Neural, Parallel, and Scientific Computations 22 (2014) 149-158

CHANGE POINT ANALYSIS IN LINEAR REGRESSION MODELS

REZA HABIBI

Department of Statistics, Central Bank of Iran
Ferdowsi Ave., Postal Code: 1135931496, Tehran, Iran

ABSTRACT. This paper is concerned with the change point detection in the linear regression models. Test procedures considered are the incomplete U-process and the quasi-Bayesian test procedures. The asymptotic null distribution of test statistics are proposed in terms of supremum of the Guassian process and the stochastic integral with respect to the Kiefer process.

Keywords: Change Point; Kiefer Process; Linear Regression; Quasi-Bayesian; Stochastic Integral; U-Process

1. Introduction

Let $X_1, \ldots, X_n$ be realizations of the following linear regression model

$$X_i = \alpha\gamma\left(\frac{i}{n}\right) + \varepsilon_i,$$

$i = 1, 2, \ldots, n$ where α, the regression coefficient, is an unknown parameter and function $\gamma(\cdot)$ has known functional form with $\int_0^1 \gamma^2(t)dt = c \in (0, \infty)$. Define function $\lambda_\gamma(t)$ as follows

$$\lambda_\gamma(t) = \int_0^t \gamma(u)du - t\int_0^1 \gamma(u)du.$$

It is assumed that the residuals $\varepsilon_1, \ldots, \varepsilon_n$ are independent zero mean random variables whose distribution functions are $F_i(\cdot)$, $i = 1, 2, \ldots, n$. The null hypothesis of no change point H_0 specifies the assumption

$$H_0 : F_1 = \cdots = F_n = F,$$

(F unknown) which under the local alternative hypothesis there exists an unknown change point $t_0 \in (0, 1)$ such that

$$H_1 : F_i = \begin{cases} F & i = 1, 2, \ldots, [nt_0], \\ (1 - \frac{\delta}{\sqrt{n}})F + \frac{\delta}{\sqrt{n}}G & i = [nt_0] + 1, \ldots, n, \end{cases}$$

for some distribution function G such that $F \neq G$ and $\delta \in (0, 1)$. That is, under H_1, the random variables ε_i, $i \geq [nt_0] + 1$ have a mixture distribution. Let $\sigma^2 =$

Received August 16, 2013

$\int x^2 dF(x) < \infty$. The least square estimate of α, $\widehat{\alpha}$, is given by

$$\widehat{\alpha} - \alpha = \frac{\sum_{i=1}^{n} \gamma(\frac{i}{n})\varepsilon_i}{\sum_{i=1}^{n} \gamma^2(\frac{i}{n})}.$$

It can be shown that, under H_0, as $n \to \infty$, then $\widehat{\alpha} - \alpha = O_p(n^{-1/2})$, that is

$$\widehat{v}_n = \sqrt{n}(\widehat{\alpha} - \alpha) \xrightarrow{d} U = c^{-1} \int_0^1 \gamma(t) dW(t),$$

where $W(\cdot)$ is the standard Brownian motion on $[0,1]$ and U is distributed as $N(0, c^{-1})$.

During the last four decades, we have witnessed many exciting developments to check the stability of a parametric models over a period of time. Chernoff and Zacks (1964) introduced the quasi-Bayesian test statistics for detecting shifts in the mean of normal observations. Their results were extended to the exponential family by Kander and Zacks (1966). Haccou, Meelis and van de Geer (1988) considered the likelihood ratio test for a change in a sequence of independent exponentially distributed random variables. Csörgő and Horváth (1986) proposed a test based on supremum of U-statistics. The prototypical Kolmogorov-Smirnov test statistics can be found in Carlstein (1988) and Csörgő and Horváth (1987). Einmahl and McKeague (2003) used empirical likelihood to introduce a test for a change in distribution function. Zarepour and Habibi (2006) extended Kander and Zacks (1966) results in exponential family to a general class of distributions. An excellent reference in the change point analysis is Csörgő and Horváth (1997). The purpose of this paper is to provide tools for the instability in linear regressions. This paper is organized as follows. Section 2 contains the incomplete U-process test procedure. The test statistic is represented and its asymptotic null distribution is proposed as supremum of a Gaussian process. In section 3, the quasi-Bayesian test procedure is considered. It is shown that the limiting null distribution of the quasi-Bayesian test statistic is a stochastic integral with respect to Kiefer process.

2. **U-Process Procedure**

Csörgő and Horváth (1986) constructed a test procedure based on the supremum of the following incomplete U-process

$$\sum_{i=1}^{[nt]} \sum_{j=[nt]+1}^{n} h(\varepsilon_i, \varepsilon_j),$$

where $h(\cdot,\cdot)$ is a symmetric kernel. They showed that, under H_0, as $n \to \infty$, then

$$n^{-1/2} \sup_{0<t<1} \sum_{i=1}^{[nt]} \sum_{j=[nt]+1}^{n} \{h(\varepsilon_i, \varepsilon_j) - \theta\} \xrightarrow{d} \sup_{0<t<1} \Gamma(t),$$

where $\theta = E_{H_0}(h(\varepsilon_1, \varepsilon_2))$ and $\Gamma(t)$ is a Guassian process. During this paper, it is assumed that $h(x,y) = \rho(x-y)$, for some even loss function $\rho(\cdot)$. Some selects for

$\rho(x)$ are $|x|$ and $I(x \le s)$ and x^2. The residuals ε_i are unobservable in practice and so they are replaced by estimated residuals $e_i = \varepsilon_i - n^{-1/2}\widehat{v}_n\gamma(\frac{i}{n})$, and the estimated U-process test statistic is given by

$$\sup_{0<t<1} \sum_{i=1}^{[nt]} \sum_{j=[nt]+1}^{n} \{\rho(e_{ij}) - \theta\},$$

where $e_{ij} = e_i - e_j$. The difference stochastic process $\Lambda_n(t)$ is defined by

$$\Lambda_n(t) = \sum_{i=1}^{[nt]} \sum_{j=[nt]+1}^{n} \{\rho(e_{ij}) - \rho(\varepsilon_{ij})\} = \Lambda_n(t, \widehat{v}_n),$$

where

$$\Lambda_n(t, v) = \sum_{i=1}^{[nt]} \sum_{j=[nt]+1}^{n} \rho(\varepsilon_{ij} - n^{-1/2}\gamma_{ij}v) - \rho(\varepsilon_{ij}).$$

where $\varepsilon_{ij} = \varepsilon_i - \varepsilon_j$ and $\gamma_{ij} = \gamma(\frac{i}{n}) - \gamma(\frac{j}{n})$. Sometimes, as $n \to \infty$, under H_0, it can be shown that $\Lambda_n(\cdot,\cdot) \xrightarrow{d} \Lambda(\cdot,\cdot)$, for some two parameters convergent process $\Lambda(t, v)$, then the continuous mapping theorem can be applied to show that the U-process test statistic and estimated U-process test statistic have the same asymptotic null distributions. Suppose that the distribution function F admits a density function f ($f > 0$) and let $f^* = f * f$ be the convulsion of f.

For more illustration, for example, suppose that $\rho(x) = |x|$. Using the identity that for $x \neq 0$

$$|x - y| - |x| = -y\, sgn(x) + 2\int_0^y [I(x \le s) - I(x \le 0)]ds,$$

(see Knight (1998)) then

$$\Lambda_n(t, v) = \sum_{i=1}^{[nt]} \sum_{j=[nt]+1}^{n} \Gamma_{ij}(v),$$

where

$$\Gamma_{ij}(v) = 2\int_0^{n^{-1/2}\gamma_{ij}v} [I(\varepsilon_{ij} \le s) - I(\varepsilon_{ij} \le 0)]ds - n^{-1/2}\gamma_{ij} sgn(\varepsilon_{ij})v.$$

It can be shown that the convergent process $\Lambda(t, v)$ is given by

$$\Lambda(t, v) = 2f^*(0)\lambda_\gamma(t)v - \int_0^t \int_t^1 \gamma(x, y)W^*(dx, dy)v,$$

with $W^*(x, y) = \cdots$ and $\gamma(x, y) = \gamma(x) - \gamma(y)$. Another example of the loss function is $\rho(x) = I(x \le s)$ for some s. It can be shown that $\Lambda(t, v) = f^*(s)\lambda_\gamma(t)v$. Facing with the smooth functions $\rho(\cdot)$, for example, when $\rho(x) = x^2$ then the estimated U-process statistic can be applied to detect the change in variance of ε_i. One can show that the convergent process $\Lambda(t,v)$ can be expressed as

$$\left\{\int_0^t\int_t^1 \gamma^2(x,y)dx\,dy\right\}v^2-2\sigma\left\{\int_0^t\int_t^1 \gamma(x,y)dW^*(x,y)\right\}v.$$

As follows, we suppose that $\rho(\cdot)$ is a convex, differentiable loss function with derivative function $\psi(\cdot)$ which satisfies the conditions

$A_1)$ $E(\psi(\varepsilon_1))=0$.

$A_2)$ $E(\psi^2(\varepsilon_1))=\sigma^2<\infty$.

$A_3)$ ψ has Lipschitz-continuous derivative ψ', i.e., there exists a nonnegative constant k such that for all x and y,

$$|\psi'(x)-\psi'(y)|\le k|x-y|.$$

$A_4)$ $0<|E\psi'(\varepsilon_1)|=|\eta|<\infty$.

As follows, mimicking Knight (1989), we show that the $\Lambda(t,v)$ is

$$\frac{\eta^2}{2}\left\{\int_0^t\int_t^1 \gamma^2(x,y)dx\,dy\right\}v^2-\sigma\left\{\int_0^t\int_t^1 \gamma(x,y)dW^*(x,y)\right\}v.$$

3. **Quasi-Bayesian Procedure**

Here, based on the quasi-Bayesian method of Kander and Zacks (1966) in Bernoulli distribution, we derive new version of weighted Kolmogorov-Smirnov type test statistic to test the null hypothesis of no change point. To do so, for any fixed x, let $\zeta_i=I(\varepsilon_i\le x)$, $p_i=F_i(x)$, $i=1,2,\ldots,n$ and assume that $p=F(x)$ and $p_0=G(x)$. The ζ_is are independent Bernoulli random variables with parameter of success p_i. The hypothesis testing problem reduces to

$$H_0: p_1=\cdots=p_n=p,$$

(p unknown) against the local alternative

$$H_1: p_i=\begin{cases}p & i=1,2,\ldots,[nt_0],\\ (1-\frac{\delta}{\sqrt{n}})p+\frac{\delta}{\sqrt{n}}p_0 & i=[nt_0]+1,\ldots,n.\end{cases}$$

Consider t_0 as a random variable with the prior density $\pi(\cdot)$ on $(0,1)$ and let $\Pi(t)=\int_0^t \pi(x)dx$ with $\Pi(0)=0$ and $\Pi(1)=1$. As $\delta(p-p_0)\to 0$, the quasi-Bayesian test statistic (Zarepour and Habibi (2006)) is given by $\sum_{i=1}^n \Pi(\frac{i-1}{n})(\zeta_i-\overline{\zeta}_n)$, which is

$$\sum_{i=1}^n \Pi\left(\frac{i-1}{n}\right)\{I(\varepsilon_i\le x)-F_n(x)\}.$$

To remove the effect of x, the weighted Kolmogorov-Smirnov type test statistic is given as follows

$$T_n=\sup_x\sum_{i=1}^n \Pi\left(\frac{i-1}{n}\right)\{I(\varepsilon_i\le x)-F_n(x)\}.$$

To study the null limiting behavior of T_n, let

$$K_n(t,x) = n^{-1/2}\sum_{i=1}^{[nt]}(I(\varepsilon_i \le x) - F_n(x)),$$

and notice that

$$n^{-1/2}T_n = \sup_x \int_0^1 \Pi(t)K_n(dt,x).$$

One can show that, as $n \to \infty$, under H_0,

$$K_n(\cdot,\cdot) \xrightarrow{d} \overline{K}(\cdot, F(\cdot)),$$

where $\overline{K}(t,x)$ is the Kiefer bridge defined based on the Kiefer process $K(t,x)$ as follows

$$\overline{K}(t,x) = K(t,x) - tK(1,x).$$

The continuous mapping theorem implies that

$$n^{-1/2}T_n \xrightarrow{d} \sup_x \int_0^1 \Pi(t)\overline{K}(dt, F_0(x)).$$

The Darling-Anderson type test statistic is given by $\int_{-\infty}^{\infty}\int_0^1 \Pi(t)K_n(dt,x)dx$, which as $n \to \infty$, converges to the $\int_{-\infty}^{\infty}\int_0^1 \Pi(t)\overline{K}(dt,F_0(x))dx$.

Remark 1. Test Procedures can be obtained under the random exchangeable weights (Zarepour and Habibi (2006)). For example, for $i = 1, \ldots, n-1$ let

$$\Pi\left(\frac{i}{n}\right) - \Pi\left(\frac{i-1}{n}\right) = \frac{G_{i(n-1)}}{\sum_{i=1}^n G_{i(n-1)}},$$

for a sequence of *iid* random variables $\{G_{in}\}$ satisfying

$$nP(G_{1n} \in dx) \xrightarrow{v} \alpha\frac{e^{-x}}{x}dx,$$

and then the test statistic is given by

$$T_n = \frac{\sum_{i=1}^n(\sum_{j=1}^{i-1} G_{i(n-1)})\{I(\varepsilon_i \le x) - F_n(x)\}}{\sum_{i=1}^n G_{i(n-1)}},$$

and as $n \to \infty$, then

$$n^{-1/2}T_n \xrightarrow{d} \sup_x \int_0^1 \frac{S(t)}{S(1)}\overline{K}(dt, F_0(x)),$$

where $S(t)$ is gamma process (Ferguson and Klass (1972)). On the other hand, let $\{G_i\}$ be a sequence of *iid* random variables such that there exists a sequence of positive constants a_n such that

$$nP(a_n^{-1}G_i \in dx) \xrightarrow{v} \alpha x^{-\alpha-1}I(x>0)dx,$$

and let $G_{in} = \frac{G_i}{a_n}$, and $n^{-1/2}$ times the test statistic converges to the

$$\sup_x \int_0^1 \frac{S(t)}{S(1)}\overline{K}(dt, F_0(x)),$$

where $S(t)$ is stable process (Resnick (1987)). Another choice for $\Pi(\frac{i}{n})$ is

$$\Pi\left(\frac{i}{n}\right) = U_{i:n},$$

where $U_{i:n}$ are the order statistics of a size of n sample of iid uniform random variables on $(0,1)$ with $U_{0:n}=0$ and $U_{n:n}=1$ (see Rubin (1981) and Lo (1987) and Shao (1995) in the Bayesian bootstrap setting). Notice that

$$(U_{1:n}, U_{2:n}, \ldots, U_{n-1:n}) \stackrel{d}{=} \left(\frac{S_1}{S_n}, \frac{S_2}{S_n}, \ldots, \frac{S_{n-1}}{S_n}\right),$$

where $S_i = \sum_{j=1}^{i} E_j$ for $E_1, \ldots, E_n$ a sequence of iid random variables with $\exp(1)$ distribution. It is easy to show that

$$n^{-1/2} T_n \stackrel{d}{\to} \sup_x \int_0^1 t\overline{K}(dt, F_0(x)),$$

which corresponds to the result when $\Pi(t) = t$.

Remark 2. Einmahal and Mckeague (2003) tested the change point using the empirical likelihood. One can show that the quasi-Bayesian empirical likelihood test statistic would reject the null hypothesis of no change point whenever

$$T_n^{**} = -2 \int_{-\infty}^{\infty} \Pi(t) \log R(t,x) dt,$$

is large (see Einmahal and Mckeague (2003)). It can be shown that under the null hypothesis as $n \to \infty$,

$$T_n^{**} \stackrel{d}{\to} \int_0^1 \int_0^1 \Pi(t) \frac{W_0(t,y)}{t(1-t)y(1-y)} dy\, dt,$$

where

$$W_0(t,y) = W(t,y) - tW(1,y) - yW(t,1) + tyW(1,1),$$

which $W(\cdot,\cdot)$ is a standard Bivariate Wienner process.

In practice, using the estimated residuals e_i, the estimated quasi-Bayes test statistic $\widehat{T}_n$ is given by

$$\widehat{T}_n = \sup_x \sum_{i=1}^{n} \Pi\left(\frac{i-1}{n}\right) \{I(e_i \le x) - F_n(x)\}.$$

It can be shown that under H_0

$$n^{-1/2} \sup_{x,t} \left| \sum_{i=1}^{[nt]} I(e_i \le x) - I(\varepsilon_i \le x) - (\widehat{\alpha} - \alpha)\gamma\left(\frac{i}{n}\right) f(x) \right| = o_p(1),$$

as $n \to \infty$. Then

$$n^{-1/2} \left\{ \sum_{i=1}^{0[nt]} I(e_i \le x) - F_0(x) \right\} \stackrel{d}{\to} U^*(t,x),$$

where

$$U^*(t,x) = K(t,F_0(x)) + \frac{\int_0^1 \gamma(t)dW(t)}{\int_0^1 \gamma^2(t)dt}\int_0^t \gamma(x)dx,$$

and let

$$V(t,x) = U^*(t,x) - tU^*(1,x).$$

One can show that

$$n^{-1/2}\widehat{T}_n \xrightarrow{d} \sup_x \int_0^1 \Pi(t)V(dt,x).$$

This event happens in the other statistical models. Let us consider the following example.

Example. In the ARMA time series models, Bai (1994) proved that, under the some mild conditions,

$$n^{-1/2}\sup_{x,s}\Big|\sum_{i=1}^{[ns]} I(e_i \le x) - I(\varepsilon_i \le x)\Big| = o_p(1),$$

as $n \to \infty$, where $\{e_i\}$ and $\{\varepsilon_i\}$ are the estimated residuals and residuals of ARMA models. The the limiting behavior of $\widehat{T}_n$, obtained by substituting X_i by e_i, is the same as T_n, that is

$$n^{-1/2}\widehat{T}_n \xrightarrow{d} \sup_x \int_0^1 \Pi(t)\overline{K}(dt,F_0(x)).$$

Ling (1998) proved that in the unit root models $X_i = X_{i-1} + \varepsilon_i$, then

$$n^{-1/2}\sup_{x,t}\Big|\sum_{i=1}^{[nt]} I(e_i \le x) - I(\varepsilon_i \le x) - (\widehat{\alpha}-1)X_{i-1}f(x)\Big| = o_p(1),$$

where $f(x)$ is the density function of ε_i and $\widehat{\alpha}$ is the least square estimation of α in $X_i = \alpha X_{i-1} + \varepsilon_i$, that is

$$\widehat{\alpha} = \frac{\sum_{i=2}^n X_{i-1}X_i}{\sum_{i=2}^n X_{i-1}^2}.$$

Then

$$n^{-1/2}\widehat{T}_n \xrightarrow{d} \sup_x \int_0^1 \Pi(t)K^*(dt,x),$$

where

$$K^*(t,x) = \overline{K}(t,F_0(x)) + U(t,x),$$

with

$$U(t,x) = \frac{\int_0^1 W(t)dW(t)}{\int_0^1 W^2(t)dt}\left(\int_0^t W(u)du - t\int_0^1 W(u)du\right)f(x).$$

4. **Application: Change detection in distribution**

By applying the quasi-Bayesian method of Kander and Zacks (1966) in Bernoulli distribution, we derive new version of weighted Kolmogorov-Smirnov type test statistic to test the null hypothesis of no change point in distribution. To do so, for any fixed x, let $\zeta_i = I(\varepsilon_i \leq x)$, $p_i = F_i(x)$, $i = 1, 2, \ldots, n$ and assume that $p = F(x)$ and $p_0 = G(x)$. The ζ_is are independent Bernoulli random variables with parameter of success p_i. The hypothesis testing problem reduces to

$$H_0 : p_1 = \cdots = p_n = p,$$

(p unknown) against the local alternative

$$H_1 : p_i = \begin{cases} p & i = 1, 2, \ldots, [nt_0], \\ (1 - \frac{\delta}{\sqrt{n}})p + \frac{\delta}{\sqrt{n}} p_0 & i = [nt_0] + 1, \ldots, n. \end{cases}$$

Consider t_0 as a random variable with the prior density $\pi(\cdot)$ on $(0, 1)$ and let $\Pi(t) = \int_0^t \pi(x)dx$ with $\Pi(0) = 0$ and $\Pi(1) = 1$. As $\delta(p - p_0) \to 0$, the quasi-Bayesian test statistic (Zarepour and Habibi (2006)) is given by $\sum_{i=1}^n \Pi(\frac{i-1}{n})(\zeta_i - \overline{\zeta}_n)$, which is

$$\sum_{i=1}^n \Pi\left(\frac{i-1}{n}\right) \{I(\varepsilon_i \leq x) - F_n(x)\}.$$

To remove the effect of x, the weighted Kolmogorov-Smirnov type test statistic is given as follows

$$T_n = \sup_x \sum_{i=1}^n \Pi\left(\frac{i-1}{n}\right) \{I(\varepsilon_i \leq x) - F_n(x)\}.$$

To study the null limiting behavior of T_n, let

$$K_n(t, x) = n^{-1/2} \sum_{i=1}^{[nt]} (I(\varepsilon_i \leq x) - F_n(x)),$$

and notice that

$$n^{-1/2} T_n = \sup_x \int_0^1 \Pi(t) K_n(dt, x).$$

One can show that, as $n \to \infty$, under H_0,

$$K_n(\cdot, \cdot) \xrightarrow{d} \overline{K}(\cdot, F(\cdot)),$$

where $\overline{K}(t, x)$ is the Kiefer bridge defined based on the Kiefer process $K(t, x)$ as follows

$$\overline{K}(t, x) = K(t, x) - tK(1, x).$$

The continuous mapping theorem implies that

$$n^{-1/2} T_n \xrightarrow{d} \sup_x \int_0^1 \Pi(t) \overline{K}(dt, F_0(x)).$$

The Darling-Anderson type test statistic is given by $\int_{-\infty}^{\infty}\int_0^1 \Pi(t)K_n(dt,x)dx$, which as $n \to \infty$, converges to the $\int_{-\infty}^{\infty}\int_0^1 \Pi(t)\overline{K}(dt,F_0(x))dx$.

Remark 3. Test Procedures can be obtained under the random exchangeable weights (Zarepour and Habibi (2006)). For example, for $i = 1, \dots, n-1$ let

$$\Pi\left(\frac{i}{n}\right) - \Pi\left(\frac{i-1}{n}\right) = \frac{G_{i(n-1)}}{\sum_{i=1}^{n} G_{i(n-1)}},$$

for a sequence of *iid* random variables $\{G_{in}\}$ satisfying

$$nP(G_{1n} \in dx) \xrightarrow{v} \alpha\frac{e^{-x}}{x}dx,$$

and then the test statistic is given by

$$T_n = \frac{\sum_{i=1}^{n}(\sum_{j=1}^{i-1} G_{i(n-1)})\{I(\varepsilon_i \le x) - F_n(x)\}}{\sum_{i=1}^{n} G_{i(n-1)}},$$

and as $n \to \infty$, then

$$n^{-1/2}T_n \xrightarrow{d} \sup_x \int_0^1 \frac{S(t)}{S(1)}\overline{K}(dt,F_0(x)),$$

where $S(t)$ is gamma process (Ferguson and Klass (1972)). On the other hand, let $\{G_i\}$ be a sequence of *iid* random variables such that there exists a sequence of positive constants a_n such that

$$nP(a_n^{-1}G_i \in dx) \xrightarrow{v} \alpha x^{-\alpha-1}I(x>0)dx,$$

and let $G_{in} = \frac{G_i}{a_n}$, and $n^{-1/2}$ times the test statistic converges to the

$$\sup_x \int_0^1 \frac{S(t)}{S(1)}\overline{K}(dt,F_0(x)),$$

where $S(t)$ is stable process (Resnick (1987)). Another choice for $\Pi(\frac{i}{n})$ is

$$\Pi\left(\frac{i}{n}\right) = U_{i:n},$$

where $U_{i:n}$ are the order statistics of a size of n sample of *iid* uniform random variables on $(0,1)$ with $U_{0:n} = 0$ and $U_{n:n} = 1$ (see Rubin (1981) and Lo (1987) and Shao (1995) in the Bayesian bootstrap setting). Notice that

$$(U_{1:n}, U_{2:n}, \dots, U_{n-1:n}) \stackrel{d}{=} \left(\frac{S_1}{S_n}, \frac{S_2}{S_n}, \dots, \frac{S_{n-1}}{S_n}\right),$$

where $S_i = \sum_{j=1}^{i} E_j$ for $E_1, \dots, E_n$ a sequence of *iid* random variables with $\exp(1)$ distribution. It is easy to show that

$$n^{-1/2}T_n \xrightarrow{d} \sup_x \int_0^1 t\overline{K}(dt,F_0(x)),$$

which corresponds to the result when $\Pi(t) = t$.

REFERENCES

[1] Bai, J. (1994). Weak convergence of sequential empirical process of residuals in ARMA models. *Annals of Statistics* **22**, 2051–2061.

[2] Carlstein, E. (1988). Nonparametric change point estimation. *Annals of Statistics* **16**, 188–197.

[3] Chernoff, H. and Zacks, S. (1964). Estimating the current mean of a normal distribution which is subjected to changes in time. *Annals of Mathematical Statististics* **35,** 999–1018.

[4] Csörgő, M. and Horváth, L. (1986)

[5] Csörgő, M. and Horváth, L. (1987). Nonparametric tests for the change point problem. *Journal of Statistical Planning and Inference*, 17–19.

[6] Csörgő, M. and Horváth, L. (1997). *Limit Theorems in Change-Point Analysis* Wiley. New York.

[7] Einmahl, J. H. J. and McKeague, I. W. (2003). Empirical likelihood based hypothesis testing. *Bernoulli* **9**, No. 2. 267–290.

[8] Ferguson, T. S. and Klass, M, J. (1972). A representation of independent increment processes without Guassian components. *Annals of Mathematical Statististics* **43**. 1634–1643.

[9] Haccou, P. and Meelis, E. and van der Geer, S. (1988). Likelihood ratio tests for the change point problem for exponentially distributed random variables. *Stochastic Processes and their Applications* **27**, 121–139.

[10] Kander, Z. and Zacks, S. (1966). Test procedures for possible changes in parameters of statistical distributions occurring at unknown time points. *Annals of Mathematical Statististics* **37** 1196–1210.

[11] Knight, K. (1989). Limit theory for autoregressive parameter estimates in an infinite variance random walk. *Canadian Journal of Statistics* **17**, 261–278.

[12] Knight, K. (1998). Limiting distributions for L_1 regression estimators under general conditions. *Annals of Statistics* **26** 755–770.

[13] Ling, S. (1998). Weak convergence of the sequential empirical processes of residuals in nonstationary autoregressive models. *Annals of Statistics* **26**, 741–754.

[14] Lo, A. Y. (1993). A Bayesian method for weighted sampling. *Annals of Statistics* **21**, 2138–2148.

[15] Resnick, S. I. (1987). *Regular Variation, Extremes and Point processes.* Springer.

[16] Rubin, D. B. (1981). The Bayesian bootstrap. *Annals of Statistics* **9**, 130–134.

[17] Shao, J. (1995). *The Jackknife and Bootstrap* Springer.

[18] Zarepour, M. and Habibi, R. (2006). A quasi-Bayesian change point detection with exchangeable weights. *Submitted.*