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EXISTENCE RESULTS FOR NONLINEAR PERTURBATIONS OF LINEAR MAXIMAL MONOTONE OPERATORS

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ABSTRACT. Let X be a real reflexive Banach space with its dual X^*. Let $L: X \supset D(L) \to X^*$ be densely defined linear maximal monotone, $T: X \to 2^{X^*}$ bounded maximal monotone, and $C: X \supset D(C) \to X^*$ bounded demicontinuous of type (S_+) w.r.t. $D(L)$. An eigenvalue problem of the type $Lx+Tx+C(\lambda,x) \ni 0$ is solved. Here, T satisfies $0 \in T(0)$ and $C(\lambda,\cdot)$, $\lambda > 0$, is like C above with $C(0,x)=0$. In addition, an open mapping theorem is established for $L+T+C$. The topological degree theory developed by Addou and Mermri is used along with the methodology of Berkovits and Mustonen and recent invariance of domain and eigenvalue results by Kartsatos and Skrypnik.

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1. Introduction and Preliminaries

Let X be a real reflexive Banach space with norm $\|\cdot\|$ with the dual space X^*. In order to emphasize that the norm is of a Banach space Y, we write $\|\cdot\|_Y$. The symbol $B_r(x)$ denotes the open ball of radius $r>0$ with center at x. We denote by $\langle x^*, x\rangle$ the value of the functional $x^* \in X^*$ at $x \in X$. If $\{x_n\}$ is a sequence in X, we denote its strong convergence to x_0 in X by $x_n \to x_0$ and its weak convergence in X by $x_n \rightharpoonup x_0$. The term "continuous" means "strongly continuous". An operator $T: X \supset D(T) \to Y$, with Y another Banach space, is said to be "bounded" if it maps bounded subsets of the domain $D(T)$ onto bounded subsets of Y. The symbols $\mathbf{R}$ and $\mathbf{R}_+$ denote $(-\infty,\infty)$ and $[0,\infty)$, respectively. Also, the symbols ∂D and $\overline{D}$ denote the strong boundary and closure of the set D, respectively. The normalized duality mapping $J: X \supset D(J) \to 2^{X^*}$ is defined by

$$Jx = \{x^* \in X^* : \langle x^*, x\rangle = \|x\|^2,\ \|x^*\| = \|x\|\},\ x \in X.$$

The Hahn-Banach theorem ensures that $D(J) = X$ and therefore $J: X \to 2^{X^*}$.

By a well-known renorming theorem due to Trojanski [16], one can always renorm a reflexive Banach space X with an equivalent norm so that both X and X^* are locally uniformly convex (therefore strictly convex). Henceforth we assume that X and X^* are locally uniformly convex reflexive. In this setting, the normalized duality mapping J is single-valued homeomorphism from X onto X^* and satisfies $J(\alpha x) = \alpha J(x), \quad (\alpha, x) \in \mathbf{R}_+ \times X$.

For a multi-valued operator T from X to X^* we write $T: X \supset D(T) \to 2^{X^*}$, where $D(T) = \{x \in X : Tx \neq \emptyset\}$ is the effective domain of T. We denote by $G(T)$ the graph of T, i.e., $G(T) =$

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$\{(x,y)\ :\ x \in D(T),\ y \in Tx\}$. An operator $T : X \supset D(T) \to 2^{X^*}$ is called "monotone" if for every $(x,u),(y,v) \in G(T)$ we have

$$\langle u - v, x - y\rangle \geq 0.$$

A monotone operator T is "maximal monotone" if $G(T)$ is maximal in $X \times X^*$, in view of the Zorn's lemma, when $X \times X^*$ is partially ordered by inclusion.

In a reflexive Banach space X, it is known that a monotone operator T is maximal monotone if and only if $R(T + \lambda J) = X^*$ for all $\lambda \in (0,\infty)$ (equivalently for some $\lambda > 0$). If T is maximal monotone, then it is is a well-known fact that the Yosida approximant $T_s := (T^{-1} + sJ^{-1})^{-1} : X \to X^*$ is bounded, demicontinuous, maximal monotone and satisfies $T_s x \rightharpoonup T^{\{0\}}x$ as $s \to 0^+$ for every $x \in D(T)$, where $T^{\{0\}}x$ denotes the element $y^* \in Tx$ of minimum norm, i.e. $\|T^{\{0\}}x\| = \inf\{\|y^*\|\ :\ y^* \in Tx\}$. In our setting this infimum is always attained, $D(T^{\{0\}}) = D(T)$. Also, $T_s x = \frac{1}{s}J(x - J_s x)$ and $T_s x \in TJ_s x$, where $J_s := I - sJ^{-1}T_s : X \to X$ and satisfies $\lim_{s\to 0} J_s x = x$ for all $x \in \overline{\mathrm{co}D(T)}$, where $\mathrm{co}A$ denotes the convex hull of the set A. In addition, $x \in D(T)$ and $s_0 > 0$ imply $\lim\limits_{s\to s_0} T_s x = T_{s_0}x$, and for every $s > 0$, we have $\|T_s x\| \leq \|w\|$ for all $w \in Tx$. The operators T_s and J_s were introduced by Brézis, Crandall and Pazy in [8]. Since X and X^* are locally uniformly reflexive spaces one has $T_s x \to T^{\{0\}}x$ as $s \to 0^+$ for every $x \in D(T)$. For some basic properties, we refer the reader to [8, 4] as well as Pascali and Sburlan [14, pp. 128-130].

If T satisfies either (i) or (ii) of the following lemma, then we say that T is demiclosed. The lemma can be found in [17, p. 915].

Lemma 1.1. *Let $T : X \supset D(T) \to 2^{X^*}$ be maximal monotone. Then the following are true:*

(i) $\{x_n\} \subset D(T)$, $x_n \to x_0$ *and* $Tx_n \ni y_n \rightharpoonup y_0$ *imply* $x_0 \in D(T)$ *and* $y_0 \in Tx_0$.
(ii) $\{x_n\} \subset D(T)$, $x_n \rightharpoonup x_0$ *and* $Tx_n \ni y_n \to y_0$ *imply* $x_0 \in D(T)$ *and* $y_0 \in Tx_0$.

Definition 1.2. An operator $f : \overline{G} \to X^*$, $G \subset X$, is said to be demicontinuous on $\overline{G}$ if for every sequence $\{x_n\} \in \overline{G}$ with $x_n \to x_0$ in $\overline{G}$, we have $Cx_n \rightharpoonup Cx_0$ in X^*.

Definition 1.3. Let $L : X \supset D(L) \to X^*$ be a densely defined linear maximal monotone operator and G a bounded open subset of X. A bounded demicontinuous operator $C : \overline{G} \to X^*$ is said to be of type (S_+) w.r.t. $D(L)$ if for every sequence $\{x_n\} \subset D(L) \cap \overline{G}$ with $x_n \rightharpoonup x_0$ in X, $Lx_n \rightharpoonup Lx_0$ in X^* and

$$\limsup_{n\to\infty} \langle Cx_n, x_n - x_0\rangle \leq 0,$$

we have $x_n \to x_0 \in\ \overline{D(L)} = X$.

Since the graph $G(L)$ of L is closed in $X \times X^*$, the space $Y = D(L)$ associated with the graph norm $\|\cdot\|_Y$ defined by

$$\|x\|_Y = \|x\|_X + \|Lx\|_{X^*}, \quad x \in Y,$$

becomes a real reflexive Banach space. We may assume that Y and its dual Y^* are locally uniformly convex. Let $j : Y \to X$ be the natural embedding and $j^* : X^* \to Y^*$ its adjoint. Since $j : Y \to X$ is continuous, $j^{-1}(\overline{G}) = \overline{G} \cap D(L)$ is closed in $D(L)$ and $j^{-1}(G) = G \cap D(L)$ is open in $D(L)$ for any open set $G \subset X$. Moreover,

$$\overline{j^{-1}(G)} \subset j^{-1}(\overline{G}), \quad \text{and} \quad \partial(j^{-1}(G)) \subset j^{-1}(\partial G).$$

Definition 1.4. Let $L : X \supset D(L) \to X^*$ be a densely defined linear maximal monotone operator, G a bounded open subset of X and $C(t) : \overline{G} \to X^*,\ t \in [0,1]$, a one-parameter family of operators.

The family $\{C(t)\}_{t\in[0,1]}$ is said to be a "homotopy of type (S_+) w.r.t. $D(L)$" if for every pair of sequences $\{x_n\}\subset D(L)\cap\overline{G}$ and $\{t_n\}\subset[0,1]$ with $x_n\rightharpoonup x_0$ in X, $Lx_n\rightharpoonup Lx_0$ in X^*, $t_n\to t_0$, and

$$\limsup_{n\to\infty}\langle C(t_n)x_n, x_n-x_0\rangle\le 0,$$

we have $x_n\to x_0$ in X and $C(t_n)x_n\rightharpoonup C(t_0)x_0$ in X^*.

Definition 1.5. A family $T(t): X\to 2^{X^*}$, $t\in[0,1]$, of maximal monotone operators is said to be bounded if the set $\bigcup_{t\in[0,1]}T(t)(G)$ is bounded in X^* for every bounded subset G of X.

Definition 1.6. Let $T(t): X\to 2^{X^*}, t\in[0,1]$, be a family of maximal monotone operators with $0\in T_t(0)$ for all t. Then $\{T_t\}$ is called a "pseudomonotone homotopy" if for every $\{t_n\}\subset[0,1]$ with $t_n\to t_0$ and $\{(x_n,y_n)\}\subset G(T_{t_n})$ with $x_n\rightharpoonup x_0$ in X and $y_n\rightharpoonup y_0\in X^*$ such that

$$\limsup_{n\to\infty}\langle y_n,x_n\rangle\le\langle y_0,x_0\rangle,$$

we have $(x_0,y_0)\in G(T_{t_0})$ and $\lim\limits_{n\to\infty}\langle y_n,x_n\rangle=\langle y_0,x_0\rangle$.

Browder [6] gave the concept of a pseudomonotone homotopy of maximal monotone operators with 4 equivalent conditions one of which is in Definition 1.6. For the remaining three conditions, the reader is referred to [6].

Let $\mathcal{H}_G$ denote the set of all operators of the form $L+T(t)+C(t)$, where L and $C(t)$ are as above and $T(t)$ is a bounded pseudomonotone homotopy of maximal monotone operators from X into 2^{X^*}. For $F(t):=L+T(t)+C(t)\in\mathcal{H}_G$, we define

$$\hat{L}:=j^*\circ L\circ j: Y\to Y^*,\qquad \hat{C}(t):=j^*\circ C(t)\circ j: j^{-1}(\overline{G})\to Y^*,$$

and for every $s>0$

$$\hat{T}_s(t):=j^*\circ T_s(t)\circ j: Y\to Y^*,$$

where $T_s(t)$ is the Yosida approximant of $T(t)$. We also define $M: Y\to Y^*$ by

$$(Mx,y)=\langle Ly, J^{-1}(Lx)\rangle,\quad x,y\in Y.$$

Here, the duality pair $(\cdot,\cdot)$ is in $Y^*\times Y$ and J^{-1} is the inverse of the duality map $J: X\to X^*$. In particular, for every $x\in Y$ such that $Mx\in j^*(X^*)$, we have $J^{-1}(Lx)\in D(L^*)$ and

$$Mx=j^*\circ L^*\circ J^{-1}(Lx).$$

The operator M is monotone because

$$(Mx-My,x-y)=\langle Lx-Ly, J^{-1}(Lx)-J^{-1}(Ly)\rangle\ge 0$$

for all $x,y\in Y$. One can see that J^{-1} coincides with the normalized duality map from X^* to X^{**} when we identify X and X^{**}.

The following lemma whose proof can be found in [1] will be needed later on.

Lemma 1.7. *If $F(t)\in\mathcal{H}_G$ and $s>0$, then $\hat{F}_s(t):=\hat{L}+\hat{T}_s(t)+\hat{C}(t)+sM$ is a bounded homotopy of type (S_+) from $j^{-1}(\overline{G})\subset Y$ to Y^*. Moreover, for every continuous $h:[0,1]\to X^*$, the set $\{u\in j^{-1}(\overline{G}):\hat{F}_s(t)(u)=j^*h(t), t\in[0,1]\}$ is bounded in Y.*

The full proof of Lemma 1.8 below can be found in [3]. The reader is also referred to Brézis, Crandall and Pazy [8].

Lemma 1.8. *Assume that the operators $T : X \supset D(T) \to 2^{X^*}$ and $S : X \supset D(S) \to 2^{X^*}$ are maximal monotone with $0 \in D(T) \cap D(S)$ and $0 \in S(0) \cap T(0)$. Assume, further, that $T + S$ is maximal monotone and that there is a sequence $\{s_n\} \subset (0, \infty)$ such that $s_n \downarrow 0$, and a sequence $\{x_n\} \subset D(S)$ such that $x_n \rightharpoonup x_0 \in X$ and $T_{s_n} x_n + w_n^* \rightharpoonup y_0^* \in X^*$, where $w_n^* \in S x_n$. Then the following are true:*

(i) *the inequality*

$$\lim_{n\to\infty} \langle T_{s_n} x_n + w_n^*, x_n - x_0 \rangle < 0 \tag{1.1}$$

is impossible;

(ii) *if*

$$\lim_{n\to\infty} \langle T_{s_n} x_n + w_n^*, x_n - x_0 \rangle = 0, \tag{1.2}$$

then $x_0 \in D(T + S)$ and $y_0^* \in (T + S)x_0$.

Remark 1.9. Lemma 1.8 is also valid if we replace T_{s_n} with T (in the absence of the sequence $\{s_n\}$) and make necessary notational changes.

Lemma 1.10 below from Kartsatos and Skrypnik [11] is needed in the proof of Theorem 2.3 in Section 2.

Lemma 1.10. *Let $T : X \supset D(T) \to 2^{X^*}$ be maximal monotone and such that $0 \in D(T)$ and $0 \in T(0)$. Then the mapping $(t, x) \to T_t x$ is continuous on the set $(0, \infty) \times X$.*

Addou and Mermri [1] extended the Berkovits-Mustonen [5] degree for $L + C$ to the triplet $L + T + C$, with a bounded maximal monotone operator T. In this paper we consider an eigenvalue problem and an open mapping theorem for nonlinear perturbations of linear densely defined maximal monotone operators using the methodology of the construction of the degree theories by Berkovits and Mustonen [5] and by Addou and Mermri [1]. These results generalize similar ones of Kartsatos and Skrypnik in [12, 13].

Kartsatos [9] established invariance of domain theorems for maximal monotone operators whose domain do not necessarily contain any open sets. Kartsatos and Skrypnik [13] have extended the well-known invariance of domain theorem of Schauder about injective operators of the type $I + C$ with C compact to the operators of the form $T + C$ with T maximal monotone and C bounded demicontinuous of type (S_+) using the topological degrees of Browder and Skrypnik. In addition, Kartsatos and Skrypnik [13] gave invariance of domain theorems for the operators of the form $T + C$ with both T, C densely defined and T single-valued. These results make use of the topological degree theory developed by the authors for the sum $T + C$, where T is single-valued maximal monotone T and C satisfies conditions like quasiboundedness and (S_+) w.r.t T. Recent results on open balls in the ranges of nonlinear operators were obtained by Kartsatos and Quarcoo in [10].

The author and Kartsatos created a new degree theory in [2] for $L + T + C$, where L is densely defined linear maximal monotone, T (possibly nonlinear) maximal monotone and strongly quasibounded (a notion more general than boundedness, cf. [7]), and C bounded, demicontinuous and of type (S_+) w.r.t. the set $D(L)$. This degree theory for the case $T = 0$ reduces to the degree theory of Berkovits and Mustonen [5]. For a bounded T, it reduced to the degree theory of Addou and Mermri [1]. As in [5], the construction of the degree mappings in [1, 2] uses the graph norm topology of the space $Y = D(L)$. It should be noted that there is a large class of strongly quasibounded maximal monotone operators. For example, Browder and Hess [7] have shown that a maximum monotone operator with zero in the interior of its domain is strongly quasibounded.

2. **Existence of Eigenvalues**

In this section we utilize various topological degree theories to solve the implicit eigenvalue problem $Lx+Tx+C(\lambda,x)\ni 0$, which generalizes the eigenvalue problem $Tx+C(\lambda,x)=0$ considered in Kartsatos and Skrypnik [12]. Here, $T:X\supset D(T)\to X^*$ is maximal monotone with $0\in D(T)$ and $0\in T(0)$, and $C:[0,\Lambda]\times\overline{G}\to X^*$, $G\subset X$ bounded open and $0\in G$, is bounded demicontinuous of type (S_+).

The following definition is a variant of one in [12, p. 3854] and will be needed for an eigenvalue problem which generalizes a similar result of Kartsatos and Skrypnik in [12].

Definition 2.1. Let $G\subset X$ be open and bounded, $\Lambda>0$. An operator $C:[0,\Lambda]\times\overline{G}\to X^*$ is said to be demicontinuous if $\{(t_n,x_n)\}\subset[0,\Lambda]\times\overline{G}$ such that $(t_n,x_n)\to(t_0,x_0)\in[0,\lambda]\times\overline{G}$ implies $C(t_n,x_n)\rightharpoonup C(t_0,x_0)$ in X^*. A demicontinuous operator $C(t,x)$ is said to be continuous in t uniformly w.r.t. $x\in\overline{G}$ if $\{t_n\}\subset[0,\Lambda]$ with $t_n\to t_0\in[0,\lambda]$ implies $C(t_n,x)\to C(t_0,x)$ for all $x\in\overline{G}$. A demicontinuous operator $C:[0,\Lambda]\times\overline{G}\to X^*$ is said to be of type (S_+) w.r.t $D(L)$ if for every sequence $\{x_n\}\in D(L)$ and every $\lambda\in(0,\Lambda]$ with $x_n\rightharpoonup x_0$ in X, $Lx_n\rightharpoonup Lx_0$ in X^* and

$$\limsup_{n\to\infty}\langle C(\lambda,x_n),x_n-x_0\rangle\le 0,$$

we have $x_n\to x_0$ in X.

Definition 2.2. An operator $T:X\supset D(T)\to 2^{X^*}$ is said to satisfy condition "(S_q)" on a set $A\subset D(T)$ if for every sequence $\{x_n\}\subset A$ with $x_n\rightharpoonup x_0$ in X and any $x_n^*\in Tx_n$ with $x_n^*\to x^*$ for some $x^*\in X^*$, we have $x_n\to x_0$ in X.

We next have an eigenvalue result for our setting of L, T and C by employing the Browder and Skrypnik degree theory as well as the methodology established by Kartsatos and Skrypnik in [12].

Theorem 2.3. *Let $G\subset X$ be open, bounded and $0\in G$. Let $L:X\supset D(L)\to X^*$ be a densely defined linear maximal monotone operator and $T:X\to 2^{X^*}$ a bounded maximal monotone operator with $0\in D(T)$ and $0\in T(0)$. Let $C:[0,\Lambda]\times\overline{G}\to X^*$ be a bounded demicontinuous operator of type (S_+) w.r.t. to $D(L)$. Assume that $C(0,x)=0, x\in\overline{G}$, and $C(t,x)$ is continuous in t uniformly w.r.t. $x\in\overline{G}$. Let ϵ,ϵ_0 be positive numbers. Assume further that*
(**P**) *there exists $\lambda\in(0,\Lambda]$ such that the inclusion*

$$Lx+Tx+C(\lambda,x)+\epsilon Jx\ni 0$$

has no solution in $x\in D(L)\cap G$. Then
(i) *there exists $(\lambda_0,x_0)\in(0,\Lambda]\times(D(L)\cap\partial G)$ such that*

$$Lx_0+Tx_0+C(\lambda_0,x_0)+\epsilon Jx_0\ni 0; \tag{2.1}$$

(ii) *if $0\notin(L+T)(D(L)\cap\partial G)$, $L+T$ satisfies condition (S_q) on $D(L)$, and property (**P**) is satisfied for every $\epsilon'\in(0,\epsilon_0]$, then there exists $(\lambda_0,x_0)\in(0,\Lambda]\times(D(L)\cap\partial G)$ such that $Lx_0+Tx_0+C(\lambda_0,x_0)\ni 0$.*

Proof: Assume that (2.1) is not true. Then the inclusion

$$H(\lambda,x):=Lx+Tx+C(\lambda,x)+\epsilon Jx\ni 0$$

has no solution on $D(L)\cap\partial G$ for every $\lambda\in[0,\Lambda]$. Here, $L+T+\epsilon J$ is strictly monotone and so the assumption is obviously true for $\lambda=0$. Thus,

$$H(\lambda,D(L)\cap\partial G)\not\ni 0,\quad \lambda\in[0,\Lambda]. \tag{2.2}$$

Let $Y = D(L)$ be equipped with the graph norm. We are now going to show that there exist $s_0 > 0$, $\lambda_0 \in (0, \Lambda]$ such that for every $s \in (0, s_0]$ and $\lambda \in (0, \lambda_0]$, the equation

$$H_1(s, \lambda, x) := \hat{L}x + \hat{T}_s x + \hat{C}(\lambda, x) + \epsilon \hat{J}x + sMx = 0 \tag{2.3}$$

has no solution $x \in \partial G_R(Y)$, where $G_R(Y) = j^{-1}(G) \cap B_Y(0, R)$. Here, $B_Y(0, R) := \{y \in Y : \|y\|_Y < R\}$. By Lemma 1.7, the set of solutions of (2.3) in $j^{-1}(\overline{G})$ is bounded in Y and $0 \in j^{-1}(\overline{G})$, and therefore such an $R > 0$ exists. Moreover, $G_R(Y)$ is bounded and open in Y. We also note that $\partial(j^{-1}(G)) \subset j^{-1}(\partial G)$.

Assume that the assertion about (2.3) is not true. Then there exist $s_n \downarrow 0$, $\lambda_n \downarrow 0$, $x_n \in \partial(j^{-1}(G))$, $x_0 \in Y$, such that $x_n \rightharpoonup x_0$ in Y (Y is reflective in the graph norm) and

$$\hat{L}x_n + \hat{T}_{s_n} x_n + \hat{C}(\lambda_n, x_n) + \epsilon \hat{J}x_n + s_n M x_n = 0, \tag{2.4}$$

where $\hat{J} = j^* \circ J \circ j$. The definitions of $\hat{L}$, $\hat{T}_s$, $\hat{C}$, $\hat{J}$, M, monotonicity of J, and (2.4) imply

$$\begin{aligned} (\hat{L}x_n, x_n - x_0) &= -(\hat{C}(\lambda_n, x_n), x_n - x_0) - (\hat{T}_{s_n} x_n, x_n - x_0) \\ &\quad -\epsilon(\hat{J}x_n, x_n - x_0) - s_n(Mx_n, x_n - x_0) \\ &= -\langle C(\lambda_n, x_n), x_n - x_0\rangle - \langle T_{s_n} x_n, x_n - x_0\rangle \\ &\quad -\epsilon\langle Jx_n, x_n - x_0\rangle - s_n\langle Lx_n - Lx_0, J^{-1}(Lx_n)\rangle \\ &\le \|C(\lambda_n, x_n)\|\|x_n - x_0\| - \langle T_{s_n} x_0, x_n - x_0\rangle - \epsilon\langle Jx_0, x_n - x_0\rangle \\ &\quad -s_n\langle Lx_n - Lx_0, J^{-1}(Lx_n)\rangle. \end{aligned} \tag{2.5}$$

We note that $x_n \rightharpoonup x_0$ in Y implies $x_n \rightharpoonup x_0$ in X and $Lx_n \rightharpoonup Lx_0$ in X^*. Since $T_{s_n} x_0 \to y_0 := T^{\{0\}} x_0$, where $T^{\{0\}} x_0$ is the element of minimum norm in the closed convex set Tx_0, we obtain

$$\begin{aligned} \limsup_{n\to\infty} (\hat{L}x_n, x_n - x_0) &\le \lim_{n\to\infty} [\|C(\lambda_n, x_n)\|\|x_n - x_0\|] - \epsilon \lim_{n\to\infty} \langle Jx_0, x_n - x_0\rangle \\ &\quad - \lim_{n\to\infty} \langle T_{s_n} x_0, x_n - x_0\rangle - \lim_{n\to\infty} s_n\langle Lx_n - Lx_0, J^{-1}(Lx_n)\rangle \\ &= 0. \end{aligned}$$

Here, we have also used the continuity of $C(\lambda, x)$ in λ uniformly w.r.t. $x \in \overline{G}$, the fact that $\partial(j^{-1}(G)) \subset j^{-1}(\partial G) = \partial G \cap Y \subset \overline{G}$, and the boundedness of J^{-1}. Also, the monotonicity of $\hat{L}$ implies

$$\liminf_{n\to\infty} (\hat{L}x_n, x_n - x_0) \ge \lim_{n\to\infty} (\hat{L}x_0, x_n - x_0) \ge 0.$$

Therefore

$$\lim_{n\to\infty} (\hat{L}x_n, x_n - x_0) = 0.$$

This along with (2.5) gives

$$\limsup_{n\to\infty} \langle T_{s_n} x_n + \epsilon J x_n, x_n - x_0\rangle = 0.$$

If

$$\limsup_{n\to\infty} \langle Jx_n, x_n - x_0\rangle > 0,$$

then it follows, for a subsequence of $\{n\}$ again denoted by $\{n\}$, that

$$\liminf_{n\to\infty} \langle T_{s_n} x_n, x_n - x_0\rangle < 0.$$

This is impossible by Lemma 1.8(i) by using an appropriate further subsequence of $\{n\}$. This shows that we must have

$$\limsup_{n\to\infty} \langle Jx_n, x_n - x_0\rangle \le 0.$$

Since it is a well-known fact that J is of type (S_+) (cf., [6]), we have $x_n \to x_0$ in X and $x_0 \in \partial G \cap Y = j^{-1}(\partial G)$. The continuity of $\hat{J}$ implies $\hat{J}x_n \to \hat{J}x_0$. Since, for each $s > 0$ and $x \in D(T)$, $\|T_s x\| \le \|w\|$ for all $w \in Tx$ and T is bounded, $\{T_{s_n} x_n\}$ is bounded, and therefore we may assume that $T_{s_n} x_n \rightharpoonup w_0 \in Tx_0$ for a subsequence of $\{n\}$ again denoted by $\{n\}$. Then (2.4) implies $\hat{L}x_n + \hat{T}_{s_n} x_n + \epsilon \hat{J}x_n \rightharpoonup \hat{L}x_0 + j^* w + \epsilon \hat{J}x_0 = 0$.

Let $y \in Y$. Then

$$(\hat{L}x_0 + j^* w_0 + \epsilon \hat{J}x_0, y) = 0,$$

which implies

$$\begin{aligned}\langle Lx_0 + w_0 + \epsilon Jx_0, y\rangle &= ((j^* \circ L \circ j)x_0 + j^* w_0 + \epsilon (j^* \circ J \circ j)x_0, y) \\ &= (\hat{L}x_0 + j^* w_0 + \epsilon \hat{J}x_0, y) \\ &= 0.\end{aligned}$$

Since Y is dense in X, we have that $Lx_0 + w_0 + \epsilon Jx_0 = 0$ which is a contradiction because $L + T + \epsilon J$ is strictly monotone and $0 \in G \cap Y$. Therefore the assertion about (2.3) is true.

Now, we fix $s \in (0, s_0]$ and $\lambda \in (0, \lambda_0]$ and consider the homotopy function

$$H_2(t,x) := \hat{L}x + \hat{T}_s x + \hat{C}(t\lambda, x) + \epsilon \hat{J}x + sMx, \quad (t,x) \in [0,1] \times \overline{j^{-1}(G)}. \tag{2.6}$$

By a similar argument as used previously, we can show that $0 \notin H_2(t, \partial G_R(Y))$ for all $t \in [0,1]$ and for possibly a bigger $R > 0$. Obviously, we can use this R hereafter.

Set $S(t) = C(t\lambda, \cdot) + \epsilon J$ and $T(t) = T_s$. Then $\hat{S}(t) = \hat{C}(t\lambda, \cdot) + \epsilon \hat{J}$ and $\hat{T}_t = \hat{T}_s$. In order to show that $H_2(t,x)$ is an admissible homotopy for the Browder and Skrypnik degree, in view of Lemma 1.7, it suffices to show that $S(t)$ is a bounded homotopy of type (S_+) with respect to $D(L)$ and T_t is a bounded pseudomonotone homotopy of maximal monotone operators. The latter follows immediately from the former because T_s is bounded whenever T is bounded. Let $\{x_n\} \subset D(L)$ be such that $x_n \rightharpoonup x_0$ in X, $Lx_n \rightharpoonup Lx_0$ in X^*, $t_n \to t \in [0,1]$ and

$$\limsup_{n\to\infty} \langle S(t_n)x_n, x_n - x_0\rangle \le 0. \tag{2.7}$$

We observe that

$$\begin{aligned}\langle S(t_n)x_n, x_n - x_0\rangle &= \langle C(t_n\lambda, x_n), x_n - x_0\rangle + \epsilon\langle Jx_n, x_n - x_0\rangle \\ &= \langle C(t_n\lambda, x_n), x_n - x_0\rangle + \epsilon\langle Jx_n - Jx_0, x_n - x_0\rangle \\ &\quad + \epsilon\langle Jx_0, x_n - x_0\rangle \\ &\ge \langle C(t_n\lambda, x_n), x_n - x_0\rangle + \epsilon\langle Jx_0, x_n - x_0\rangle.\end{aligned} \tag{2.8}$$

Using this with (2.7) we get

$$\limsup_{n\to\infty} \langle C(t_n\lambda, x_n), x_n - x_0\rangle \le 0. \tag{2.9}$$

If $t = 0$, then $C(t_n\lambda, x_n) \to 0$ and

$$\lim_{n\to\infty} \langle C(t_n\lambda, x_n), x_n - x_0\rangle = 0.$$

Using this in (2.9), we obtain

$$\limsup_{n\to\infty} \langle Jx_n, x_n - x_0\rangle \le 0. \tag{2.10}$$

Since J is of type (S_+), we obtain $x_n \to x_0$. This implies $C(t_n\lambda, x_n) \to C(0, x_0) = 0$ and $Jx_n \to Jx_0$. It follows that $S(t_n)x_n = C(t_n\lambda, x_n) + \epsilon Jx_n \to \epsilon Jx_0 = S(0, x_0)$.

We next consider the case $t > 0$ and observe that

$$\langle C(t_n\lambda, x_n), x_n - x_0\rangle = \langle C(t_n\lambda, x_n) - C(t\lambda, x_n), x_n - x_0\rangle + \langle C(t\lambda, x_n), x_n - x_0\rangle. \tag{2.11}$$

Using

$$\lim_{n\to\infty}[C(t_n\lambda, x_n) - C(t\lambda, x_n)] = 0$$

and (2.9) in (2.11) we obtain

$$\limsup_{n\to\infty}\langle C(t\lambda, x_n), x_n - x_0\rangle \le 0.$$

The (S_+) property of C w.r.t. $D(L)$ implies $x_n \to x_0$ in X. So, $Jx_n \to Jx_0$ and by the demicontinuity of C we have

$$S(t_n)x_n = C(t_n\lambda, x_n) + \epsilon Jx_n \rightharpoonup C(t\lambda, x_0) + \epsilon Jx_0 = S(t)x_0.$$

This establishes the admissibility of the homotopy $H_2(t, x)$ according to the Skrypnik degree d_S (cf. [15]). Therefore, by the homotopy invariance of this degree, we have

$$\begin{aligned} d_S(H_2(t,\cdot), G_R(Y), 0) &= d_S(H_2(1,\cdot), G_R(Y), 0) \\ &= d_S(H_2(0,\cdot), G_R(Y), 0) \\ &= d_S(\hat{L} + \hat{T}_s + \epsilon\hat{J} + sM, G_R(Y), 0). \end{aligned} \tag{2.12}$$

Consider another homotopy function

$$H_0(t, x) := t(\hat{L} + \hat{T}_s + \epsilon\hat{J} + sM) + (1-t)(\hat{L} + \epsilon\hat{J} + sM), \quad (t, x) \in [0,1] \times \overline{j^{-1}(G)}.$$

Since $T(t) = tT$ is a bounded pseudomonotone homotopy, it is clear that $H_0(t, x)$ is a homotopy of type (S_+) from $j^{-1}(\overline{G}) \subset Y$ to Y^*. By using the techniques in the proof of Lemma 1.1 which is given in [5], one can show that the set of solutions of $H_0(t, x) = 0$ is bounded in Y. Choose the number $R > 0$ bigger enough so that all the solutions of $H_0(t, x) = 0$ are contained in $B_Y(0, R)$ so that $H_0(t, x) = 0$ has no solution $(t, x) \in [0, 1] \times \partial G_R(Y)$. Otherwise, for some $(t_0, x_0) \in [0, 1] \times \partial G_R(Y)$, we have

$$\hat{L} + t\hat{T}_s + \epsilon\hat{J} + sM = 0.$$

Consequently,

$$(\hat{L}x_0, x_0) + t_0(\hat{T}_s x_0, x_0) + \epsilon(\hat{J}x_0, x_0) + s(Mx_0, x_0) = 0$$

which implies $x_0 = 0$. But $x_0 \in \partial(j^{-1}(G))$ which implies $0 = x_0 \in \partial G$. This is a contradiction. Thus, by the invariance under homotopy of the degree, we have

$$d_S(\hat{L} + \hat{T}_s + \epsilon\hat{J} + sM, G_R(Y), 0) = d_S(\hat{L} + \epsilon\hat{J} + sM, G_R(Y), 0). \tag{2.13}$$

The topological degree, d, developed in [1] is based on the methodology of degree developed in [5] by Berkovits and Mustonen and the degree is the limit

$$\begin{aligned} d(H(\lambda,\cdot), G, 0) &= \lim_{s\downarrow 0} d_S(H_1(s,\lambda,\cdot), G_R(Y), 0) \\ &= \lim_{s\downarrow 0} d_S(H_2(1,\cdot), G_R(Y), 0) \\ &= \lim_{s\downarrow 0} d_S(\hat{L} + \hat{T}_s + \epsilon\hat{J} + sM, G_R(Y), 0) \\ &= \lim_{s\downarrow 0} d_S(\hat{L} + \epsilon\hat{J} + sM, G_R(Y), 0) \\ &= d(L + \epsilon J, G, 0) \\ &= 1. \end{aligned}$$

Here, we have used (2.12) and (2.13) and Corollary 1 in [5, p.611]. Therefore there exists $x \in G \cap D(L)$ such that

$$Lx + Tx + C(\lambda, x) + \epsilon Jx \ni 0.$$

This contradicts our assumption (**P**).

(ii) In view of (i), for each positive integer n, there exist $\{x_n\} \subset \partial G \cap D(L)$, $x_n^* \in Tx_n$, $\lambda_n \in (0, \Lambda]$ such that

$$Lx_n + x_n^* + C(\lambda_n, x_n) + \frac{1}{n} Jx_n = 0. \tag{2.14}$$

We may assume that $\lambda_n \to \lambda_0 \in [0, \Lambda]$, $C(\lambda_n, x_n) \rightharpoonup c^*$ and $Jx_n \rightharpoonup p$. Since T is bounded, (2.14) implies $\{Lx_n\}$ is bounded in X^*. Since $\{x_n\}$ is bounded in X, it follows that $\{x_n\}$ is bounded in $Y = D(L)$ with the graph norm. Since Y is reflexive, we may assume that $x_n \rightharpoonup x_0$ in Y. Therefore, we have $x_n \rightharpoonup x_0$ in X and $Lx_n \rightharpoonup Lx_0$ in X^*.

We now consider two cases: (a) $\lambda_0 = 0$; (b) $\lambda_0 > 0$.

(a) Since

$$Lx_n + x_n^* = -C(\lambda_n, x_n) - \frac{1}{n} Jx_n \to 0$$

and $L + T$ satisfies (S_q) on $D(L)$, we have $x_n \to x_0 \in \partial G$ in X. Since $L + T$ is maximal monotone, by Lemma 1.1 we have $x_0 \in D(L)$ and $0 \in Lx_0 + Tx_0$, which contradicts $0 \notin (L + T)(\partial G \cap D(L))$.

(b) We first assert that

$$\limsup_{n\to\infty} \langle C(\lambda_n, x_n), x_n - x_0\rangle \le 0. \tag{2.15}$$

Assume that it is not true. Then there is a subsequence of $\{x_n\}$, which we again denote by $\{x_n\}$, such that

$$\lim_{n\to\infty} \langle C(\lambda_n, x_n), x_n - x_0\rangle = q > 0. \tag{2.16}$$

Since $Lx_n + x_n^* \rightharpoonup -c^*$, we invoke (2.14) and (2.16) to obtain

$$\lim_{n\to\infty} \langle Lx_n + x_n^*, x_n - x_0\rangle < 0,$$

which is impossible by Lemma 1.8. Therefore, (2.15) is true. Using (2.15), $C(\lambda_n, x_n) - C(\lambda_0, x_n) \to 0$, and

$$\langle C(\lambda_0, x_n), x_n - x_0\rangle = \langle C(\lambda_0, x_n) - C(\lambda_n, x_n), x_n - x_0\rangle + \langle C(\lambda_n, x_n), x_n - x_0\rangle,$$

we obtain

$$\limsup_{n\to\infty} \langle C(\lambda_0, x_n), x_n - x_0\rangle \le 0.$$

Since C is a homotopy of class (S_+) w.r.t. Y, we obtain $x_n \to x_0$ in X. Since C is demicontinuous, $C(\lambda_n, x_n) \rightharpoonup C(\lambda_0, x_0) = c^*$. Thus, $Lx_n + x_n^* \rightharpoonup -C(\lambda_0, x_0)$. Since $L + T$ is maximal monotone, it is demiclosed by Lemma 1.1. Thus, $Lx_0 + Tx_0 + C(\lambda_0, x_0) \ni 0$ and the proof of the theorem is complete.

3. **Open Mapping Theorem**

In this section, we obtain an open mapping theorem for the triplet $L+T+C$ with the operators L, T, C as previously considered. Unlike in Theorem 2.3, we need not assume $0 \in D(T)$ and $0 \in T(0)$.

Definition 3.1. An operator $T : X \supset D(T) \to Y$, with Y another real Banach space, is "injective" if for every $x_1,\ x_2 \in D(T)$ with $Tx_1 \cap Tx_2 \neq \emptyset$ we have $x_1 = x_2$. Given an operator $T : X \supset D(T) \to 2^{X^*}$, we say that T is "locally injective" on $G \subset X$ if for every $x_0 \in D(T) \cap G$ there exists a ball $B_r(x_0) \subset X$ such that T is injective on $D(T) \cap \overline{B_r(x_0)}$. If $G = X$, then we simply say that T "locally injective"

The following open mapping theorem generalizes a similar result of Kartsatos and Skrypnik in [13] for $T + C$, where $T : X \supset D(T) \to 2^{X^*}$ is maximal monotone and $C : \overline{G} \to X^*$ demicontinuous, bounded and locally of type (S_+) with $G \subset X$ open and bounded.

Theorem 3.2. *Let $L : X \supset D(L) \to X^*$ be a densely defined linear maximal monotone operator, $T : X \to 2^{X^*}$ a bounded maximal monotone operator and $C : \overline{G} \to X^*$ a bounded demicontinuous operator of type (S_+) with respect to $D(L)$, where $G \subset X$ is an open bounded subset of X. Assume that $L + T + C + \epsilon J$ is locally injective on G for all $\epsilon \geq 0$. Then $(L + T + C)(D(L) \cap G)$ is open in X^*.*

Proof: Let $u^* \in (L + T + C)(D(L) \cap G)$. We may assume without loss of generality that $u^* = 0$, $0 \in D(L) \cap G$, $0 \in T(0)$, $0 \in C(0)$. Since $L + T + C$ is locally injective on G, choose $q > 0$ such that $\overline{B_q(0)} \subset G$ and $L+T+C$ is locally injective on $\overline{B_q(0)}$. It is sufficient to show the existence an $r > 0$ such that $B_r(0) \subset (L + T + C)(D(L) \cap B_q(0))$.

We claim that there is $r > 0$ such that $(L + T + C)(D(L) \cap \partial B_q(0)) \cap B_r(0) = \emptyset$. Suppose that the contrary is true. Then there exists a sequence $\{r_n\}$, $r_n \downarrow 0$ and $\{x_n\} \subset D(L) \cap \overline{B_q(0)}$ and $p_n^* \in B_{r_n}(0)$, $v_n^* \in Tx_n$ such that

$$Lx_n + v_n^* + Cx_n = p_n^*. \tag{3.1}$$

Let $Y = D(L)$ with the graph norm. Since T and C are bounded, it follows that $\{\|Lx_n\|\}$ is bounded and hence $\{\|x_n\|_Y\}$ is bounded. Since Y is reflexive, we may assume that $x_n \rightharpoonup x_0$ in Y, which implies $x_n \rightharpoonup x_0$ in X and $Lx_n \rightharpoonup Lx_0$ in X^*. We are now going to show that

$$\limsup_{n\to\infty} \langle Cx_n, x_n - x_0\rangle \leq 0. \tag{3.2}$$

If (3.2) is not true, we may assume that

$$\lim_{n\to\infty} \langle Cx_n, x_n - x_0\rangle > 0. \tag{3.3}$$

In view of (3.1) and (3.3), we obtain

$$\lim_{n\to\infty} \langle Lx_n + v_n^*, x_n - x_0\rangle < 0,$$

which is impossible by Remark 1.9 after Lemma 1.8 since $L + T$ is maximal monotone. Thus (3.2) is true. Since C is of type (S_+) w.r.t. L, we obtain $x_n \to x_0 \in \partial B_q(0)$ in X. By the demicontinuity of C, we get $Cx_n \to Cx_0$ and therefore $Lx_n + v_n^* \to -Cx_0$. Since $L + T$ is demiclosed as given in Lemma 1.1, we obtain $0 \in (L + T + C)(x_0)$ which is a contradiction to the injectivity of $L + T + C$ on $\overline{B_q(0)}$ and our claim is proved.

We now fix $p^* \in B_r(0)$ and define $f(t) = tp^*$, $t \in [0, 1]$. Clearly, $f(t)$ lies in $B_r(0)$ for all $t \in [0, 1]$. We next claim that there exist an integer $n_0 > 0$ and a number $s_0 > 0$ such that

$$\hat{L}x + \hat{T}_s x + \hat{C}x + sMx + \frac{1}{n}\hat{J}x = j^*(f(t)), \quad s \in (0, s_0],\ n \geq n_0, \tag{3.4}$$

has no solution $x \in \partial G_R(Y)$, where $G_R(Y) = j^{-1}(B_q(0)) \cap B_Y(0, R)$. Here, $B_Y(0, R) := \{y \in Y : \|y\|_Y < R\}$. By using techniques in the proof of Lemma 1.1 which is given in [5], one can see that the set of solutions of (3.4) in $j^{-1}(\overline{G})$ is bounded in Y uniformly with respect to large n and small s, and therefore such a number $R > 0$ exists. We recall that $\partial(j^{-1}(B_q(0))) \subset j^{-1}(\partial B_q(0))$.

Assume that our claim is not true. Then there exist sequences, $\{t_n\} \subset [0,1]$, $s_n \downarrow 0$, $x_n \in \partial(j^{-1}(B_q(0)))$, $x_0 \in Y$, $t_0 \in [0,1]$ with $t_n \to t_0$ and $x_n \rightharpoonup x_0$ in Y such that

$$\hat{L}x_n + \hat{T}_{s_n}x_n + \hat{C}x_n + s_nMx_n + \frac{1}{n}\hat{J}x_n = j^*(f(t_n)). \tag{3.5}$$

We are going to show that

$$\limsup_{n\to\infty}\langle Cx_n, x_n - x_0\rangle \le 0. \tag{3.6}$$

Suppose that this is not true. Then we may assume that

$$\lim_{n\to\infty}\langle Cx_n, x_n - x_0\rangle > 0. \tag{3.7}$$

We observe that

$$\begin{aligned} \langle Lx_n + T_{s_n}x_n, x_n - x_0\rangle &= (\hat{L}x_n + \hat{T}_{s_n}x_n, x_n - x_0) \\ &= -(\hat{C}x_n, x_n - x_0) - s_n(Mx_n, x_n - x_0) - \frac{1}{n}(\hat{J}x_n, x_n - x_0) \\ &\quad +(j^*(f(t_n)), x_n - x_0) \\ &= \langle Cx_n, x_n - x_0\rangle - s_n\langle Lx_n - Lx_0, J^{-1}(Lx_n)\rangle \\ &\quad -\frac{1}{n}\langle Jx_n, x_n - x_0\rangle + \langle (f(t_n)), x_n - x_0\rangle, \end{aligned} \tag{3.8}$$

which implies

$$\lim_{n\to\infty}\langle Lx_n + T_{s_n}x_n, x_n - x_0\rangle < 0.$$

This is impossible by Lemma 1.8(i), and therefore (3.6) is true. Since C is of type (S_+) w.r.t. L, we get $x_n \to x_0 \in \partial B_q(0)$ in X. Since C is demicontinuous, we have $Cx_n \rightharpoonup Cx_0$ in X^*. Since, for each $s > 0$ and $x \in D(T)$, $\|T_s x\| \le \|w\|$ for all $w \in Tx$ and T is bounded, $\{T_{s_n}x_n\}$ is bounded, and therefore we may assume that $T_{s_n}x_n \rightharpoonup w \in Tx_0$ for a subsequence of $\{n\}$ again denoted by $\{n\}$. Then (3.5) implies $\hat{L}x_n + \hat{T}_{s_n}x_n + \hat{C}x_n \rightharpoonup \hat{L}x_0 + j^*w + \hat{C}x_0 = j^*(f(t_0))$.

For all $v \in Y$, we have

$$\langle Lx_0 + w + Cx_0, v\rangle = (\hat{L}x_0 + j^*w + \hat{C}x_0, v) = (j^*(f(t_0)), v) = \langle f(t_0), v\rangle.$$

Since Y is dense in X, we obtain $Lx_0 + w + Cx_0 = f(t_0)$ which implies $f(t_0) \in (L + T + C)(D(L) \cap \partial B_q(0))$. Since $x_0 \in D(L) \cap \partial B_q(0)$ and $f(t_0) \in B_r(0)$, we have a contradiction to $(L + T + C)(D(L) \cap \partial B_q(0)) \cap B_r(0) = \emptyset$.

We now consider the homotopy function

$$H(s,t,x,n) := t\left(\hat{L}x + \hat{T}_s x + \hat{C}x + \frac{1}{n}\hat{J}x\right) + sMx + (1-t)\hat{J}x, \tag{3.9}$$

where $(t,x) \in [0,1] \times j^{-1}(\overline{B_q(0)})$. As pointed out several times previously in other similar situations, one can use techniques used in the proof of Lemma 1.1 which is given in [5] to prove the existence of a number $s_0 > 0$ and an integer $n_0 > 0$ such that the solutions of $H(s,t,x,n) = 0$, $s_0 \in (0, s_0]$, $n \ge n_0$, are bounded Y uniformly with respect to $t \in [0,1]$.

Let $R > 0$ be such that $B_Y(0,R) := \{y \in Y : \|y\|_Y < R\}$ contains all the solutions of the equation of $H(s,t,x,n) = 0$ for all $s \in (0, s_0]$ and $n \ge n_0$.

Let $G_R(Y) = j^{-1}(B_q(0)) \cap B_Y(0,R)$. We are going to show that there exist an integer $n_1 > 0$ and a number $s_1 > 0$ such that (3.9) has no solution $x \in \partial G_R(Y)$ for any $s \in (0, s_1]$, $n \ge n_1$ and

$t \in [0,1]$. Assuming that the contrary is true, let there be sequences $\{x_n\} \subset \partial G_R(Y)$, $\{s_n\} \subset (0,\infty)$, and $\{t_n\} \subset [0,1]$ such that $x_n \rightharpoonup x_0$ in Y, $s_n \to 0$, $t_n \to t_0$ and

$$t_n\left(\hat{L}x_n + \hat{T}_{s_n}x_n + \hat{C}x_n + \frac{1}{n}\hat{J}x_n\right) + s_n M x_n + (1-t_n)\hat{J}x_n = 0. \tag{3.10}$$

If $t_n = 0$ for all n, then

$$s_n M x_n + \hat{J}x_n = 0,$$

which implies $x_n = 0$ for all n, and this is a contradiction to the choice of $\{x_n\}$. Also, if $t_n = 1$ for all n, we again have a contradiction by the argument as in the previous part with $j^*(f(t)) = 0$. Thus, we may assume that $t_n \in (0,1)$. Consider the cases: (a) $t_0 = 0$; (b) $t_0 > 0$.

Case (a): Since $x_n \rightharpoonup x_0$ in Y, it follows that $x_n \rightharpoonup x_0$ in X and $Lx_n \rightharpoonup Lx_0$ in X^*. In particular, $\{\|x_n\|\}$ is bounded. By the boundedness of C, $\{Cx_n\}$ is also bounded. Now,

$$t_n\hat{L}x_n + t_n\hat{T}_{s_n}x_n + \hat{J}x_n = -t_n\hat{C}x_n - t_n\left(\frac{1}{n}-1\right)\hat{J}x_n - s_n M x_n$$

implies

$$\begin{aligned} t_n(\hat{L}x_n, x_n) + t_n(\hat{T}_{s_n}x_n, x_n) + (\hat{J}x_n, x_n) \quad &= \quad -t_n(\hat{C}x_n, x_n) - t_n\left(\frac{1}{n}-1\right)(\hat{J}x_n, x_n) \\ &\quad -s_n(Mx_n, x_n). \end{aligned}$$

By the monotonicity of L and T_s, we get

$$\langle Jx_n, x_n\rangle \le -t_n\langle Cx_n, x_n\rangle - t_n\left(\frac{1}{n}-1\right)\langle Jx_n, x_n\rangle - s_n\langle Lx_n, J^{-1}(Lx_n)\rangle \to 0.$$

This shows that $x_0 = 0$, i.e., a contradiction because $\{x_n\} \subset \partial B_q(0)$. This completes Case (a).

Case (b): If $t_0 = 1$, let $d_n = \dfrac{1}{t_n} - 1$. Clearly, $d_n > 0$ and $d_n J x_n \to 0$. Also, from (3.10), we have

$$\hat{L}x_n + \hat{T}_{s_n}x_n + \hat{C}x_n + \left(\frac{1}{n} + d_n\right)\hat{J}x_n + \frac{s_n}{t_n}Mx_n = 0.$$

This equation is similar to (3.5) with $f(t) \equiv 0$. This shows that the case $t_0 = 1$ is also impossible. Assume now that $t_0 \in (0,1)$. Put

$$e_n = \frac{1}{t_n} + \frac{1}{n} - 1.$$

We may assume that $e_n > 0$ for all n. From (3.10), we have

$$\hat{L}x_n + \hat{T}_{s_n}x_n + \hat{C}x_n + e_n\hat{J}x_n + \frac{s_n}{t_n}Mx_n = 0. \tag{3.11}$$

We are now going to show that (3.6) is true. Assuming the contrary, suppose that (3.7) holds true. We observe that

$$\begin{aligned} \langle Lx_n + T_{s_n}x_n, x_n - x_0\rangle \quad &= \quad (\hat{L}x_n + \hat{T}_{s_n}x_n, x_n - x_0) \\ &= \quad -(\hat{C}x_n, x_n - x_0) - \frac{s_n}{t_n}(Mx_n, x_n - x_0) - e_n(\hat{J}x_n, x_n - x_0) \\ &= \quad -\langle Cx_n, x_n - x_0\rangle - \frac{s_n}{t_n}\langle Lx_n - Lx_0, J^{-1}(Lx_n)\rangle \\ &\quad -e_n\langle Jx_n, x_n - x_0\rangle, \end{aligned} \tag{3.12}$$

which implies

$$\lim_{n\to\infty}\langle Lx_n + T_{s_n}x_n, x_n - x_0\rangle < 0,$$

and this is impossible by Lemma 1.8(i). Thus (3.6) is true. Since C is of type (S_+) w.r.t. L, we get $x_n \to x_0 \in \partial B_q(0)$ in X. Since C is demicontinuous, we have $Cx_n \rightharpoonup Cx_0$ in X^*. As previously noted, we may assume that $T_{s_n}x_n \rightharpoonup w \in Tx_0$ for a subsequence of $\{n\}$ again denoted by $\{n\}$. Then (3.11) implies

$$\hat{L}x_n + \hat{T}_{s_n}x_n + \hat{C}x_n \rightharpoonup \hat{L}x_0 + j^*w + \hat{C}x_0 = -\frac{1-t_0}{t_0}\hat{J}x_0.$$

For all $v \in Y$, we have

$$\left\langle Lx_0 + w + Cx_0 + \frac{1-t_0}{t_0}Jx_0, v\right\rangle = \left(\hat{L}x_0 + j^*w + \hat{C}x_0 + \frac{1-t_0}{t_0}\hat{J}x_0, v\right) = 0.$$

Since Y is dense in X, we obtain

$$Lx_0 + w + Cx_0 + \frac{1-t_0}{t_0}Jx_0 = 0$$

which implies

$$0 \in \left(L + T + C + \frac{1-t_0}{t_0}J\right)(D(L) \cap \partial B_q(0)).$$

Since $x_0 \in D(L) \cap \partial B_q(0)$ and $0 \in B_q(0)$, we have a contradiction to the injectivity of

$$L + T + C + \frac{1-t_0}{t_0}J$$

on $D(L) \cap \overline{B_q(0)}$. This concludes Case (b).

Thus the homotopy equation (3.9) has no solution on $\partial G_R(Y)$ for all large n, and for all $s \in (0, s_0]$ for some $s_0 > 0$, and for all $t \in [0,1]$. Since $H(s,t,x,n)$ is an affine homotopy of bounded demicontinuous operators of type (S_+) from $j^{-1}(\overline{B_q(0)}) \subset Y$ to Y^*, it is a bounded homotopy of type (S_+) from $j^{-1}(\overline{B_q(0)}) \subset Y$ to Y^*. The homotopy invariance of the degree for (S_+) (cf. [15]) implies

$$\begin{aligned} d_S(H(s,1,\cdot,n), G_R(Y), 0) &= d_S(H(s,0,\cdot,n), G_Y(R), 0) \\ &= d_S(\hat{J} + sM, G_Y(R), 0) \\ &= 1. \end{aligned}$$

Next, we consider the homotopy equation

$$H_1(s,t,x,n) := \hat{L}x + \hat{T}_s x + \hat{C}x + sMx + \frac{1}{n}\hat{J}x - j^*(f(t)),$$

where $f(t) = tp^*$ with $p^* \in B_r(0)$ and $t \in [0,1]$. We have already seen that the equation $H_1(s,t,x,n) = 0$ has no solution $x \in \partial G_Y(R)$. We notice that $H_1(s,t,x,n)$ is admissible for the Skrypnik degree for (S_+) mappings. By the invariance property of the degree, we obtain

$$\begin{aligned} d_S(H_1(s,t,\cdot,n), G_Y(R), 0) &= d_S(H_1(s,0,\cdot,n), G_Y(R), 0) \\ &= d_S\left(\hat{L} + \hat{T}_s + \hat{C} + \frac{1}{n}\hat{J} + sM, G_Y(R), 0\right) \\ &= d_S(H(s,1,\cdot,n), G_Y(R), 0) \\ &= 1. \end{aligned}$$

Since

$$d_S(H_1(s,1,\cdot,n), G_Y(R), 0) = d_S(\hat{L} + \hat{T}_s + \hat{C} + sM + \frac{1}{n}\hat{J} - j^*(f(t)), G_Y(R), 0),$$

we have that the degree of $L + T + C + \frac{1}{n}J$ as in [1] satisfies

$$\begin{aligned} & d(L + T + C + \frac{1}{n}J - f(t), B_q(0), 0) \\ = \;& \lim_{s\to 0} d_S(\hat{L} + \hat{T}_s + \hat{C} + sM + \frac{1}{n}\hat{J} - j^*(f(t)), G_Y(R), 0) \\ = \;& 1, \end{aligned}$$

for all $t \in [0,1]$ and for all $n \geq n_0$. Thus, for all $n \geq n_0$, we have

$$B_r(0) \subset \left(L + T + C + \frac{1}{n}J\right)(B_q(0) \cap D(L))$$

so that, for each $n \geq n_0$, there exists $x_n \in B_q(0) \cap D(L)$ such that

$$p^* = Lx_n + w_n + Cx_n + \frac{1}{n}Jx_n \tag{3.13}$$

for some $w_n \in Tx_n$. Since T and C are bounded, we have that $\{Lx_n\}$ is bounded and hence we may assume that $x_n \rightharpoonup x_0$ in Y. We claim that

$$\limsup_{n\to\infty} \langle Cx_n, x_n - x_0\rangle \leq 0. \tag{3.14}$$

If (3.14) is not true, then we may assume that

$$\lim_{n\to\infty} \langle Cx_n, x_n - x_0\rangle > 0. \tag{3.15}$$

From (3.13) and 3.15), it follows that

$$\lim_{n\to\infty} \langle Lx_n + w_n, x_n - x_0\rangle < 0,$$

which is impossible by Remark 1.9 after Lemma 1.8, and therefore we conclude that

$$\limsup_{n\to\infty} \langle Cx_n, x_n - x_0\rangle \leq 0.$$

By the (S_+)-property of C with respect to $D(L)$, we get $x_n \to x_0$ in X and hence by the demicontinuity of C, we get $Cx_n \rightharpoonup Cx_0$ in X^*. Therefore $w_n \rightharpoonup p^* - Lx_0 - Cx_0$ in X^*. Since T is demiclosed by Lemma 1.1, we have $p^* - Lx_0 - Cx_0 \in Tx_0$, which means that $p^* \in (L+T+C)x_0$. We note that $x_0 \in D(L) \cap \overline{B_q(0)}$. Since $(L+T+C)(D(L) \cap \partial B_q(0)) \cap B_r(0) = \emptyset$ and $p^* \in B_r(0)$, it follows that

$$p^* \in (L+T+C)(D(L) \cap B_q(0)).$$

Since $p^* \in B_r(0)$ arbitrary, we have

$$B_r(0) \subset (L+T+C)(D(L) \cap B_q(0)).$$

This completes the proof.

4. **Examples**

Addou and Mermri [1] have considered parabolic time-dependent problems that involve the operators considered in this paper. We now present the operators for the purpose of completeness. Eigenvalue problems and open mapping theorems in the spirit of the results obtained in this paper can then be stated in terms of the operators.

Let Ω be a bounded open set in $\mathbf{R}^N$ with smooth boundary, $m \geq 1$ an integer, and $a > 0$. Set $Q = \Omega \times [0,a]$. We consider the differential operator

$$\begin{aligned} &\frac{\partial u(x,t)}{\partial t} + \sum_{|\alpha|\leq m} (-1)^{|\alpha|} D^\alpha A_\alpha(x,t,u(x,t),Du(x,t),...,D^m u(x,t)) \\ &+ \sum_{|\alpha|\leq m} (-1)^{|\alpha|} D^\alpha B_\alpha(x,t,u(x,t),Du(x,t),...,D^m u(x,t)) \end{aligned} \tag{4.1}$$

in Q. The coefficients $A_\alpha = A_\alpha(x,t,\xi)$, are defined for $(x,t) \in Q$, $\xi = \{\xi_\gamma,\ |\gamma| \leq m\} = (\eta,\zeta) \in \mathbf{R}^{N_0}$ with $\eta = \{\eta_\gamma,\ |\gamma| \leq m-1\} \in \mathcal{R}^{N_1}$, $\zeta = \{\zeta_\gamma,\ |\gamma| = m\} \in \mathbf{R}^{N_2}$, and $N_1 + N_2 = N_0$. We assume that each coefficient $A_\alpha(x,t,\xi)$ satisfies the usual Carathéodory conditions. Let the following standard conditions be satisfied.

(A_1) **(Continuity)** For some $p \geq 2,\ c_1 > 0,\ g \in L^q(Q)$ with $q = p/(p-1)$, we have

$$|A_\alpha(x,t,\eta,\zeta)| \leq c_1(|\zeta|^{p-1} + |\eta|^{p-1} + g(x,t)),$$
$$(x,t) \in Q,\ \xi = (\eta,\zeta) \in \mathbf{R}^{N_0},\ |a| \leq m.$$

(A_2) **(Monotonicity)**

$$\sum_{|\alpha|\leq m} (A_\alpha(x,t,\xi_1) - A_\alpha(x,t,\xi_2))(\xi_{1_\gamma} - \xi_{2_\gamma}) \geq 0, \quad (x,t) \in Q,\ \xi_1,\xi_2 \in \mathbf{R}^{N_0}.$$

(A_3) **(Leray-Lions)**

$$\sum_{|\alpha|=m} (A_\alpha(x,t,\eta,\zeta) - A_\alpha(x,t,\eta,\zeta^*))(\zeta_\gamma - \zeta^*_\gamma) > 0,$$
$$(x,t) \in Q,\ \eta \in \mathcal{R}^{N_1},\ \zeta,\zeta^* \in \mathbf{R}^{N_2}.$$

(A_4) **(Coercivity)** There exist $c_0 > 0$ and $h \in L^1(Q)$ such that

$$\sum_{|a|\leq m} A_\alpha(x,t,\xi) \geq c_0|\xi|^p - h(x,t), \quad (x,t) \in Q,\ \xi \in \mathbf{R}^{N_0}.$$

Put $V = W_0^{m,p}(\Omega)$ and $X = L^p(0,a;V)$. Then $X^* = L^q(0,a;V^*)$. Under the condition (A_1), the second term of (4.1) induces a continuous bounded operator $T : X \to X^*$ given by

$$\langle Tu, v\rangle = \sum_{|\alpha|\leq m} \int_Q A_\alpha(x,t,u,Du,...,D^m u) D^\alpha v, \quad u,v \in X.$$

This operator is also maximal monotone under the condition (A_2). Under (A_1), (A_3) and (A_4) (with "A" replaced by "B" and the other necessary changes) the third term of (4.1) induces a continuous, bounded operator C which satisfies the condition (S_+) w.r.t. $D(L)$, where the operator L is defined below. The operator C is given by

$$\langle Cu, v\rangle = \sum_{|\alpha|\leq m} \int_Q B_\alpha(x,t,u,Du,...,D^m u) D^\alpha v, \quad u,v \in X.$$

The operator $\partial/\partial t$ induces the operator $L : X \supset D(L) \to X^*$, where

$$D(L) = \{v \in X \mid v' \in X^*,\ v(0) = 0\},$$

via the equality

$$\langle Lu, v\rangle = \int_0^a \langle u'(t), v(t)\rangle dt, \quad u \in D(L),\ v \in X.$$

The symbol $u'(t)$ above is the generalized derivative of $u(t)$, i.e.

$$\int_0^a u'(t)\varphi(t)\,dt = -\int_0^a u(t)\varphi'(t)\,dt, \quad \varphi \in C_0^\infty(0,a).$$

One can verify, as in Zeidler [17], that L is a linear, closed, densely defined maximal monotone operator.

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