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A NEW METHOD FOR APPROXIMATE SOLUTIONS OF FRACTIONAL ORDER BOUNDARY VALUE PROBLEMS

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ABSTRACT. In this paper we offer an approximate solution of a fractional order two-point boundary value problem (FBVP). We use the reproducing kernel method (RKM) that has been employed for the fractional order differential equations rarely. In order to illustrate the applicability and accuracy of the present method, the method is applied to some examples. The results are compared with the ones obtained by the Cubic splines (CS) and sinc-Galerkin method (SGM). Because there are only a few studies regarding the application of reproducing kernel method to fractional order differential equations, this study is going to be a new contribution and highly useful for the researchers in fractional calculus area of scientific research. Results of numerical examples show that this method is simple, effective, and easy to use.

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1. Introduction

Fractional calculus is one of the most novel types of calculus having a broad range of applications in many different scientific and engineering disciplines. Order of the derivatives in the fractional calculus might be any real number which separates the fractional calculus from the ordinary calculus where the derivatives are allowed only positive integer numbers. Therefore fractional calculus might be considered as an extension of ordinary calculus. Fractional calculus is a highly useful tool in the modelling of many sorts of scientific phenomena including image processing, earthquake engineering, biomedical engineering and physics. In the references [1] and [2] (amongst many others), fundamental concepts of fractional calculus and applications of it to different scientific and engineering areas are studied.

Even though fractional calculus is a highly useful and important topic, a general solution method which could be used at almost every sorts of problems has not yet been established. Most of the solution techniques in this area have been developed for particular sorts of problems. As a result, a single standard method for problems regarding fractional calculus has not emerged. Therefore, finding reliable and accurate solution techniques along with fast implementation methods is useful and active

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research area. Some well-known methods for the analytical and numerical solutions of fractional differential and integral equations might be listed as power series method [3], differential transform [4] and [5], homotopy analysis method [6], variational iteration method [7], homotopy perturbation method [8] and sinc–Galerkin method [9].

The aim of our work is to investigate the effectiveness of the RKM to solve FBVP. To demonstrate this, we solve several examples in the succeeding sections. The theory of reproducing kernels [10] was used for the first time at the beginning of the 20th century by Zaremba in his work on boundary value problems for harmonic and biharmonic functions. Reproducing kernel theory has important applications in numerical analysis, differential equations, probability and statistics [11, 12]. Recently, using the RKM, some authors discussed fractional differential equations, nonlinear oscillators with discontinuity, singular nonlinear two-point periodic boundary value problems, integral equations and nonlinear partial differential equations [10, 13, 14, 15, 16, 17, 18, 19, 20, 21, 11, 12, 22, 23, 24, 25, 26, 27, 28, 29].

In this paper, we consider the numerical solution of the following equation:

$$y''(x) - g(x)y'(x) + A\,{}_0^C D_x^a y(x) + By(x) = f(x), \tag{1.1}$$

subject to boundary conditions

$$y(0) = m, \quad y(1) = b, \tag{1.2}$$

where B, m and b are real constants; $f(x)$ and $g(x)$ are continuous on the interval $[0,1]$.

This paper is organized as follows. Section 2 reviews the underlying ideas and basic theorems of fractional calculus. Section 3 introduces several reproducing kernel spaces. A reproducing kernel function satisfying FBVP is constructed. The representation in $W_2^3[0,1]$ and a related linear operator are presented in Section 4. Section 5 provides the main results. The exact and approximate solutions of (1.1)–(1.2) are given in this section. An iterative method are developed for the kind of problems in the reproducing kernel space. We prove that the approximate solution converges uniformly to the exact solution. Numerical experiments are illustrated in Section 6. Some conclusions are given in the last section.

2. **Fractional Calculus**

In this section, we give the definitions of the Riemann-Liouville and the Caputo of fractional derivative.

Definition 2.1 (See [9])**.** Let $f : [a,b] \to \mathbb{R}$ be a function, α be a positive real number, n be the integer satisfying $n-1 \leq \alpha \leq n$, and Γ be the Euler gamma function. Then:

i. The left and right Riemann-Liouville fractional derivatives of order α of $f(x)$ are given as

$$ {}_aD_x^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_a^x (x-t)^{n-\alpha-1} f(t)\mathrm{d}t \tag{2.1} $$

and

$$ {}_xD_b^\alpha f(x) = \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_x^b (t-x)^{n-\alpha-1} f(t)\mathrm{d}t \tag{2.2} $$

respectively.

ii. The left and right Caputo fractional derivatives of order α of $f(x)$ are given as

$$ {}_a^CD_x^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-t)^{n-\alpha-1} f^{(n)}(t)\mathrm{d}t \tag{2.3} $$

and

$$ {}_x^CD_b^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \int_x^b (-1)^n (t-x)^{n-\alpha-1} f^{(n)}(t)\mathrm{d}t \tag{2.4} $$

respectively.

Definition 2.2 (See [9])**.** If $0 < \alpha < 1$ and f is a function such that $f(a) = f(b) = 0$, we can write

$$ \int_a^b g(x) {}_a^CD_x^\alpha f(x)\mathrm{d}x = \int_a^b f(x) {}_xD_b^\alpha g(x)\mathrm{d}x \tag{2.5} $$

and

$$ \int_a^b g(x) {}_x^CD_b^\alpha f(x)\mathrm{d}x = \int_a^b f(x) {}_aD_x^\alpha g(x)\mathrm{d}x. \tag{2.6} $$

3. **Reproducing Kernel Spaces**

In this section, we define some useful reproducing kernel spaces.

Definition 3.1 (Reproducing kernel function)**.** Let $E \neq \emptyset$. A function $K : E \times E \to \mathbb{C}$ is called a *reproducing kernel function* of the Hilbert space H if and only if

a) $K(\cdot, t) \in H$ for all $t \in E$,
b) $\langle \varphi, K(\cdot, t)\rangle = \varphi(t)$ for all $t \in E$ and all $\varphi \in H$.

The last condition is called "the reproducing property" as the value of the function φ at the point t is reproduced by the inner product of φ with $K(\cdot, t)$.

Definition 3.2 (Reproducing kernel Hilbert space)**.** A Hilbert space H which is defined on a nonempty set E is called a *reproducing kernel Hilbert space* if there exists a reproducing kernel function $K : E \times E \to \mathbb{C}$.

Definition 3.3. We define the space $W_2^1[0,1]$ by

$$W_2^1[0,1] = \{u \in AC[0,1] :\ u' \in L^2[0,1]\}.$$

The inner product and the norm in $W_2^1[0,1]$ are defined by

$$\langle u, g\rangle_{W_2^1} = \int_0^1 u(x)g(x) + u'(x)g'(x)\mathrm{d}x, \quad u, g \in W_2^1[0,1] \tag{3.1}$$

and

$$\|u\|_{W_2^1} = \sqrt{\langle u, u\rangle_{W_2^1}}, \quad u \in W_2^1[0,1]. \tag{3.2}$$

The space $W_2^1[0,1]$ is a reproducing kernel space, and its reproducing kernel function T_x is given by [13]

$$T_x(y)= \frac{1}{2\sinh(1)}\left[\cosh(x+y-1) + \cosh(|x-y|-1)\right]. \tag{3.3}$$

Definition 3.4. We define the space $W_2^3[0,1]$ by

$$W_2^3[0,1] = \{u \in AC[0,1] :\ u', u'' \in AC[0,1],\ u^{(3)} \in L^2[0,1],\ u(0) = u(1) = 0\}.$$

The inner product and the norm in $W_2^3[0,1]$ are defined by

$$\langle u, v\rangle_{W_2^3} = \sum_{i=0}^{2} u^{(i)}(0)v^{(i)}(0) + \int_0^1 u^{(3)}(x)v^{(3)}(x)\mathrm{d}x, \quad u, v \in W_2^3[0,1]$$

and

$$\|u\|_{W_2^3} = \sqrt{\langle u, u\rangle_{W_2^3}}, \quad u \in W_2^3[0,1].$$

Theorem 3.5. *The space $W_2^3[0,1]$ is a reproducing kernel space, and its reproducing kernel function R_y is given by*

$$R_y(x) = \begin{cases} \sum_{i=1}^{5} c_i(y)x^i, & 0 \le x \le y \le 1, \\ \sum_{i=0}^{5} d_i(y)x^i, & 0 \le y < x \le 1, \end{cases} \tag{3.4}$$

where

$$\begin{aligned}
c_1(y) &= -\frac{1}{156}y^5 + \frac{5}{156}y^4 - \frac{5}{78}y^3 - \frac{5}{26}y^2 + \frac{3}{13}y, \\
c_2(y) &= -\frac{1}{624}y^5 + \frac{5}{624}y^4 - \frac{5}{312}y^3 + \frac{21}{104}y^2 - \frac{5}{26}y, \\
c_3(y) &= -\frac{1}{1872}y^5 + \frac{5}{1872}y^4 - \frac{5}{936}y^3 + \frac{7}{104}y^2 - \frac{5}{78}y, \\
c_4(y) &= \frac{1}{3744}y^5 - \frac{5}{3744}y^4 + \frac{5}{1872}y^3 + \frac{5}{624}y^2 - \frac{1}{104}y, \\
c_5(y) &= -\frac{1}{18720}y^5 + \frac{1}{3744}y^4 - \frac{1}{1872}y^3 - \frac{1}{624}y^2 - \frac{1}{156}y + \frac{1}{120}, \\
d_0(y) &= \frac{1}{120}y^5, \\
d_1(y) &= -\frac{1}{156}y^5 - \frac{1}{104}y^4 - \frac{5}{78}y^3 - \frac{5}{26}y^2 + \frac{3}{13}y, \\
d_2(y) &= -\frac{1}{624}y^5 + \frac{5}{624}y^4 + \frac{7}{104}y^3 + \frac{21}{104}y^2 - \frac{5}{26}y, \\
d_3(y) &= -\frac{1}{1872}y^5 + \frac{5}{1872}y^4 - \frac{5}{936}y^3 - \frac{5}{312}y^2 - \frac{5}{78}y, \\
d_4(y) &= \frac{1}{3744}y^5 - \frac{5}{3744}y^4 + \frac{5}{1872}y^3 + \frac{5}{624}y^2 + \frac{5}{156}y, \\
d_5(y) &= -\frac{1}{18720}y^5 + \frac{1}{3744}y^4 - \frac{1}{1872}y^3 - \frac{1}{624}y^2 - \frac{1}{156}y.
\end{aligned}$$

Proof. Let $u \in W_2^3[0,1]$ and $0 \le y \le 1$. Define R_y by (3.4). Note that

$$R'_y(x) = \begin{cases} \sum_{i=0}^4 (i+1)c_{i+1}(y)x^i, & 0 \le x < y \le 1, \\ \sum_{i=0}^4 (i+1)d_{i+1}(y)x^i, & 0 \le y < x \le 1, \end{cases}$$

$$R''_y(x) = \begin{cases} \sum_{i=0}^3 (i+1)(i+2)c_{i+2}(y)x^i, & 0 \le x < y \le 1, \\ \sum_{i=0}^3 (i+1)(i+2)d_{i+2}(y)x^i, & 0 \le y < x \le 1, \end{cases}$$

$$R^{(3)}_y(x) = \begin{cases} \sum_{i=0}^2 (i+1)(i+2)(i+3)c_{i+3}(y)x^i, & 0 \le x < y \le 1, \\ \sum_{i=0}^2 (i+1)(i+2)(i+3)d_{i+3}(y)x^i, & 0 \le y < x \le 1, \end{cases}$$

$$R^{(4)}_y(x) = \begin{cases} \sum_{i=0}^1 (i+1)(i+2)(i+3)(i+4)c_{i+4}(y)x^i, & 0 \le x < y \le 1, \\ \sum_{i=0}^1 (i+1)(i+2)(i+3)(i+4)d_{i+4}(y)x^i, & 0 \le y < x \le 1, \end{cases}$$

and

$$R^{(5)}_y(x) = \begin{cases} 120c_5(y), & 0 \le x < y \le 1, \\ 120d_5(y), & 0 \le y < x \le 1. \end{cases}$$

By Definition 3.4 and integrating by parts two times, we obtain

$$\begin{aligned}
\langle u, R_y\rangle_{W_2^3} &= \sum_{i=0}^{2} u^{(i)}(0)R_y^{(i)}(0) + \int_0^1 u^{(3)}(x)R_y^{(3)}(x)\mathrm{d}x \\
&= u'(0)R_y'(0) + u''(0)R_y''(0) + u''(1)R_y^{(3)}(1) - u''(0)R_y^{(3)}(0) \\
&\quad -u'(1)R_y^{(4)}(1) + u'(0)R_y^{(4)}(0) + \int_0^1 u'(x)R_y^{(5)}(x)\mathrm{d}x \\
&= c_1(y)u'(0) + 2c_2(y)u''(0) \\
&\quad +6(d_3(y) + 4d_4(y) + 10d_5(y))u''(1) - 6c_3(y)u''(0) \\
&\quad -24(d_4(y) + 5d_5(y))u'(1) + 24c_4(y)u'(0) \\
&\quad + \int_0^y 120c_5(y)u'(x)\mathrm{d}x + \int_y^1 120d_5(y)u'(x)\mathrm{d}x \\
&= (c_1(y) + 24c_4(y))u'(0) + 2(c_2(y) - 3c_3(y))u''(0) \\
&\quad +6(d_3(y) + 4d_4(y) + 10d_5(y))u''(1) - 24(d_4(y) + 5d_5(y))u'(1) \\
&\quad +120(c_5(y) - d_5(y))u(y) \\
&= u(y).
\end{aligned}$$

This completes the proof. □

4. **Solution Representation in** $W_2^3[0,1]$

In this section, the solution of (1.1)–(1.2) is given in the reproducing kernel space $W_2^3[0,1]$. On defining the linear operator $L : W_2^3[0,1] \to W_2^1[0,1]$ by

$$Lu = u''(x) - g(x)u'(x) + A\,{}_0^C D_x^a u(x) + Bu(x), \quad u \in W_2^3[0,1], \tag{4.1}$$

after homogenizing the boundary conditions, model problem (1.1)–(1.2) changes to the problem

$$\begin{cases} Lu = z(x), \\ u(0) = 0, \quad u(1) = 0. \end{cases} \tag{4.2}$$

Theorem 4.1. *The linear operator L defined by* (4.1) *is a bounded linear operator.*

Proof. We only need to prove $\|Lu\|_{W_2^1}^2 \le M\,\|u\|_{W_2^3}^2$, where $M > 0$ is a positive constant. By (3.1) and (3.2), we get

$$\|Lu\|_{W_2^1}^2 = \langle Lu, Lu\rangle_{W_2^1} = \int_0^1 [Lu(x)]^2 + [Lu'(x)]^2\,\mathrm{d}x.$$

By reproducing property, we have

$$u(x) = \langle u(\cdot), R_x(\cdot)\rangle_{W_2^3},$$

and

$$Lu(x) = \langle u(\cdot), LR_x(\cdot)\rangle_{W_2^3},$$

so

$$|Lu(x)| \leq \|u\|_{W_2^3} \|LR_x\|_{W_2^3} = M_1 \|u\|_{W_2^3},$$

where $M_1 > 0$ is a positive constant, thus

$$\int_0^1 [(Lu)(x)]^2 \, \mathrm{d}x \leq M_1^2 \|u\|_{W_2^3}^2 .$$

Since

$$(Lu)'(x) = \langle u(\cdot), (LR_x)'(\cdot)\rangle_{W_2^3},$$

then

$$|(Lu)'(x)| \leq \|u\|_{W_2^3} \|(LR_x)'\|_{W_2^3} = M_2 \|u\|_{W_2^3},$$

where $M_2 > 0$ is a positive constant so, we have

$$[(Lu)'(t)]^2 \leq M_2^2 \|u\|_{W_2^3}^2,$$

and

$$\int_0^1 [(Lu)'(x)]^2 \, \mathrm{d}x \leq M_2^2 \|u\|_{W_2^3}^2,$$

that is

$$\|Lu\|_{W_2^1}^2 \leq \int_0^1 \left([(Lu)(x)]^2 + [(Lu)'(x)]^2 \right) \mathrm{d}x \leq \left(M_1^2 + M_2^2\right) \|u\|_{W_2^3}^2 = M \|u\|_{W_2^3}^2,$$

where $M = M_1^2 + M_2^2 > 0$ is a positive constant. □

5. The Structure of the Solution and the Main Results

Put $\varphi_i(x) = T_{x_i}(x)$ and $\psi_i(x) = L^*\varphi_i(x)$, where L^* is conjugate operator of L. The orthonormal system $\left\{\widehat{\Psi}_i(x)\right\}_{i=1}^{\infty}$ of $W_2^3[0,1]$ can be derived from Gram-Schmidt orthogonalization process of $\{\psi_i(x)\}_{i=1}^{\infty}$ and

$$\widehat{\psi}_i(x) = \sum_{k=1}^{i} \beta_{ik}\psi_k(x), \quad (\beta_{ii} > 0, \quad i = 1, 2, \ldots). \tag{5.1}$$

Theorem 5.1. *Let* $\{x_i\}_{i=1}^{\infty}$ *be dense in* $[0,1]$ *and* $\psi_i(x) = L_y R_x(y)\big|_{y=x_i}$. *Then the sequence* $\{\psi_i(x)\}_{i=1}^{\infty}$ *is a complete system in* $W_2^3[0,1]$.

Proof. We have

$$\psi_i(x) = (L^*\varphi_i)(x) = \langle (L^*\varphi_i)(y), R_x(y)\rangle = \langle (\varphi_i)(y), LyR_x(y)\rangle = L_y R_x(y)\big|_{y=x_i}.$$

The subscript y by the operator L indicates that the operator L applies to the function of y. Clearly, $\psi_i(x) \in W_2^3[0,1]$. For each fixed $u(x) \in W_2^3[0,1]$, let $\langle u(x), \psi_i(x)\rangle = 0$, $(i = 1, 2, \ldots)$, which means that,

$$\langle u(x), (L^*\varphi_i)(x)\rangle = \langle Lu(\cdot), \varphi_i(\cdot)\rangle = (Lu)(x_i) = 0.$$

Note that, $\{x_i\}_{i=1}^{\infty}$ is dense in $[0,1]$, hence, $(Lu)(x) = 0$. It follows that $u \equiv 0$ from the existence of L^{-1}. So the proof of Theorem (5.1) is complete. □

Theorem 5.2. *If $u(x)$ is the exact solution of* (4.2)*, then*

$$u(x) = L^{-1}z(x) = \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} z(x_k)\widehat{\Psi}_i(x). \tag{5.2}$$

where $\{(x_i)\}_{i=1}^{\infty}$ is dense in $[0,1]$.

Proof. From the (5.1) and uniqueness of solution of (4.2) we have

$$\begin{aligned} u(x) &= \sum_{i=1}^{\infty} \left\langle u(x), \widehat{\Psi}_i(x) \right\rangle_{W_2^3} \widehat{\Psi}_i(x) \\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} \left\langle u(x), \Psi_k(x) \right\rangle_{W_2^3} \widehat{\Psi}_i(x) \\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} \left\langle u(x), L^*\varphi_k(x) \right\rangle_{W_2^3} \widehat{\Psi}_i(x) \\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} \left\langle Lu(x), \varphi_k(x) \right\rangle_{W_2^1} \widehat{\Psi}_i(x) \\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} \left\langle z(x), T_{x_k} \right\rangle_{W_2^1} \widehat{\Psi}_i(x) \\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} z(x_k)\widehat{\Psi}_i(x). \end{aligned}$$

This completes the proof. □

Now the approximate solution $u_n(x)$ can be obtained from the n- term intercept of the exact solution u and

$$u_n(x) = \sum_{i=1}^{n}\sum_{k=1}^{i} \beta_{ik} z(x_k)\widehat{\Psi}_i(x). \tag{5.3}$$

Theorem 5.3. *If $u \in W_2^3[0,1]$ then*

$$\|u_n - u\|_{W_2^3} \to 0, \quad n \to \infty.$$

Moreover a sequence $\|u_n - u\|_{W_2^3}$ is monotonically decreasing in n.

Proof. From (5.2) and (5.3), it follows that

$$\|u_n - u\|_{W_2^3} = \left\| \sum_{i=n+1}^{\infty}\sum_{k=1}^{i} \beta_{ik} z(x_k)\widehat{\Psi}_i \right\|_{W_2^3}.$$

Thus

$$\|u_n - u\|_{W_2^3} \to 0, \quad n \to \infty.$$

In addition

$$\begin{aligned}\|u_n - u\|^2_{W_2^3} &= \left\| \sum_{i=n+1}^{\infty} \sum_{k=1}^{i} \beta_{ik} z(x_k) \widehat{\Psi}_i \right\|^2_{W_2^3} \\ &= \sum_{i=n+1}^{\infty} \left(\sum_{k=1}^{i} \beta_{ik} z(x_k) \widehat{\Psi}_i \right)^2 .\end{aligned}$$

Clearly, $\|u_n - u\|_{W_2^3}$ is monotonically decreasing in n. □

6. Numerical Results

In this section, three problems that have homogeneous and nonhomogeneous boundary conditions will be tested by using RKM. The numerical solutions which are obtained by using the RKM for these problems are presented in Tables 1–6.

Example 6.1. We start with the linear fractional boundary value problem

$$y''(x) + {}_0^C D_x^{0.5} y(x) = f(x), \tag{6.1}$$

with the boundary conditions

$$y(0) = 0 = y(1), \tag{6.2}$$

where $f(x) = 2 + \frac{1}{\Gamma(0.5)}(2.65x^{1.5} - 2x^{0.5})$. The exact solution of (6.1)–(6.2) is given as [9]

$$y(x) = x(x-1).$$

Using the above method, we obtain Table 1 and Table 2.

Example 6.2. We consider the linear fractional boundary value problem

$$y''(x) + 0.5 {}_0^C D_x^{0.3} y(x) + y(x) = f(x), \tag{6.3}$$

with the boundary conditions

$$y(0) = 0 = y(1), \tag{6.4}$$

where $f(x) = 4x^2(5x-3) + 0.5x^{3.7}\left(\frac{120}{\Gamma(5.7)}x - \frac{24}{\Gamma(4.7)}\right) + x^4(x-1)$. The exact solution of (6.3)–(6.4) is given as [9]

$$y(x) = x^4(x-1).$$

Using the above method, we obtain Table 3 and Table 4.

Example 6.3. We take notice of the linear fractional boundary value problem

$$y''(x) - xy'(x) + {}_0^C D_x^{0.5} y(x) = f(x), \tag{6.5}$$

with the nonhomogeneous boundary conditions

$$y(0) = 0, \quad y(1) = 2, \tag{6.6}$$

where $f(x) = -3x^3 - 2x^2 + 6x + 2 + \frac{1}{\Gamma(0.5)}(2.67x^{1.5} + 3.2x^{2.5})$. The exact solution of (6.5)–(6.6) is given as [9]

$$y(x) = x^2(x+1).$$

We use the transformation $u(x) = y(x) - 2x$ to homogenize the nonhomogeneous boundary conditions. Then we obtain

$$u''(x) - xu'(x) + {}_0^C D_x^{0.5} u(x) = -3x^3 - 2x^2 + 8x + 2 + \frac{1}{\Gamma(0.5)}(2.67x^{1.5} + 3.2x^{2.5} - 4x^{0.5}), \tag{6.7}$$

with the homogeneous boundary conditions

$$u(0) = 0 = u(1). \tag{6.8}$$

After homogenizing the boundary conditions and using the above method, we obtain Table 5 and Table 6.

Remark 6.4. In Tables 1–6, we abbreviate the exact solution and the approximate solution by AS and ES, respectively. AE stands for the absolute error, i.e., the absolute value of the difference of the exact solution and the approximate solution, while RE indicates the relative error, i.e., the absolute error divided by the absolute value of the exact solution.

x	ES	AS	AE	RE
0.1	−0.09	−0.0900000037926874219158	3.792×10^{-8}	4.214×10^{-7}
0.2	−0.16	−0.1599999349864336603	6.501×10^{-8}	4.063×10^{-7}
0.3	−0.21	−0.21000008124579205921	8.124×10^{-8}	3.868×10^{-7}
0.4	−0.24	−0.23999999946480029926	5.351×10^{-10}	2.229×10^{-9}
0.5	−0.25	−0.25000045192082762963	4.519×10^{-7}	1.807×10^{-6}
0.6	−0.24	−0.23999999946480029926	5.351×10^{-10}	2.229×10^{-9}
0.7	−0.21	−0.21000008124579205921	8.124×10^{-8}	3.868×10^{-7}
0.8	−0.16	−0.1599999349864336603	6.501×10^{-8}	4.063×10^{-7}
0.9	−0.09	−0.0900000037926874219158	3.792×10^{-8}	4.214×10^{-7}

TABLE 1. Numerical results for Example 6.1.

x	[9] $L = 5,\ M = 5$	[9] $L = 40,\ M = 100$	RKM
0.1	7.79×10^{-4}	1.15×10^{-6}	3.79×10^{-8}
0.2	2.34×10^{-3}	1.50×10^{-6}	6.50×10^{-8}
0.3	1.74×10^{-3}	1.85×10^{-6}	8.12×10^{-8}
0.4	4.25×10^{-4}	1.43×10^{-6}	5.35×10^{-10}
0.5	1.72×10^{-3}	1.04×10^{-6}	4.51×10^{-7}
0.6	1.19×10^{-3}	1.27×10^{-6}	5.35×10^{-10}
0.7	5.87×10^{-4}	5.20×10^{-7}	8.12×10^{-8}
0.8	1.55×10^{-3}	1.59×10^{-7}	6.50×10^{-8}
0.9	4.46×10^{-4}	3.50×10^{-7}	3.79×10^{-8}

TABLE 2. Comparision of absolute error for Example 6.1.

x	ES	AS	AE	RE
0.125	−0.000213623046875	−0.000213623238185920018 1	1.913×10^{-10}	8.955×10^{-7}
0.250	−0.0029296875	−0.00292968664148474342	8.585×10^{-10}	2.930×10^{-7}
0.375	−0.012359619140625	−0.01235961965183831573	5.112×10^{-10}	4.136×10^{-8}
0.500	−0.03125	−0.03125000139491155761 4	1.394×10^{-9}	4.463×10^{-8}
0.625	−0.057220458984375	−0.05722045968551878641 8	7.011×10^{-10}	1.225×10^{-8}
0.750	−0.0791015625	−0.07910156285552529958 1	3.555×10^{-10}	4.494×10^{-9}
0.875	−0.073272705078125	−0.07327270508626339431 9	8.138×10^{-12}	1.110×10^{-10}

TABLE 3. Numerical results for Example 6.2.

x	[9] $L=5,\ M=5$ SGM	[9] $L=5,\ M=5$ CS	[9] $L=40,\ M=100$ SGM	RKM
0.125	2.80×10^{-4}	1.99×10^{-3}	2.06×10^{-9}	1.91×10^{-10}
0.250	4.72×10^{-3}	4.08×10^{-3}	2.19×10^{-9}	8.58×10^{-10}
0.375	4.28×10^{-3}	5.83×10^{-3}	3.61×10^{-9}	5.11×10^{-10}
0.500	3.00×10^{-3}	6.85×10^{-3}	5.73×10^{-9}	1.39×10^{-9}
0.625	3.17×10^{-3}	5.83×10^{-1}	5.40×10^{-9}	7.01×10^{-10}
0.750	1.85×10^{-3}	5.56×10^{-3}	2.47×10^{-9}	3.55×10^{-10}
0.875	3.09×10^{-3}	3.26×10^{-3}	2.61×10^{-11}	8.13×10^{-12}

TABLE 4. Comparision of absolute error for Example 6.2.

x	ES	AS	AE	RE
0.1	0.011	0.010999354265061277546	6.4573×10^{-7}	5.8703×10^{-5}
0.2	0.048	0.04799979716411731643	2.0283×10^{-7}	4.2257×10^{-6}
0.3	0.117	0.11699764761663712586	2.3523×10^{-6}	2.0105×10^{-5}
0.4	0.224	0.22400650686738379444	6.5068×10^{-6}	2.9048×10^{-5}
0.5	0.375	0.37499233675382009937	7.6632×10^{-7}	2.0435×10^{-5}
0.6	0.576	0.57600049159156832193	4.9159×10^{-6}	8.5345×10^{-7}
0.7	0.833	0.83299966665968847031	3.3334×10^{-7}	4.0016×10^{-7}
0.8	1.152	1.1520031386360497278	3.1386×10^{-6}	2.7245×10^{-6}
0.9	1.539	1.5389933357337636439	6.6642×10^{-6}	4.3302×10^{-6}

TABLE 5. Numerical results for Example 6.3.

x	[9] $L = 5,\ M = 5$	[9] $L = 50,\ M = 100$	RKM
0.1	5.14×10^{-3}	4.32×10^{-5}	6.45×10^{-7}
0.2	1.97×10^{-3}	8.58×10^{-5}	2.02×10^{-7}
0.3	5.34×10^{-3}	1.30×10^{-4}	2.35×10^{-6}
0.4	3.13×10^{-3}	1.69×10^{-4}	6.50×10^{-6}
0.5	1.40×10^{-4}	2.01×10^{-4}	7.66×10^{-7}
0.6	8.77×10^{-6}	2.24×10^{-4}	4.91×10^{-6}
0.7	2.94×10^{-3}	2.48×10^{-4}	3.33×10^{-7}
0.8	4.64×10^{-3}	2.40×10^{-4}	3.13×10^{-6}
0.9	1.03×10^{-3}	1.61×10^{-4}	6.66×10^{-6}

TABLE 6. Comparision of absolute error for Example 6.3.

7. Conclusion

In this study, fractional order boundary value problems were solved by the reproducing kernel method. We described the method and used it in some test examples in order to show its applicability and validity in comparison with exact and other numerical solutions. The obtained results show that this approach can solve the problem effectively with few computations. The results are satisfactory. The results that we obtained were compared with the results that were obtained in [9]. Numerical experiments on test examples show that our proposed schemes are of high accuracy, supporting the theoretical results. It has been shown that the obtained results are uniformly convergent and the operator that was used is a bounded linear operator.

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