q\delta_q\mu_r\delta_r\tilde{\mathbf{h}}^T$$

$$+\mu_r\delta_r\tilde{\mathbf{h}}\mu_r\delta_r\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_q\delta_q\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T\Big]$$

$$+\sum_{\substack{p,q=1\\p\neq q}}^{n}\sigma_2\sigma_4\Big[\mu_p\delta_p\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_q\delta_q\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T+\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_q\delta_q\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T$$

$$+\mu_p\delta_p\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_p\delta_p\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\Big]$$

$$+\sum_{\substack{p,q,r=1\\p\neq q\neq r}}^{n}\sigma_2^3\Big[\mu_q\delta_q\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T\mu_q\delta_q\tilde{\mathbf{h}}\delta_r^2\tilde{\mathbf{h}}^T+\mu_r\delta_r\tilde{\mathbf{h}}\mu_q\delta_q\mu_r\delta_r\tilde{\mathbf{h}}^T\mu_q\delta_q\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T$$

$$+\mu_p\delta_p\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_r\delta_r\tilde{\mathbf{h}}\mu_q\delta_q\mu_r\delta_r\tilde{\mathbf{h}}^T\Big]$$

$$+\sum_{\substack{p,q=1\\p\neq q}}^{n}\sigma_2\sigma_4\Big[\mu_p\delta_p\tilde{\mathbf{h}}\mu_q\delta_q\mu_p\delta_p\tilde{\mathbf{h}}^T\mu_p\delta_p\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T$$

$$+\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T+\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_p\delta_p\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T\Big]$$

$$+\sum_{\substack{p,q=1\\p\neq q}}^{n}\sigma_2\sigma_4\Big[\mu_p\delta_p\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T+\mu_q\delta_q\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T$$

$$+\mu_p\delta_p\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_p\delta_p\tilde{\mathbf{h}}\mu_q\delta_q\mu_p\delta_p\tilde{\mathbf{h}}^T\Big]$$

$$+\sum_{\substack{p,q,r=1\\p\neq q\neq r}}^{n}\sigma_2^3\Big[\mu_p\delta_p\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_r\delta_r\tilde{\mathbf{h}}\mu_r\delta_r\mu_q\delta_q\tilde{\mathbf{h}}^T$$

$$+\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_r\delta_r\tilde{\mathbf{h}}\mu_p\delta_p\mu_r\delta_r\tilde{\mathbf{h}}^T+\mu_r\delta_r\tilde{\mathbf{h}}\mu_q\delta_q\mu_r\delta_r\tilde{\mathbf{h}}^T\mu_p\delta_p\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\Big]$$

$$+\sum_{\substack{p,q,r=1\\p\neq q\neq r}}^{n}\sigma_2^3\mu_r\delta_r\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_r\delta_r\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T+\sum_{p=1}^{n}\sigma_6\mu_p\delta_p\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T\delta_p^2\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T$$

$$+\sum_{\substack{p,q=1\\p\neq q}}^{n}\sigma_2\sigma_4\Big[\mu_q\delta_q\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T\delta_q^2\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T+\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T\delta_q^2\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T$$

$$+\mu_q\delta_q\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T\delta_p^2\tilde{\mathbf{h}}\delta_q^2\tilde{\mathbf{h}}^T\Big]$$

$$+\sum_{\substack{p,q=1\\p\neq q}}^{n}\sigma_2\sigma_4\Big[\mu_q\delta_q\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T\delta_p^2\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T+\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T\delta_p^2\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T$$

$$+\mu_p\delta_p\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\delta_q^2\tilde{\mathbf{h}}^T\Big]$$

$$+\sum_{\substack{p,q,r=1\\p\neq q\neq r}}^{n}\sigma_2^3\Big[\mu_q\delta_q\tilde{\mathbf{h}}\mu_r\delta_r\tilde{\mathbf{h}}^T\delta_p^2\tilde{\mathbf{h}}\mu_r\delta_r\mu_q\delta_q\tilde{\mathbf{h}}^T+\mu_q\delta_q\tilde{\mathbf{h}}\mu_r\delta_r\tilde{\mathbf{h}}^T\delta_p^2\tilde{\mathbf{h}}\mu_q\delta_q\mu_r\delta_r\tilde{\mathbf{h}}^T$$

$$+\mu_r\delta_r\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_r\delta_r\mu_q\delta_q\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T\Big]$$

$$+\sum_{\substack{p,q=1\\p\neq q}}^{n}\sigma_2\sigma_4\Big[\mu_p\delta_p\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T+\mu_q\delta_q\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T$$

$$+\mu_p\delta_p\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\Big]$$

$$+\sum_{\substack{p,q,r=1\\p\neq q\neq r}}^{n}\sigma_2^3\Big[\mu_q\delta_q\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T\delta_p^2\tilde{\mathbf{h}}\delta_r^2\tilde{\mathbf{h}}^T+\mu_r\delta_r\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_q\delta_q\mu_r\delta_r\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T$$

$$+\mu_p\delta_p\tilde{\mathbf{h}}\mu_r\delta_r\tilde{\mathbf{h}}^T\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_q\delta_q\mu_r\delta_r\tilde{\mathbf{h}}^T\Big]$$

$$+ \sum_{\substack{p,q=1\\ p\neq q}}^{n}\sigma_2\sigma_4\Big[\mu_p\delta_p\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T\mu_q\delta_q\mu_p\delta_p\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T$$

$$+\mu_q\delta_q\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T + \mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T\Big]$$

$$+ \sum_{\substack{p,q=1\\ p\neq q}}^{n}\sigma_2\sigma_4\Big[\mu_p\delta_p\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T\delta_p^2\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T + \mu_q\delta_q\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T\delta_p^2\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T$$

$$+\mu_p\delta_p\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_q\delta_q\mu_p\delta_p\tilde{\mathbf{h}}^T\Big]$$

$$+ \sum_{\substack{p,q,r=1\\ p\neq q\neq r}}^{n}\sigma_2^3\Big[\mu_p\delta_p\tilde{\mathbf{h}}\mu_r\delta_r\tilde{\mathbf{h}}^T\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_r\delta_r\mu_q\delta_q\tilde{\mathbf{h}}^T$$

$$+\mu_q\delta_q\tilde{\mathbf{h}}\mu_r\delta_r\tilde{\mathbf{h}}^T\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\mu_r\delta_r\tilde{\mathbf{h}}^T + \mu_r\delta_r\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T\mu_q\delta_q\mu_r\delta_r\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\Big]\Big)$$

$$+ \frac{\sigma_2 R}{h^2}\sum_{p=1}^{n}\Big[2\mu_p\delta_p\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T + \mu_p\delta_p\tilde{\mathbf{h}}^T\mu_p\delta_p\tilde{\mathbf{h}}\Big]$$

$$+\frac{2R}{4h^4}\left[\sum_{p=1}^{n}\sigma_4\delta_p^2\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T + \sigma_2^2\sum_{\substack{p,q=1\\ p\neq q}}^{n}\delta_p^2\tilde{\mathbf{h}}\delta_q^2\tilde{\mathbf{h}}^T + \mu_p\delta_p\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\right]$$

$$+\frac{\sigma_2}{h^2}\sum_{p=1}^{n}\mu_p\delta_p\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T\left[\sum_{i=1}^{n}(I_{n,n}R_{i,i}) + 2R\right]$$

$$+ \frac{R}{4h^4}\left[\sum_{p=1}^{n}\sigma_4\delta_p^2\tilde{\mathbf{h}}^T\delta_p^2\tilde{\mathbf{h}} + \sigma_2^2\sum_{\substack{p,q=1\\ p\neq q}}^{n}\delta_p^2\tilde{\mathbf{h}}^T\delta_q^2\tilde{\mathbf{h}} + \mu_p\delta_p\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_p\delta_p\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\right]$$

$$+ \frac{2}{4h^4}\left[\sum_{p=1}^{n}\sigma_4\delta_p^2\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T + \sigma_2^2\sum_{\substack{p,q=1\\ p\neq q}}^{n}\delta_p^2\tilde{\mathbf{h}}\delta_q^2\tilde{\mathbf{h}}^T + \mu_p\delta_p\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\right]R$$

$$+3RR^T.$$

$$\mathbb{E}\left[\mathcal{A}\mathcal{A}^T\mathcal{A}C^T \mid Y_{t_{k-1}}\right] =$$

$$S_x\left(\frac{\sigma_4}{2h^4}\sum_{p=1}^{n}\mu_p\delta_p\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T\delta_p^2\tilde{\mathbf{h}} + \frac{\sigma_2^2}{2h^4}\sum_{\substack{p,q=1\\ p\neq q}}^{n}\mu_p\delta_p\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T\delta_q^2\tilde{\mathbf{h}}\right.$$

$$+\frac{\sigma_2^2}{2h^4}\sum_{\substack{p,q=1\\ p\neq q}}^{n}\mu_p\delta_p\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}} + \frac{\sigma_4}{2h^4}\sum_{p=1}^{n}\mu_p\delta_p\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T\mu_p\delta_p\tilde{\mathbf{h}}$$

$$+\frac{\sigma_2^2}{2h^4}\sum_{\substack{p,q=1\\p\neq q}}^{n}(\mu_p\delta_p\tilde{\mathbf{h}}\delta_q^2\tilde{\mathbf{h}}^T\mu_p\delta_p\tilde{\mathbf{h}})+\frac{\sigma_2^2}{2h^4}\sum_{\substack{p,q=1\\p\neq q}}^{n}(\mu_p\delta_p\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_q\delta_q\tilde{\mathbf{h}})$$

$$+\quad\frac{\sigma_2^2}{2h^4}\sum_{\substack{p,q=1\\p\neq q}}^{n}(\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_p\delta_p\tilde{\mathbf{h}})+\frac{\sigma_4}{2h^4}\sum_{p=1}^{n}\delta_p^2\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T\mu_p\delta_p\tilde{\mathbf{h}}$$

$$+\frac{\sigma_2^2}{2h^4}\sum_{\substack{p,q=1\\p\neq q}}^{n}\delta_p^2\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_q\delta_q\tilde{\mathbf{h}})+\frac{\sigma_2^2}{2h^4}\sum_{\substack{p,q=1\\p\neq q}}^{n}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T\mu_q\delta_q\tilde{\mathbf{h}}$$

$$+\frac{\sigma_2^2}{2h^4}\sum_{\substack{p,q=1\\p\neq q}}^{n}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_p\delta_p\tilde{\mathbf{h}}+\frac{\sigma_6}{8h^6}\sum_{p=1}^{n}(\delta_p^2\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T\delta_p^2\tilde{\mathbf{h}})$$

$$+\quad\frac{\sigma_4\sigma_2}{8h^6}\sum_{\substack{p,q=1\\p\neq q}}^{n}\delta_p^2\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T\delta_q^2\tilde{\mathbf{h}}+\frac{\sigma_4\sigma_2}{8h^6}\sum_{\substack{p,q=1\\p\neq q}}^{n}\delta_p^2\tilde{\mathbf{h}}\delta_q^2\tilde{\mathbf{h}}^T\delta_p^2\tilde{\mathbf{h}}$$

$$+\frac{\sigma_4\sigma_2}{8h^6}\sum_{\substack{p,q=1\\p\neq q}}^{n}\delta_p^2\tilde{\mathbf{h}}\delta_q^2\tilde{\mathbf{h}}^T\delta_q^2\tilde{\mathbf{h}}+\frac{\sigma_4\sigma_2}{8h^6}\sum_{\substack{p,q,r=1\\p\neq q\neq r}}^{n}\delta_p^2\tilde{\mathbf{h}}\delta_q^2\tilde{\mathbf{h}}^T\delta_r^2\tilde{\mathbf{h}}$$

$$+\frac{\sigma_4\sigma_2}{8h^6}\sum_{\substack{p,q=1\\p\neq q}}^{n}\delta_p^2\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}+\frac{\sigma_4\sigma_2}{8h^6}\sum_{\substack{p,q=1\\p\neq q}}^{n}\delta_q^2\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}$$

$$+\quad\frac{\sigma_4\sigma_2}{8h^6}\sum_{\substack{p,q=1\\p\neq q}}^{n}\delta_q^2\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\mu_q\delta_q\mu_p\delta_p\tilde{\mathbf{h}}+\frac{\sigma_2^3}{8h^6}\sum_{\substack{p,q,r=1\\p\neq q\neq r}}^{n}\delta_p^2\tilde{\mathbf{h}}\mu_q\delta_q\mu_r\delta_r\tilde{\mathbf{h}}^T\mu_q\delta_q\mu_r\delta_r\tilde{\mathbf{h}}$$

$$+\frac{\sigma_4\sigma_2}{8h^6}\sum_{\substack{p,q=1\\p\neq q}}^{n}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}$$

$$+\quad\frac{\sigma_4\sigma_2}{8h^6}\sum_{\substack{p,q=1\\p\neq q}}^{n}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T\mu_q\delta_q\mu_p\delta_p\tilde{\mathbf{h}}+\frac{\sigma_4\sigma_2}{8h^6}\sum_{\substack{p,q=1\\p\neq q}}^{n}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\delta_q^2\tilde{\mathbf{h}}^T\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}$$

$$+\frac{\sigma_4\sigma_2}{8h^6}\sum_{\substack{p,q=1\\p\neq q}}^{n}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\delta_q^2\tilde{\mathbf{h}}^T\mu_q\delta_q\mu_p\delta_p\tilde{\mathbf{h}}+\frac{\sigma_2^3}{8h^6}\sum_{\substack{p,q,r=1\\p\neq q\neq r}}^{n}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\delta_r^2\tilde{\mathbf{h}}$$

$$+\quad\frac{\sigma_2^3}{8h^6}\sum_{\substack{p,q,r=1\\p\neq q\neq r}}^{n}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\delta_r^2\tilde{\mathbf{h}}^T\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}+\frac{\sigma_2^3}{8h^6}\sum_{\substack{p,q,r=1\\p\neq q\neq r}}^{n}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\delta_r^2\tilde{\mathbf{h}}^T\mu_q\delta_q\mu_p\delta_p\tilde{\mathbf{h}}$$

$$+\quad\frac{\sigma_4\sigma_2}{8h^6}\sum_{\substack{p,q=1\\p\neq q}}^{n}\Big(\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\delta_p^2\tilde{\mathbf{h}}+\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\delta_q^2\tilde{\mathbf{h}}\Big)$$

$$+ \quad \frac{\sigma_2^3}{8h^6}\sum_{\substack{p,q,r=1\\p\neq q\neq r}}^{n}\Big(\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_q\delta_q\mu_r\delta_r\tilde{\mathbf{h}}^T\mu_p\delta_p\mu_r\delta_r\tilde{\mathbf{h}}+\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_q\delta_q\mu_r\delta_r\tilde{\mathbf{h}}^T\mu_r\delta_r\mu_p\delta_p\tilde{\mathbf{h}}\Big)$$

$$+ \quad \frac{\sigma_2^3}{8h^6}\sum_{\substack{p,q,r=1\\p\neq q\neq r}}^{n}\Big(\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\mu_r\delta_r\tilde{\mathbf{h}}^T\mu_q\delta_q\mu_r\delta_r\tilde{\mathbf{h}}+\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\mu_r\delta_r\tilde{\mathbf{h}}^T\mu_r\delta_r\mu_q\delta_q\tilde{\mathbf{h}}\Big)$$

$$+\frac{\sigma_2}{4h^2}\sum_{p=1}^{n}\left[\delta_p^2h_ie_jR_{j,j}+\delta_p^2h_je_iR_{i,i}\right]\Bigg)C^T+\frac{\sigma_2}{2h^2}\sum_{p=1}^{n}\delta_p^2\tilde{\mathbf{h}}\delta_{i,j}R_{i,j}$$

$$\mathbb{E}\left[\mathcal{A}\mathcal{A}^TC\mathcal{A}^T\mid Y_{t_{k-1}}\right]=$$

$$S_x\Bigg(\frac{\sigma_4}{2h^4}\sum_{p=1}^{n}\mu_p\delta_p\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^TC\delta_p^2\tilde{\mathbf{h}}^T+\frac{\sigma_2^2}{2h^4}\sum_{\substack{p,q=1\\p\neq q}}^{n}\mu_p\delta_p\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^TC\delta_q^2\tilde{\mathbf{h}}^T$$

$$+\frac{\sigma_2^2}{2h^4}\sum_{\substack{p,q=1\\p\neq q}}^{n}\mu_p\delta_p\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^TC\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T+\frac{\sigma_4}{2h^4}\sum_{p=1}^{n}\mu_p\delta_p\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^TC\mu_p\delta_p\tilde{\mathbf{h}}^T$$

$$+\frac{\sigma_2^2}{2h^4}\sum_{\substack{p,q=1\\p\neq q}}^{n}\mu_p\delta_p\tilde{\mathbf{h}}\delta_q^2\tilde{\mathbf{h}}^TC\mu_p\delta_p\tilde{\mathbf{h}}^T+\mu_p\delta_p\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^TC\mu_q\delta_q\tilde{\mathbf{h}}^T$$

$$+ \quad \frac{\sigma_2^2}{2h^4}\sum_{\substack{p,q=1\\p\neq q}}^{n}(\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^TC\mu_p\delta_p\tilde{\mathbf{h}}^T)+\frac{\sigma_4}{2h^4}\sum_{p=1}^{n}\delta_p^2\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^TC\mu_p\delta_p\tilde{\mathbf{h}}^T$$

$$+\frac{\sigma_2^2}{2h^4}\sum_{\substack{p,q=1\\p\neq q}}^{n}\left[\delta_p^2\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^TC\mu_q\delta_q\tilde{\mathbf{h}}^T+\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^TC\mu_q\delta_q\tilde{\mathbf{h}}^T\right]$$

$$+ \quad \frac{\sigma_2^2}{2h^4}\sum_{\substack{p,q=1\\p\neq q}}^{n}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^TC\mu_p\delta_p\tilde{\mathbf{h}}^T+\frac{\sigma_6}{8h^6}\sum_{p=1}^{n}\delta_p^2\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^TC\delta_p^2\tilde{\mathbf{h}}^T$$

$$+ \quad \frac{\sigma_4\sigma_2}{8h^6}\sum_{\substack{p,q=1\\p\neq q}}^{n}\left[\delta_p^2\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^TC\delta_q^2\tilde{\mathbf{h}}^T+\delta_p^2\tilde{\mathbf{h}}\delta_q^2\tilde{\mathbf{h}}^TC\delta_p^2\tilde{\mathbf{h}}^T+\delta_p^2\tilde{\mathbf{h}}\delta_q^2\tilde{\mathbf{h}}^TC\delta_q^2\tilde{\mathbf{h}}^T\right]$$

$$+ \quad \frac{\sigma_4\sigma_2}{8h^6}\sum_{\substack{p,q,r=1\\p\neq q\neq r}}^{n}\delta_p^2\tilde{\mathbf{h}}\delta_q^2\tilde{\mathbf{h}}^TC\delta_r^2\tilde{\mathbf{h}}^T+\frac{\sigma_4\sigma_2}{8h^6}\sum_{\substack{p,q=1\\p\neq q}}^{n}\delta_p^2\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^TC\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T$$

$$+ \quad \frac{\sigma_4\sigma_2}{8h^6}\sum_{\substack{p,q=1\\p\neq q}}^{n}\delta_q^2\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^TC\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T+\frac{\sigma_2^3}{8h^6}\sum_{\substack{p,q,r=1\\p\neq q\neq r}}^{n}\delta_p^2\tilde{\mathbf{h}}\mu_q\delta_q\mu_r\delta_r\tilde{\mathbf{h}}^TC\mu_q\delta_q\mu_r\delta_r\tilde{\mathbf{h}}^T$$

$$
\begin{aligned}
&+ \quad \frac{\sigma_4\sigma_2}{8h^6}\sum_{\substack{p,q=1\\p\neq q}}^{n}\left[\delta_q^2\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T C\mu_q\delta_q\mu_p\delta_p\tilde{\mathbf{h}}^T + \mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T C\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\right] \\
&+ \quad \frac{\sigma_4\sigma_2}{8h^6}\sum_{\substack{p,q=1\\p\neq q}}^{n}\left[\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T C\mu_q\delta_q\mu_p\delta_p\tilde{\mathbf{h}}^T + \mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\delta_q^2\tilde{\mathbf{h}}^T C\mu_q\delta_q\mu_p\delta_p\tilde{\mathbf{h}}^T\right. \\
&\qquad \left. +\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\delta_q^2\tilde{\mathbf{h}}^T C\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T\right] \\
&+ \quad \frac{\sigma_2^3}{8h^6}\sum_{\substack{p,q,r=1\\p\neq q\neq r}}^{n}\left[\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\delta_r^2\tilde{\mathbf{h}}^T C\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T + \mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\delta_r^2\tilde{\mathbf{h}}^T C\mu_q\delta_q\mu_p\delta_p\tilde{\mathbf{h}}^T\right. \\
&\qquad \left. +\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T C\delta_r^2\tilde{\mathbf{h}}^T\right] \\
&+ \quad \frac{\sigma_4\sigma_2}{8h^6}\sum_{\substack{p,q=1\\p\neq q}}^{n}\left(\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T C\delta_p^2\tilde{\mathbf{h}}^T + \mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T C\delta_q^2\tilde{\mathbf{h}}^T\right) \\
&+ \quad \frac{\sigma_2^3}{8h^6}\sum_{\substack{p,q,r=1\\p\neq q\neq r}}^{n}\left(\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_q\delta_q\mu_r\delta_r\tilde{\mathbf{h}}^T C\mu_p\delta_p\mu_r\delta_r\tilde{\mathbf{h}}^T\right. \\
&\qquad \left. +\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_q\delta_q\mu_r\delta_r\tilde{\mathbf{h}}^T C\mu_r\delta_r\mu_p\delta_p\tilde{\mathbf{h}}^T + \mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\mu_r\delta_r\tilde{\mathbf{h}}^T C\mu_r\delta_r\mu_q\delta_q\tilde{\mathbf{h}}^T\right) \\
&+ \quad \frac{\sigma_2^3}{8h^6}\sum_{\substack{p,q,r=1\\p\neq q\neq r}}^{n}\left(\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\mu_r\delta_r\tilde{\mathbf{h}}^T C\mu_q\delta_q\mu_r\delta_r\tilde{\mathbf{h}}^T\right) \\
&\qquad +\frac{\sigma_2}{4h^2}\sum_{p=1}^{n}\left(\delta_p^2\tilde{\mathbf{h}}\sum_{i=1}^{n}R_{i,i} + 2\delta_p^2 R\tilde{\mathbf{h}}\right)\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}\left[\mathcal{A}C^T\mathcal{A}\mathcal{A}^T \mid Y_{t_{k-1}}\right] &= \mathbb{E}\left[\mathcal{A}\mathcal{A}^T C\mathcal{A}^T \mid Y_{t_{k-1}}\right] \\
\mathbb{E}\left[\mathcal{A}C^T\mathcal{A}C^T \mid Y_{t_{k-1}}\right] &= \mathbb{E}\left[\mathcal{A}C^T C\mathcal{A}^T \mid Y_{t_{k-1}}\right] \\
\mathbb{E}\left[\mathcal{A}C^T C\mathcal{A}^T \mid Y_{t_{k-1}}\right] &= \sum_{p=1}^{n}\frac{\sigma_2}{h^2}\mu_p\delta_p\tilde{\mathbf{h}}(\hat{z})C^T C\mu_p\delta_p\tilde{\mathbf{h}}^T + \frac{\sigma_4}{4h^4}\delta_p^2\tilde{\mathbf{h}}C^T C\delta_p^2\tilde{\mathbf{h}}^T \\
&+ \quad \frac{\sigma_2^2}{4h^4}\sum_{\substack{p,q=1\\q\neq p}}^{n}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}C^T C\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}} + \delta_p^2\tilde{\mathbf{h}}C^T C\delta_q^2\tilde{\mathbf{h}}^T + RC^T C
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}\left[\mathcal{A}\mathcal{A}^T CC^T \mid Y_{t_{k-1}}\right] &= \left[\sum_{p=1}^{n}\frac{\sigma_2}{h^2}\mu_p\delta_p\tilde{\mathbf{h}}(\hat{z})\mu_p\delta_p\tilde{\mathbf{h}}^T + \frac{\sigma_4}{4h^4}\left(\delta_p^2\tilde{\mathbf{h}}(\hat{z})\right)\left(\delta_p^2\tilde{\mathbf{h}}(\hat{z})\right)^T\right] CC^T \\
&+ \quad \left[\frac{\sigma_2^2}{4h^4}\sum_{\substack{p,q=1\\q\neq p}}^{n}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}} + \delta_p^2\tilde{\mathbf{h}}\delta_q^2\tilde{\mathbf{h}}^T + R\right] CC^T
\end{aligned}
$$

$$\mathbb{E}\left[\mathcal{A}C^TCC^T \mid Y_{t_{k-1}}\right] = CC^TCC^T$$

$$\mathbb{E}\left[\mathcal{A}C^T\mathcal{A}\mathcal{A}^T \mid Y_{t_{k-1}}\right] = S_x\Bigg(\frac{\sigma_4}{2h^4}\sum_{p=1}^{n}\mu_p\delta_p\tilde{\mathbf{h}}C^T\mu_p\delta_p\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T$$

$$+ \frac{\sigma_2^2}{2h^4}\sum_{\substack{p,q=1\\p\neq q}}^{n}\mu_p\delta_p\tilde{\mathbf{h}}C^T\mu_p\delta_p\tilde{\mathbf{h}}\delta_q^2\tilde{\mathbf{h}}^T + \frac{\sigma_2^2}{2h^4}\sum_{\substack{p,q=1\\p\neq q}}^{n}\mu_p\delta_p\tilde{\mathbf{h}}C^T\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T$$

$$+ \frac{\sigma_4}{2h^4}\sum_{p=1}^{n}\mu_p\delta_p\tilde{\mathbf{h}}C^T\delta_p^2\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T + \frac{\sigma_2^2}{2h^4}\sum_{\substack{p,q=1\\p\neq q}}^{n}(\mu_p\delta_p\tilde{\mathbf{h}}C^T\delta_q^2\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T)$$

$$+\frac{\sigma_2^2}{2h^4}\sum_{\substack{p,q=1\\p\neq q}}^{n}\left[\mu_p\delta_p\tilde{\mathbf{h}}C^T\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T + \mu_q\delta_q\tilde{\mathbf{h}}C^T\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T\right]$$

$$+ \frac{\sigma_4}{2h^4}\sum_{p=1}^{n}\delta_p^2\tilde{\mathbf{h}}C^T\mu_p\delta_p\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T + \frac{\sigma_2^2}{2h^4}\sum_{\substack{p,q=1\\p\neq q}}^{n}\delta_p^2\tilde{\mathbf{h}}C^T\mu_q\delta_q\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T$$

$$+ \frac{\sigma_2^2}{2h^4}\sum_{\substack{p,q=1\\p\neq q}}^{n}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}C^T\mu_p\delta_p\tilde{\mathbf{h}}\mu_q\delta_q\tilde{\mathbf{h}}^T + \frac{\sigma_2^2}{2h^4}\sum_{\substack{p,q=1\\p\neq q}}^{n}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}C^T\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T$$

$$+ \frac{\sigma_6}{8h^6}\sum_{p=1}^{n}(\delta_p^2\tilde{\mathbf{h}}C^T\delta_p^2\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T) + \frac{\sigma_4\sigma_2}{8h^6}\sum_{\substack{p,q=1\\p\neq q}}^{n}\delta_p^2\tilde{\mathbf{h}}C^T\delta_p^2\tilde{\mathbf{h}}\delta_q^2\tilde{\mathbf{h}}^T$$

$$+ \frac{\sigma_4\sigma_2}{8h^6}\sum_{\substack{p,q=1\\p\neq q}}^{n}\delta_p^2\tilde{\mathbf{h}}C^T\delta_q^2\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T + \frac{\sigma_4\sigma_2}{8h^6}\sum_{\substack{p,q=1\\p\neq q}}^{n}\delta_p^2\tilde{\mathbf{h}}C^T\delta_q^2\tilde{\mathbf{h}}\delta_q^2\tilde{\mathbf{h}}^T$$

$$+ \frac{\sigma_4\sigma_2}{8h^6}\sum_{\substack{p,q,r=1\\p\neq q\neq r}}^{n}\delta_p^2\tilde{\mathbf{h}}C^T\delta_q^2\tilde{\mathbf{h}}\delta_r^2\tilde{\mathbf{h}}^T + \frac{\sigma_4\sigma_2}{8h^6}\sum_{\substack{p,q=1\\p\neq q}}^{n}\delta_p^2\tilde{\mathbf{h}}C^T\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T$$

$$+\frac{\sigma_4\sigma_2}{8h^6}\sum_{\substack{p,q=1\\p\neq q}}^{n}\delta_q^2\tilde{\mathbf{h}}C^T\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T$$

$$+ \frac{\sigma_4\sigma_2}{8h^6}\sum_{\substack{p,q=1\\p\neq q}}^{n}\delta_q^2\tilde{\mathbf{h}}C^T\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_q\delta_q\mu_p\delta_p\tilde{\mathbf{h}}^T + \frac{\sigma_2^3}{8h^6}\sum_{\substack{p,q,r=1\\p\neq q\neq r}}^{n}\delta_p^2\tilde{\mathbf{h}}C^T\mu_q\delta_q\mu_r\delta_r\tilde{\mathbf{h}}\mu_q\delta_q\mu_r\delta_r\tilde{\mathbf{h}}^T$$

$$+ \frac{\sigma_4\sigma_2}{8h^6}\sum_{\substack{p,q=1\\p\neq q}}^{n}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}C^T\delta_p^2\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T$$

$$+ \frac{\sigma_4\sigma_2}{8h^6}\sum_{\substack{p,q=1\\p\neq q}}^{n}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}C^T\delta_p^2\tilde{\mathbf{h}}\mu_q\delta_q\mu_p\delta_p\tilde{\mathbf{h}}^T + \frac{\sigma_4\sigma_2}{8h^6}\sum_{\substack{p,q=1\\p\neq q}}^{n}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}C^T\delta_q^2\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T$$

$$
\begin{aligned}
&+ \frac{\sigma_2^3}{8h^6} \sum_{\substack{p,q,r=1 \\ p\neq q\neq r}}^{n} \mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}C^T\delta_r^2\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T + \frac{\sigma_2^3}{8h^6} \sum_{\substack{p,q,r=1 \\ p\neq q\neq r}}^{n} \mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}C^T\delta_r^2\tilde{\mathbf{h}}\mu_q\delta_q\mu_p\delta_p\tilde{\mathbf{h}}^T \\
&+ \frac{\sigma_2^3}{8h^6} \sum_{\substack{p,q,r=1 \\ p\neq q\neq r}}^{n} \mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}C^T\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\delta_r^2\tilde{\mathbf{h}}^T + \frac{\sigma_4\sigma_2}{8h^6} \sum_{\substack{p,q=1 \\ p\neq q}}^{n} \mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}C^T\delta_q^2\tilde{\mathbf{h}}\mu_q\delta_q\mu_p\delta_p\tilde{\mathbf{h}}^T \\
&+ \frac{\sigma_4\sigma_2}{8h^6} \sum_{\substack{p,q=1 \\ p\neq q}}^{n} \left(\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}C^T\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T + \mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}C^T\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\delta_q^2\tilde{\mathbf{h}}^T\right) \\
&+ \frac{\sigma_2^3}{8h^6} \sum_{\substack{p,q,r=1 \\ p\neq q\neq r}}^{n} \Big(\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}C^T\mu_q\delta_q\mu_r\delta_r\tilde{\mathbf{h}}\mu_p\delta_p\mu_r\delta_r\tilde{\mathbf{h}}^T \\
&\quad +\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}C^T\mu_q\delta_q\mu_r\delta_r\tilde{\mathbf{h}}\mu_r\delta_r\mu_p\delta_p\tilde{\mathbf{h}}^T + \mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}C^T\mu_p\delta_p\mu_r\delta_r\tilde{\mathbf{h}}\mu_r\delta_r\mu_q\delta_q\tilde{\mathbf{h}}^T\Big) \\
&+ \frac{\sigma_2^3}{8h^6} \sum_{\substack{p,q,r=1 \\ p\neq q\neq r}}^{n} \left(\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}C^T\mu_p\delta_p\mu_r\delta_r\tilde{\mathbf{h}}\mu_q\delta_q\mu_r\delta_r\tilde{\mathbf{h}}^T\right) \\
&+ \frac{\sigma_2}{4h^2} \sum_{p=1}^{n} \left(\delta_p^2 h_i e_j R_{j,j} + \delta_p^2 h_j e_i R_{i,i}\right)\Bigg) + \frac{\sigma_2}{2h^2} \sum_{p=1}^{n} \delta_p^2\tilde{\mathbf{h}}^T\delta_{i,j}R_{i,j}
\end{aligned}
$$

Appendix B. Proof of Lemma (2.1)

Proof.

$$
\begin{aligned}
r_{0,2}(t_k \mid t_{k-1})_{\{\tilde{\mathbf{h}},\tilde{\mathbf{h}}^T\}} &= \mathbb{E}\left[(y(t_k) - \hat{y}(t_k \mid t_{k-1}))(y(t_k) - \hat{y}(t_k \mid t_{k-1}))^T \mid Y_{k-1}\right] \\
&= \mathbb{E}\Bigg[\left(\tilde{D}_{\Delta z}\tilde{\mathbf{h}} + \frac{1}{2}\tilde{D}_{\Delta z}^2\tilde{\mathbf{h}} + v - C\right) \times \\
&\quad \left(\tilde{D}_{\Delta z}\tilde{\mathbf{h}} + \frac{1}{2}\tilde{D}_{\Delta z}^2\tilde{\mathbf{h}} + v - C\right)^T \mid Y_{k-1}\Bigg] \\
&= \mathbb{E}\left[\left(\tilde{D}_{\Delta z}\tilde{\mathbf{h}} + \frac{1}{2}\tilde{D}_{\Delta z}^2\tilde{\mathbf{h}} + v\right)\left(\tilde{D}_{\Delta z}\tilde{\mathbf{h}} + \frac{1}{2}\tilde{D}_{\Delta z}^2\tilde{\mathbf{h}} + v\right)^T \mid Y_{k-1}\right] \\
&- \mathbb{E}\left(\tilde{D}_{\Delta z}\tilde{\mathbf{h}} + \frac{1}{2}\tilde{D}_{\Delta z}^2\tilde{\mathbf{h}} + v\right)C^T - C\mathbb{E}\left(\tilde{D}_{\Delta z}\tilde{\mathbf{h}} + \frac{1}{2}\tilde{D}_{\Delta z}^2\tilde{\mathbf{h}} + v\right)^T \\
&\quad +CC^T
\end{aligned}
$$

$$
\begin{aligned}
r_{1,1}(t_k \mid t_{k-1}) &= \mathbb{E}\left[(x(t_k) - \hat{x}(t_k \mid t_{k-1}))(y(t_k) - \hat{y}(t_k \mid t_{k-1}))^T \mid Y_{k-1}\right] \\
&= \mathbb{E}\left[\Delta x(t_k)(y(t_k) - \hat{y}(t_k \mid t_{k-1}))^T \mid Y_{k-1}\right] \\
&= \mathbb{E}\left[S_x\Delta z(t_k)\left(\tilde{D}_{\Delta z}\tilde{\mathbf{h}} + \frac{1}{2}\tilde{D}_{\Delta z}^2\tilde{\mathbf{h}} + v - C\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
r_{1,2}(t_k \mid t_{k-1}) &= \mathbb{E}\left[S_x \Delta z(t_k)(y(t_k)-\hat{y}(t_k \mid t_{k-1}))^T \left(\mathbb{Y}(t_k)-\hat{\mathbb{Y}}(t_k \mid t_{k-1})\right)^T\right] \\
&\quad + \mathbb{E}\left[\hat{x}(t_k \mid t_{k-1})(y(t_k)-\hat{y}(t_k \mid t_{k-1}))^T \left(\mathbb{Y}(t_k)-\hat{\mathbb{Y}}(t_k \mid t_{k-1})\right)^T\right] \\
&= \mathbb{E}\left[S_x \Delta z(t_k)\left\{(y_1(t_k)-\hat{y}_1(t_k \mid t_{k-1}))(y(t_k)-\hat{y}(t_k \mid t_{k-1}))^T, \ldots,\right.\right. \\
&\quad \left.\left.(y_n(t_k)-\hat{y}_n(t_k \mid t_{k-1}))(y(t_k)-\hat{y}(t_k \mid t_{k-1}))^T\right\}\right] \\
&+ \hat{x}(t_k \mid t_{k-1})\mathbb{E}\left[(y_1(t_k)-\hat{y}_1(t_k \mid t_{k-1}))(y(t_k)-\hat{y}(t_k \mid t_{k-1}))^T, \ldots,\right. \\
&\quad \left.(y_n(t_k)-\hat{y}_n(t_k \mid t_{k-1}))(y(t_k)-\hat{y}(t_k \mid t_{k-1}))^T\right]
\end{aligned}
$$

$$
r_{0,3}(t_k \mid t_{k-1}) = \left(\mathbb{E}\left(\Delta y_i(t_k)\Delta y_j(t_k)\Delta y(t_k)^T\right)\right)_{1\le i\le n, 1\le j\le n}
$$

$$
\begin{aligned}
r_{0,4}(t_k \mid t_{k-1}) &= \mathbb{E}\left[\left(\tilde{D}_{\Delta z}\tilde{\mathbf{h}} + \frac{1}{2}\tilde{D}^2_{\Delta z}\tilde{\mathbf{h}} + v - C\right)\left(\tilde{D}_{\Delta z}\tilde{\mathbf{h}} + \frac{1}{2}\tilde{D}^2_{\Delta z}\tilde{\mathbf{h}} + v - C\right)^T\right. \\
&\quad \left.\left(\tilde{D}_{\Delta z}\tilde{\mathbf{h}} + \frac{1}{2}\tilde{D}^2_{\Delta z}\tilde{\mathbf{h}} + v - C\right)\left(\tilde{D}_{\Delta z}\tilde{\mathbf{h}} + \frac{1}{2}\tilde{D}^2_{\Delta z}\tilde{\mathbf{h}} + v - C\right)^T \mid Y_{k-1}\right] \\
&= \mathbb{E}\left[\mathcal{A}\mathcal{A}^T\mathcal{A}\mathcal{A}^T \mid Y_{t_{k-1}}\right] - \mathbb{E}\left[\mathcal{A}\mathcal{A}^T\mathcal{A}C^T \mid Y_{t_{k-1}}\right] - \mathbb{E}\left[\mathcal{A}\mathcal{A}^T C\mathcal{A}^T \mid Y_{t_{k-1}}\right] \\
&+ \mathbb{E}\left[\mathcal{A}\mathcal{A}^T CC^T\right] - \mathbb{E}\left[\mathcal{A}C^T\mathcal{A}\mathcal{A}^T\right] + \mathbb{E}\left[\mathcal{A}C^T\mathcal{A}C^T\right] \\
&+ \mathbb{E}\left[\mathcal{A}C^T C\mathcal{A}^T\right] - \mathbb{E}\left[\mathcal{A}C^T CC^T\right]
\end{aligned}
$$

$$
\begin{aligned}
M_{0,2}(t_k \mid t_{k-1}) &= \left(\mathbb{E}\left(\Delta y(t_k)\Delta y_i(t_k)\Delta y_j(t_k)\Delta y(t_k)^T\right)\right)_{1\le i\le n, 1\le j\le n} \\
r_{2,2}(t_k \mid t_{k-1}) &= \mathbb{E}\left[\Delta x(t_k)\Delta x(t_k)^T\Delta y(t_k)\Delta y(t_k)^T\right] \\
&= \mathbb{E}\left[S_x \sum_{k=1}^{n}\left[\Delta z_i \Delta z_k S_{j,k}\right]\left(\tilde{D}_{\Delta z}\tilde{\mathbf{h}} + \frac{1}{2}\tilde{D}^2_{\Delta z}\tilde{\mathbf{h}} + v - C\right)\right. \\
&\quad \left.\left(\tilde{D}_{\Delta z}\tilde{\mathbf{h}} + \frac{1}{2}\tilde{D}^2_{\Delta z}\tilde{\mathbf{h}} + v - C\right)^T \mid Y_{k-1}\right] \\
&= S_x Q_{i,j}, \quad \text{where } Q_{i,j} \text{ is defined below} \\
r_{1,3}(t_k \mid t_{k-1}) &= \mathbb{E}\left[\Delta x(t_k)\Delta y(t_k)^T\Delta y(t_k)\Delta y(t_k)^T\right] \\
Q_{i,j} &= \frac{1}{4h^4}\left[\sigma_6 S_{j,i}\delta_i^2\tilde{\mathbf{h}}\delta_i^2\tilde{\mathbf{h}}^2 + \sum_{p=1}^{n}\sigma_2\sigma_4 S_{j,i}\delta_p^2\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T\right. \\
&\quad \left. + \sum_{p=1}^{n}\sigma_2\sigma_4 S_{j,i}\left[\delta_i^2\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T + \delta_p^2\tilde{\mathbf{h}}\delta_i^2\tilde{\mathbf{h}}^T\right] + \sum_{\substack{p,q=1 \\ p\neq q}}^{n}\sigma_2^3 S_{j,i}\delta_p^2\tilde{\mathbf{h}}\delta_q^2\tilde{\mathbf{h}}^T\right] \\
&+ \frac{\sigma_2\sigma_4}{4h^4}\sum_{p=1}^{n}S_{j,p}\left[\delta_i^2\tilde{\mathbf{h}}\mu_i\delta_i\mu_p\delta_p\tilde{\mathbf{h}}^T + \delta_p^2\tilde{\mathbf{h}}\mu_i\delta_i\mu_p\delta_p\tilde{\mathbf{h}}^T + \delta_p^2\tilde{\mathbf{h}}\mu_p\delta_p\mu_i\delta_i\tilde{\mathbf{h}}^T\right.
\end{aligned}
$$

$$+\delta_i^2\tilde{\mathbf{h}}\mu_i\delta_i\mu_p\delta_p\tilde{\mathbf{h}}^T\Big]$$

$$+ \quad \frac{\sigma_2^3}{4h^4}\sum_{\substack{p,r=1\\p\neq r}}^n S_{j,p}\Big[\delta_r^2\tilde{\mathbf{h}}\mu_i\delta_i\mu_p\delta_p\tilde{\mathbf{h}}^T + \delta_r^2\tilde{\mathbf{h}}\mu_p\delta_p\mu_i\delta_i\tilde{\mathbf{h}}^T + \mu_i\delta_i\mu_p\delta_p\tilde{\mathbf{h}}\delta_r^2\tilde{\mathbf{h}}^T$$

$$+\mu_p\delta_p\mu_i\delta_i\tilde{\mathbf{h}}\delta_r^2\tilde{\mathbf{h}}^T\Big]$$

$$+ \quad \frac{\sigma_2\sigma_4}{4h^4}\sum_{p=1}^n S_{j,p}\Big[\mu_i\delta_i\mu_p\delta_p\tilde{\mathbf{h}}\delta_i^2\tilde{\mathbf{h}}^T + \mu_i\delta_i\mu_p\delta_p\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T + \mu_p\delta_p\mu_i\delta_i\tilde{\mathbf{h}}\delta_p^2\tilde{\mathbf{h}}^T$$

$$+\mu_p\delta_p\mu_i\delta_i\tilde{\mathbf{h}}\delta_i^2\tilde{\mathbf{h}}^T\Big]$$

$$+ \quad \frac{\sigma_2^3}{4h^4}\sum_{\substack{p,q=1\\p\neq q}}^n S_{j,p}\Big[\mu_i\delta_i\mu_q\delta_q\tilde{\mathbf{h}}\mu_q\delta_q\mu_p\delta_p\tilde{\mathbf{h}}^T + \mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\mu_i\delta_i\tilde{\mathbf{h}}^T$$

$$+\mu_i\delta_i\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T + \mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_i\delta_i\mu_q\delta_q\tilde{\mathbf{h}}^T\Big]$$

$$+ \quad \frac{\sigma_2^3}{4h^4}\sum_{\substack{p,q=1\\p\neq q}}^n S_{j,q}\Big[\mu_p\delta_p\mu_i\delta_i\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T + \mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\mu_i\delta_i\tilde{\mathbf{h}}^T$$

$$+\mu_p\delta_p\mu_i\delta_i\tilde{\mathbf{h}}\mu_q\delta_q\mu_p\delta_p\tilde{\mathbf{h}}^T + \mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_i\delta_i\mu_p\delta_p\tilde{\mathbf{h}}^T\Big]$$

$$+ \quad \frac{\sigma_2^3}{4h^4}\sum_{\substack{p,q=1\\p\neq q}}^n S_{j,i}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}\mu_p\delta_p\mu_q\delta_q\tilde{\mathbf{h}}^T + \frac{\sigma_2^2}{h^2}\Bigg[\sum_{p=1}^n S_{j,i}\mu_p\delta_p\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T$$

$$+S_{j,p}\mu_i\delta_i\tilde{\mathbf{h}}\mu_p\delta_p\tilde{\mathbf{h}}^T + S_{j,p}\mu_p\delta_p\tilde{\mathbf{h}}\mu_i\delta_i\tilde{\mathbf{h}}^T\Bigg]$$

$$+ \quad \frac{\sigma_2^2}{2h^2}\sum_{p=1}^n S_{j,i}\Big[\delta_p^2\tilde{\mathbf{h}}C^T + C\delta_p^2\tilde{\mathbf{h}}^T\Big] + \frac{\sigma_2^2}{2h^2}\sum_{p=1}^n S_{j,p}\Big[\mu_i\delta_i\mu_p\delta_p\tilde{\mathbf{h}} + \mu_p\delta_p\mu_i\delta_i\tilde{\mathbf{h}}\Big]C^T$$

$$+ \quad \frac{\sigma_2^2}{2h^2}\sum_{p=1}^n S_{j,p}\Big[C\mu_i\delta_i\mu_p\delta_p\tilde{\mathbf{h}}^T + C\mu_p\delta_p\mu_i\delta_i\tilde{\mathbf{h}}^T\Big]$$

where $\Delta x(t_k) = x(t_k) - \hat{x}(t_k \mid t_{k-1})$, $\Delta y(t_k) = y(t_k) - \hat{y}(t_k \mid t_{k-1})$ and $M_{0,2}(t_k \mid t_{k-1})$ can be generated from $r_{0,4}(t_k \mid t_{k-1})$ □

Appendix C. Proof of Lemma 2.2

Proof. From (2.2) and applying Baye's rule, we have

$$P(x(t_k), y(t_k) \mid Y_{t_{k-1}}) = P(x(t_k) \mid Y_{t_k})P(y(t_k) \mid Y_{t_{k-1}}). \tag{C.1}$$

Multiplying equation (C.1) by the product of two arbitrary functions $s(x(t_k))$ and $u(y(t_k))$, and taking the expectations, we have

$$\begin{aligned}\int\int s(x)u(y)P(x,y \mid Y_{t_{k-1}})dxdy &= \int\int s(x)u(y)P(x \mid Y_{t_k})P(y \mid Y_{t_{k-1}})dxdy \\ &= \int\left[\int s(x)P(x \mid Y_{t_k})dx\right]u(y)P(y \mid Y_{t_{k-1}})dy \\ &= \mathbb{E}\left[\mathbb{E}\left[s(x(t_k)) \mid Y_{t_k}\right]u(y(t_k))Y_{t_{k-1}}\right].\end{aligned}$$

Hence,

$$\mathbb{E}\left[s(x(t_k))u(y(t_k)) \mid Y_{t_{k-1}}\right] = \mathbb{E}\left[\mathbb{E}\left[s(x(t_k)) \mid Y_{t_k}\right]u(y(t_k)) \mid Y_{t_{k-1}}\right]. \tag{C.2}$$

Equation (C.2) provides a systematic feasible procedure for solving for A_i, B_i, $i = 0, 1$, and A_2.

Substituting $s = x(t_k)$ and $u = 1$, we have

$$\mathbb{E}\left[x(t_k) \mid Y_{t_{k-1}}\right] = \mathbb{E}\left[\mathbb{E}\left[x(t_k) \mid Y_{t_k}\right] \mid Y_{t_{k-1}}\right]. \tag{C.3}$$

Hence

$$\mathbb{E}\left[x(t_k) \mid Y_{t_{k-1}}\right] = \mathbb{E}\left[\mathbb{E}\left[x(t_k) \mid Y_{t_k}\right] \mid Y_{t_{k-1}}\right].$$

This implies that

$$\begin{aligned}\hat{x}(t_k \mid t_{k-1}) &= \mathbb{E}\left[\hat{x}(t_k \mid t_k) \mid Y_{t_{k-1}}\right] \\ &= \mathbb{E}\left[A_0 + A_1(y(t_k) - \hat{y}(t_k \mid t_{k-1})) + A_2(\mathbb{Y} - \hat{\mathbb{Y}})(y(t_k) - \hat{y}(t_k \mid t_{k-1})) \mid Y_{t_{k-1}}\right].\end{aligned}$$

Thus,

$$r_{1,0}(t_k \mid t_{k-1}) = A_0(t_k \mid t_{k-1}) + A_2(t_k \mid t_{k-1})\sigma_{Y2}(t_k \mid t_{k-1}), \tag{C.4}$$

where $r_{1,0}(t_k \mid t_{k-1}) = \hat{x}(t_k \mid t_{k-1})$. Substituting $s = x(t_k)$ and $u = (y(t_k) - \hat{y}(t_k \mid t_{k-1}))^T$, we have

$$\begin{aligned}&\mathbb{E}\left[x(t_k)(y(t_k) - \hat{y}(t_k \mid t_{k-1}))^T \mid Y_{t_{k-1}}\right] \\ &= \mathbb{E}\left[\mathbb{E}\left[x(t_k) \mid Y_{t_k}\right](y(t_k) - \hat{y}(t_k \mid t_{k-1}))^T \mid Y_{t_{k-1}}\right] \\ &= \mathbb{E}\left[\mathbb{E}\left[(A_0 + A_1(y(t_k) - \hat{y}(t_k \mid t_{k-1}))\right.\right. \\ &\quad \left.\left. + A_2(\mathbb{Y} - \hat{\mathbb{Y}})(y(t_k) - \hat{y}(t_k \mid t_{k-1}))\right)(y(t_k) - \hat{y}(t_k \mid t_{k-1}))^T\right] \mid Y_{t_{k-1}}\right],\end{aligned}$$

Hence,

$$r_{1,1}(t_k \mid t_{k-1}) = A_1(t_k \mid t_{k-1})r_{0,2}(t_k \mid t_{k-1}) + A_2(t_k \mid t_{k-1})r_{0,3}^T(t_k \mid t_{k-1}). \tag{C.5}$$

Lastly, substituting $s = x(t_k)$ and $u = (y(t_k) - \hat{y}(t_k \mid t_{k-1}))^T(\mathbb{Y} - \hat{\mathbb{Y}})^T$, we have

$$\begin{aligned}&\mathbb{E}\left[x(t_k)(y(t_k) - \hat{y}(t_k \mid t_{k-1}))^T(\mathbb{Y} - \hat{\mathbb{Y}})^T \mid Y_{t_{k-1}}\right] \\ &= \mathbb{E}\left[\mathbb{E}\left[x(t_k) \mid Y_{t_k}\right](y(t_k) - \hat{y}(t_k \mid t_{k-1}))^T(\mathbb{Y} - \hat{\mathbb{Y}})^T \mid Y_{t_{k-1}}\right].\end{aligned} \tag{C.6}$$

Hence,

$$\text{(C.7)}\qquad \begin{aligned} r_{1,2}(t_k \mid t_{k-1}) &= A_0(t_k \mid t_{k-1})\sigma^T_{Y2}(t_k \mid t_{k-1}) + A_1(t_k \mid t_{k-1})r_{0,3}(t_k \mid t_{k-1}) \\ &\quad + A_2(t_k \mid t_{k-1})M_{0,2}(t_k \mid t_{k-1}). \end{aligned}$$

The result follows by solving the systems of linear equations (C.4), (C.5), (C.7). □

Appendix D. Proof of Lemma 2.4

Proof. First, we substitute $s = (x(t_k) - \hat{x}(t_k \mid t_k))(x(t_k) - \hat{x}(t_k \mid t_k))^T$ and $u = 1$ into equation (C.2) and obtain

$$\begin{aligned} &\mathbb{E}\left[(x(t_k) - \hat{x}(t_k \mid t_k))(x(t_k) - \hat{x}(t_k \mid t_k))^T \mid Y_{t_{k-1}}\right] \\ &= \mathbb{E}\left[\mathbb{E}\left[(x(t_k) - \hat{x}(t_k \mid t_k))(x(t_k) - \hat{x}(t_k \mid t_k))^T \mid Y_{t_k}\right] \mid Y_{t_{k-1}}\right] \\ &= \mathbb{E}\left[P(t_k \mid t_k)\right]. \end{aligned}$$

Hence,

$$\text{(D.1)}\qquad N_1 = B_0 + B_1 r_{0,2}(t_k \mid t_{k-1}).$$

Lastly, substituting

$$\begin{aligned} s &= (x(t_k) - \hat{x}(t_k \mid t_k))(x(t_k) - \hat{x}(t_k \mid t_k))^T \\ u &= (y(t_k) - \hat{y}(t_k \mid t_{k-1}))(y(t_k) - \hat{y}(t_k \mid t_{k-1}))^T \end{aligned}$$

into equation (C.2)

$$\text{(D.2)}\qquad N_2 = B_0 r_{0,2}(t_k \mid t_{k-1}) + B_1 r_{0,4}.$$

The fifth and upper moments of $y(t_k) - \hat{y}(t_k \mid t_{k-1})$ is neglected in N_2.

$$\begin{aligned} &\mathbb{E}\left[(x(t_k) - \hat{x}(t_k \mid t_{k-1}))(y(t_k) - \hat{y}(t_k \mid t_{k-1}))^T A_1^T (y(t_k) - \hat{y}(t_k \mid t_{k-1}))\times \right. \\ &\left. (y(t_k) - \hat{y}(t_k \mid t_{k-1}))^T \mid Y_{k-1}\right] \end{aligned}$$

can be generated from $r_{1,3}$,

$$\begin{aligned} &\mathbb{E}\left[(x(t_k \mid t_{k-1}) - A_0)(y(t_k) - \hat{y}(t_k \mid t_{k-1}))^T A_1^T (y(t_k) - \hat{y}(t_k \mid t_{k-1}))\times \right. \\ &\left. (y(t_k) - \hat{y}(t_k \mid t_{k-1}))^T \mid Y_{t_{k-1}}\right] \end{aligned}$$

can be generated from $r_{0,3}$,

$$\begin{aligned} &\mathbb{E}\left[A_1(y(t_k) - \hat{y}(t_k \mid t_{k-1}))(x(t_k) - \hat{x}(t_k \mid t_{k-1}))^T (y(t_k) - \hat{y}(t_k \mid t_{k-1}))\times \right. \\ &\left. (y(t_k) - \hat{y}(t_k \mid t_{k-1}))^T \mid Y_{t_{k-1}}\right] \end{aligned}$$

can be generated from $r_{1,3}$,

$$\begin{aligned} &\mathbb{E}\left[A_1(y(t_k) - \hat{y}(t_k \mid t_{k-1}))(x(t_k \mid t_{k-1}) - A_0)^T (y(t_k) - \hat{y}(t_k \mid t_{k-1}))\times \right. \\ &\left. (y(t_k) - \hat{y}(t_k \mid t_{k-1}))^T \mid Y_{t_{k-1}}\right] \end{aligned}$$

can be generated from $r_{0,3}$,

$$\mathbb{E}\left[A_1(y(t_k) - \hat{y}(t_k \mid t_{k-1}))(y(t_k) - \hat{y}(t_k \mid t_{k-1}))^T A_1^T (y(t_k) - \hat{y}(t_k \mid t_{k-1}))\times \right.$$
$$\left.(y(t_k) - \hat{y}(t_k \mid t_{k-1}))^T \mid Y_{t_{k-1}}\right]$$

can be generated from $r_{0,4}$, where $r_{1,3}$, $r_{0,3}$, and $r_{0,4}$ are defined in Appendix B. The conclusion of the Lemma follows by solving the systems of equation (D.1) and (D.2). □