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OPERATORS ASSOCIATED WITH SEQUENCES IN HILBERT SPACES

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ABSTRACT. The expansion formula for a signal in a Hilbert space requires a set of redundant or non-redundant vectors. Bases (non-redundant set) have been used for a long time. Over the past few years, frames (redundant or an over-complete spanning set) have been studied because of their applications. There are several desired properties of frames for specific applications. The desired frames can be constructed by using the properties of the associated linear operators. In this paper, we construct frames as a sum of two frames for a given Hilbert space by using the operators associated with them.

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1. Introduction

Let $\mathbb{H}$ be a separable real or complex Hilbert space and $\mathbb{N}$ be a countable index set. A basis $\mathbb{X} = \{x_k\}_{k,k\in\mathbb{N}}$ for $\mathbb{H}$ provides the following representation formula for $f \in \mathbb{H}$.

$$f = \sum_{k\in\mathbb{N}} c_k x_k, \tag{1.1}$$

where the coefficients c_k are real or complex numbers. There are two important operations associated with the above representation. The first is the computation of coefficients from the function (signal) f, known as *analysis* and the second is the reconstruction of the signal f from the coefficients c_k, known as *synthesis*. The well known orthogonal or orthonormal basis (ONB) provide easy implementation of the above representation; however they are not always the best as they are not very flexible. Riesz bases are also very applicable but they need another basis called the dual basis. A basis and it's dual are both needed for analysis and synthesis. Recently frames have become more popular as they provide reconstruction formula similar to (1.1), and yet, they are linearly dependent. So frames are generalizations of bases in a Hilbert space. Frames are considerably more stable than the basis upon the action of operators [8]. For example let $L : \mathbb{H} \to \mathbb{H}$ be a linear operator, $\mathbb{X} = \{x_k\}_{k\in\mathbb{N}}$ be an

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orthonormal basis, and let $L(\mathbb{X}) := \{L(x_k)\}_{k\in\mathbb{N}}$. Then $L(\mathbb{X})$ is an orthonormal basis for $\mathbb{H}$ if L is a unitary operator, $L(\mathbb{X})$ is a Riesz basis for $\mathbb{H}$ if L is a bounded bijective operator, $L(\mathbb{X})$ is a Bessel sequence in $\mathbb{H}$ if L is a bounded operator, $L(\mathbb{X})$ is a frame sequence (a frame sequence is a frame of its span) for $\mathbb{H}$ if L is bounded operator with closed range, and $L(\mathbb{X})$ is a frame for $\mathbb{H}$ if L is a bounded surjective operator.

New frames can be constructed from the given ones by using the properties of the linear operators that can be associated with them. An introduction to frames and their associated operators are provided in Section 2. Several known characterizations are summarized in the same section. In Section 3, we provide conditions under which one can obtain new frames as the sum of two frames under the action of an operator. Similar conditions were studied in [10] and [11] but we provide in a modified way. All operators involved in this paper are linear unless otherwise mentioned.

2. **Preliminaries**

The linear operators can be assigned for large classes of sequences in a Hilbert space sequences such as, Bessel, Riesz basis, ONB, frame, frame sequence, Riesz-Fisher sequence etc [1] in a Hilbert space. [1]. The issue of convergence plays a vital role in defining the operator and hence the domain of definition for the associated operators may not be all of the Hilbert space [1]. We will consider Bessel sequences and associated operators since most of issues, such as the domain of definition, range and convergence are settled once the sequence is Bessel [3].

2.1. **Bessel sequence, Frames, and Riesz basis.**

A sequence $\mathbb{X} = \{x_j\}_{j\in\mathbb{N}}$ in $\mathbb{H}$ is called a Bessel sequence if there exists a constant $B > 0$ such that for all $f \in \mathbb{H}$

$$\sum_{j\in\mathbb{N}} |\langle f, x_j\rangle|^2 \leq B||f||^2.$$

It is said to be a frame if there exists constants A, B, such that $0 < A \leq B$, and for all $f \in \mathbb{H}$,

$$A||f||^2 \leq \sum_{j\in\mathbb{N}} |\langle f, x_j\rangle|^2 \leq B||f||^2.$$

The numbers A and B are called frame bounds. If $A = B$, then $\mathbb{X}$ is called a tight frame. If $A = B = 1$, the frame is called the normalized tight frame or a Parseval frame. An orthonormal basis is an example of normalized tight frame. This is the only normalized tight frame in which the vectors have unit length. We note that the left inequality in the definition of frame implies that the sequence $\mathbb{X}$ is complete [8]. A sequence $\mathbb{X} = \{x_j\}_{j\in\mathbb{N}}$ in $\mathbb{H}$ is called a Riesz basis for $\mathbb{H}$ if $\mathbb{X}$ is complete in $\mathbb{H}$, and there exists constants $A, B > 0$ such that for every finite scalar sequences $\{c_k\}$ one has

$$A\sum_k |c_k|^2 \leq ||\sum_k c_k x_k||^2 \leq B\sum_k |c_k|^2.$$

It turns out that the Riesz basis is a of family vectors $L(\mathbb{X})$, where L is bounded bijective operator and $\mathbb{X}$ is an orthonormal basis [8].
In what follows, the sequences in Hilbert space $\mathbb{H}$ are denoted by $\mathbb{X} = \{x_k\}_{k\in\mathbb{N}}$ and the scalar sequences are denoted by $\{c_k\}$.

2.2. **Operators.** Let $\mathbb{X} = \{x_j\}_{j\in\mathbb{N}}$ be a Bessel sequence. The *analysis* and *synthesis operators*, denoted respectively by $T^*_{\mathbb{X}} : \mathbb{H} \to l_2(\mathbb{N})$ and $T_{\mathbb{X}} : l_2(\mathbb{N}) \to \mathbb{H}$, are defined respectively by,

$$T^*_{\mathbb{X}} : f \to \{\langle f, x_j\rangle\}$$

and

$$T_{\mathbb{X}} : \{c_j\} \to \sum_{j\in\mathbb{N}} c_j x_j.$$

The analysis operator is actually the Hilbert space adjoint operator of synthesis operator. Note that the analysis and synthesis operators are well-defined and bounded because $\mathbb{X}$ is a Bessel sequence [8]. These operators can be defined in suitable domains (subspaces of $\mathbb{H}$ or $l_2(\mathbb{N})$) even if $\mathbb{X}$ is not a Bessel sequence [1]. For example if $\mathbb{X}$ is an orthonormal basis, the domain of $T^*_{\mathbb{X}}$ is $\mathbb{H}$ and the domain of $T_{\mathbb{X}}$ is $l_2(\mathbb{N})$. It turns out that $\mathbb{X}$ is a frame if and only if the analysis operator is injective. Also, it is a frame if and only if the synthesis operator is surjective [8].

The *frame operator*, denoted by $S_{\mathbb{X}}$, is defined by $S_{\mathbb{X}} := T_{\mathbb{X}}T^*_{\mathbb{X}} : \mathbb{H} \to \mathbb{H}$, and is given by

$$S_{\mathbb{X}} f = \sum_j \langle f, x_j\rangle x_j. \tag{2.1}$$

It is known that if $\mathbb{X}$ is a frame, then $S_{\mathbb{X}}$ is bounded, self-adjoint, positive operator and has bounded inverse [8]. In signal processing typically one is interested in the reconstruction of a signal. Since the above operator is invertible, the signal f can be reconstructed; however the inversion is not very easy except in cases when $S_{\mathbb{X}}$ is an identity. For Parseval frame the operator $S_{\mathbb{X}}$ is an identity. For a tight frame $S_{\mathbb{X}} = AI$, where I being an identity operator. Let $\mathbb{X} = \{x_j\}_{j\in\mathbb{N}}$, and $\mathbb{Y} = \{y_j\}_{j\in\mathbb{N}}$ be two Bessel sequences in $\mathbb{H}$. If the operator, $S_{\mathbb{X},\mathbb{Y}} := T_{\mathbb{X}}T^*_{\mathbb{Y}}$ given by,

$$T_{\mathbb{X}}T^*_{\mathbb{Y}} f = \sum_j \langle f, y_j\rangle x_j,$$

is an identity, then the Bessel sequences $\mathbb{X}$ and $\mathbb{Y}$ are actually frames and are called dual frames [3]. From the equation (2.1), we can write [8]

$$f = \sum_j \left\langle f, S^{-1}_{\mathbb{X}} x_j\right\rangle x_j = \sum_j \langle f, x_j\rangle S^{-1}_{\mathbb{X}} x_j. \tag{2.2}$$

So $S^{-1}_{\mathbb{X}}(\mathbb{X}) := \{S^{-1}_{\mathbb{X}} x_k\}_{k\in\mathbb{N}}$ is a dual of $\mathbb{X}$, called the canonical dual. It is known that if $\mathbb{X}$ is a frame then $S^{-1}_{\mathbb{X}}(\mathbb{X})$ is a frame too [8]. The reconstruction formula takes the form

$$f = \sum_j \langle f, y_j \rangle x_j.$$

So $\mathbb{X}$ is dual to $\mathbb{Y}$ if and only if $T_{\mathbb{X}}T_{\mathbb{Y}}^* = I$.

The *Gramian Operator* of $\mathbb{X}$ is from $l_2(\mathbb{N}) \to l_2(\mathbb{N})$ and is given by

$$G_{\mathbb{X}} := T_{\mathbb{X}}^* T_{\mathbb{X}}.$$

Let $\mathbb{X} = \{x_k\}_{k\in\mathbb{N}}$ and $\mathbb{Y} = \{y_k\}_{k\in\mathbb{N}}$ be two Bessel sequences in the Hilbert space $\mathbb{H}$. They are said to be orthogonal if $ran(T_{\mathbb{X}}^*) \perp ran(T_{\mathbb{Y}}^*)$. The orthogonality of two frames $\mathbb{X}$ and $\mathbb{Y}$ is equivalent to the frame operator being a zero operator, namely

$$S_{\mathbb{Y},\mathbb{X}} = T_{\mathbb{Y}}T_{\mathbb{X}}^* = 0 \Leftrightarrow S_{\mathbb{X},\mathbb{Y}} = T_{\mathbb{Y}}^* T_{\mathbb{X}} = 0.$$

The last equations imply that the frames $\mathbb{X}$ and $\mathbb{Y}$ are orthogonal iff [3]

$$\sum_k \langle x, x_k \rangle y_k = 0 \text{ for all } x \in \mathbb{H} \Leftrightarrow \sum_k \langle x, y_k \rangle x_k = 0 \text{ for all } x \in \mathbb{H}.$$

This idea has been used in multi-access communication. The orthogonality has been studied in [3, 4, 5, 6] and references there in.

2.3. **Frames in $\mathbb{R}^n$.** Frames in finite dimensions can be found in many literatures, for example [1, 7, 8]. Let the vector space $\mathbb{H}$ be $\mathbb{R}^n$ or $\mathbb{C}^n$. It is known that the frames for finite dimensional space are precisely the finite spanning sets [8]. Let $\mathbb{X} = \{x_k\}_{k=1}^m$ be a frame for the vector space $\mathbb{R}^n$, $m \geq n$. Then the analysis operator $T_{\mathbb{X}}^* : \mathbb{R}^n \to \mathbb{R}^m$, is a matrix formed by taking the vectors x_k as rows and the synthesis operator $T_{\mathbb{X}} : \mathbb{R}^m \to \mathbb{R}^n$ is a matrix formed by taking the vectors x_k as the columns. Thus they are given by the matrices

$$T_{\mathbb{X}}^* = \begin{pmatrix} - & x_1 & - \\ - & x_2 & - \\ \vdots & \vdots & \vdots \\ - & x_m & - \end{pmatrix}, \quad T_{\mathbb{X}} = \begin{pmatrix} | & | & \cdots & | \\ x_1 & x_2 & \cdots & x_m \\ | & | & \cdots & | \end{pmatrix}$$

respectively. In case of $\mathbb{C}^n$, the vectors in the analysis operator need to be conjugated. We note that the analysis operator is injective and the synthesis operator is surjective. If $S_{\mathbb{X}}$ is a frame operator,

$$S_{\mathbb{X}}x = \mathbb{T}_{\mathbb{X}}T_{\mathbb{X}}^* x = \sum_{j=1}^m \langle x, x_j \rangle x_j,$$

and

$$\langle S_{\mathbb{X}}x, x \rangle = \langle T_{\mathbb{X}}^* x, T_{\mathbb{X}}^* x \rangle = ||T_{\mathbb{X}}^* x||^2 = \sum_j^m | \langle x, x_j \rangle |^2.$$

It turns out from the frame inequality that, for all $x \in \mathbb{H}$,

$$\langle Ax, x \rangle \leq \langle S_{\mathbb{X}}x, x \rangle \leq B \langle x, x \rangle \Longleftrightarrow AI \leq S \leq BI.$$

So A and B are the smallest and largest eigenvalues of $S_{\mathbb{X}}$. If $A = B$, the frame is said to be a tight frame (also known as A-tight frame). In this case, the reconstruction formula takes the form

$$x = \frac{1}{A} \sum_{j=1}^{m} \langle x, x_j \rangle \, x_j.$$

The frame is said to be normalized tight frame or Parseval frame if $A = B = 1$. So the above equation takes the form

$$x = \sum_{j=1}^{m} \langle x, x_j \rangle \, x_j.$$

Frames are therefore generalization of orthonormal bases. A frame is called equal norm if $||x_k|| = a$ for all $k \in \mathbb{N}$. Let tr denote the trace of a matrix. Then

$$tr(S_{\mathbb{X}}) = tr(T_{\mathbb{X}} T_{\mathbb{X}}^*) = tr(T_{\mathbb{X}}^* T_{\mathbb{X}}) = tr(G_{\mathbb{X}}) = \sum_{i=1}^{m} ||x_k||^2 = ma^2.$$

For a tight frame, $S_{\mathbb{X}} = AI_{n\times n}$, and so $tr(S_{\mathbb{X}}) = nA$. Thus we get, $A = \frac{ma^2}{n}$. If the frame is of unit norm and tight, then $a = 1$, and so $A = \frac{m}{n}$.

Example 2.1. Let's consider the matrix

$$\mathbb{C} = \begin{pmatrix} 0 & \sqrt{3}/2 & -\sqrt{3}/2 \\ 1 & -1/2 & -1/2 \end{pmatrix}.$$

The columns of the matrix $\mathbb{C}$ form a frame, called the Mercedez-Benz frame for $\mathbb{R}^2$. Let

$$\mathbb{X} = \left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} \sqrt{3}/2 \\ -1/2 \end{pmatrix}, \begin{pmatrix} -\sqrt{3}/2 \\ -1/2 \end{pmatrix} \right\} =: \{\phi_1, \phi_2, \phi_3\}.$$

The reconstruction formula (2.2) for this frame is given by

$$v = \frac{2}{3} \sum_{k} < v, \phi_k > \phi_k.$$

The analysis operator $T_{\mathbb{X}}^* : \mathbb{R}^2 \to \mathbb{R}^3$, is given by

$$T_{\mathbb{X}}^* = \begin{pmatrix} 0 & 1 \\ \sqrt{3}/2 & -1/2 \\ -\sqrt{3}/2 & -1/2 \end{pmatrix}.$$

The synthesis operator $T_{\mathbb{X}} : \mathbb{R}^3 \to \mathbb{R}^2$ is given by

$$T_{\mathbb{X}} = \begin{pmatrix} 0 & \sqrt{3}/2 & -\sqrt{3}/2 \\ 1 & -1/2 & -1/2 \end{pmatrix}$$

The frame operator $S_{\mathbb{X}} = T_{\mathbb{X}} T_{\mathbb{X}}^*$ is given by

$$S_{\mathbb{X}} = \frac{3}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

So $S_{\mathbb{X}} = \frac{3}{2} I_2$. The Gramian $G_{\mathbb{X}} = T_{\mathbb{X}}^* T_{\mathbb{X}}$ is

$$G_{\mathbb{X}} = \begin{pmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 1 \end{pmatrix}.$$

$G_{\mathbb{X}}$ is a rank 2 matrix.

The matrix $T_{\mathbb{X}}^*$ has a unitary completion $(\tilde{T}_{\mathbb{X}}^*)$, as given below (the vectors are normalized).

$$T_{\mathbb{X}}^* = \frac{\sqrt{2}}{\sqrt{3}} \begin{pmatrix} 0 & 1 \\ \sqrt{3}/2 & -1/2 \\ -\sqrt{3}/2 & -1/2 \end{pmatrix} \Rightarrow \tilde{T}_{\mathbb{X}}^* = \begin{pmatrix} 0 & \frac{\sqrt{2}}{\sqrt{3}} & : & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & : & \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & : & \frac{1}{\sqrt{3}} \end{pmatrix}.$$

This completion provides a pair of orthogonal frames.

Let

$$\mathbb{X} = \left\{ \begin{pmatrix} 0 \\ \frac{\sqrt{2}}{\sqrt{3}} \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{6}} \end{pmatrix}, \begin{pmatrix} \frac{-1}{\sqrt{2}} \\ \frac{-1}{\sqrt{6}} \end{pmatrix} \right\}, \quad \mathbb{Y} = \left\{ \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\}.$$

Then $\mathbb{X}$ and $\mathbb{Y}$ are a pair of orthogonal frames. $G_{\mathbb{Y}}$ is a rank 1 matrix. Indeed, the frames in $\mathbb{C}^n$ are exactly the images of the ONB in $\mathbb{C}^n$ under a surjective operator [2]. For two sequences $\mathbb{X} = \{x_k\}_{k=1}^m$, $\mathbb{Y} = \{x_k\}_{k=1}^p$, the mixed Gram matrix $G_{\mathbb{X},\mathbb{Y}}$ of size $m \times p$ is given by $(G_{\mathbb{X},\mathbb{Y}})_{i,j} = \langle x_i, y_j \rangle$. If $\mathbb{X} = \mathbb{Y}$, then this matrix is called the Gram matrix, and is denoted by $G_{\mathbb{X}}$.

2.4. **Classification of Sequences.** The authors in [1] have provided the following classifications results about the sequences and their associated operators in a Hilbert space $\mathbb{H}$. The above operators can be defined for a larger class of sequences not necessarily Bessel sequences. The following propositions are taken from [1, 8] and we assumed that the domain of $T_{\mathbb{X}}^*$, denoted by $dom(T_{\mathbb{X}}^*)$ is $\mathbb{H}$. In what follows $ran(T_{\mathbb{X}}^*)$ stands for the range of $T_{\mathbb{X}}^*$.

Proposition 2.1. *The following statements are are equivalent.*
(i) $\mathbb{X}$ *is a Bessel sequence for* $\mathbb{H}$.
(ii) $dom(T_{\mathbb{X}}^*) = \mathbb{H}$.
(iii) $dom(S_{\mathbb{X}}) = \mathbb{H}$.
(iv) $dom(T_{\mathbb{X}}) = l_2(\mathbb{N})$.

Remark 2.2. It turns out that if $\mathbb{X}$ is a Bessel sequence with bound B, then $||T_{\mathbb{X}}^*|| \leq \sqrt{B}$, $||T_{\mathbb{X}}|| \leq \sqrt{B}$, $||S_{\mathbb{X}}|| \leq B$ and $||G_{\mathbb{X}}|| \leq B$ see for example [1].

Proposition 2.2. *The following are equivalent.*
(i) $\mathbb{X}$ *is a complete sequence.*
(ii) $ran(T_{\mathbb{X}})$ *is dense in* $\mathbb{H}$.
(iii) $T_{\mathbb{X}}^*$ *is injective.*
(iv) $S_{\mathbb{X}}$ *is injective.*

Remark 2.3. It turns out that a frame is complete sequence.

Proposition 2.3. *The following are equivalent.*
(i) $\mathbb{X}$ *is a Riesz basis.*
(ii) $T_{\mathbb{X}}$ *is bijective.*
(iii) $T_{\mathbb{X}}^*$ *is bijective.*
(iv) $S_{\mathbb{X}}$ *is bijective and* $S_{\mathbb{X}}^{-1}\{x_k\}_k$ *is bi-orthogonal to* $\mathbb{X}$.
(v) $\mathbb{X}$ *is complete in* $\mathbb{H}$ *and* $G_{\mathbb{X}}$ *is a bounded invertible operator on* $l_2(\mathbb{N})$.

Proposition 2.4. *The following are equivalent.*
(i) $\mathbb{X}$ *is a frame for* $\mathbb{H}$.
(ii) $T_{\mathbb{X}}$ *is surjective.*
(iii) $T_{\mathbb{X}}^*$ *is injective and its range is closed.*
(iv) $S_{\mathbb{X}}$ *is bijective.*
(iv) $G_{\mathbb{X}}|_{ran(T_{\mathbb{X}}^*)} : ran(T_{\mathbb{X}}^*) \to ran(T_{\mathbb{X}}^*)$ *is a bounded operator with bounded inverse.*

Proposition 2.5. *The following are equivalent.*
(i) $\mathbb{X}$ *is a frame sequence in* $\mathbb{H}$.
(ii) $ran(T_{\mathbb{X}}) = ran(S_{\mathbb{X}})$.
(iii) $dom(T_{\mathbb{X}}^*) = \mathbb{H}$ *and* $ran(T_{\mathbb{X}}^*)$ *is closed.*
(iv) $dom(S_{\mathbb{X}}) = \mathbb{H}$ *and* $ran(S_{\mathbb{X}}) = \overline{span\{\mathbb{X}\}}$.
(v) $G_{\mathbb{X}}|_{ran(T_{\mathbb{X}}^*)} : ran(T_{\mathbb{X}}^*) \to ran(T_{\mathbb{X}}^*)$ *is a bounded operator with bounded inverse.*

The following are well-known results, for example [8]. As they will be used in this paper, they are provided with simple proofs.

Lemma 2.4. *Let* $\mathbb{X}$ *and* $\mathbb{Y}$ *be two frames. Then the mixed Gram matrix* $G_{\mathbb{X},\mathbb{Y}} = T_{\mathbb{X}}^* T_{\mathbb{Y}} : l_2(\mathbb{N}) \to l_2(\mathbb{N})$ *is bounded.*

Proof. Let $\{d_k\} \in ran(T_{\mathbb{Y}}^*)$, and let $f = T_{\mathbb{Y}}(\{d_k\})$, then

$$\begin{aligned} \|T_{\mathbb{X}}^* T_{\mathbb{Y}}(\{d_k\})\|^2 &= \langle T_{\mathbb{X}}^* T_{\mathbb{Y}}(\{d_k\}), T_{\mathbb{X}}^* T_{\mathbb{Y}}(\{d_k\})\rangle \\ &= \langle T_{\mathbb{Y}}(\{d_k\}), T_{\mathbb{X}} T_{\mathbb{X}}^* T_{\mathbb{Y}}(\{d_k\})\rangle \\ &= \langle T_{\mathbb{Y}}(\{d_k\}), S_{\mathbb{X}} T_{\mathbb{Y}}(\{d_k\})\rangle \\ &= \langle f, S_{\mathbb{X}} f\rangle \\ &= \textstyle\sum_i |\langle f, x_i\rangle|^2 \le B\|f\|^2 \\ &\le B\|T_{\mathbb{Y}}\|\,\|\{d_k\}\|. \end{aligned}$$

∎

Lemma 2.5. *Let $\mathbb{X}$ and $\mathbb{Y}$ be two frames and let $\tilde{\mathbb{X}}$ and $\tilde{\mathbb{Y}}$ be their dual frames. Then the mixed Gram matrix $G_{\mathbb{X},\mathbb{Y}} = T^*_{\mathbb{X}} T_{\tilde{\mathbb{Y}}} : ran(T^*_{\mathbb{Y}}) \to ran(T^*_{\mathbb{X}})$ such that*

$$T^*_{\mathbb{X}} T_{\tilde{\mathbb{Y}}}(\{\langle f, y_k \rangle\}) \to \{\langle f, x_k \rangle\}.$$

Proof. Let $\{c_k\} \in ran(T^*_{\mathbb{Y}})$. There exists an element $f \in \mathbb{H}$ such that $c_k = \langle f, y_k \rangle$.

$$\begin{aligned} T^*_{\mathbb{X}} T_{\tilde{\mathbb{Y}}}(\{c_k\}) &= T^*_{\mathbb{X}} \left(\textstyle\sum_i c_i \tilde{y}_i\right) \\ &= \{\langle \textstyle\sum_i c_i \tilde{y}_i, x_k \rangle\} \\ &= \{\langle \textstyle\sum_i \langle f, y_i \rangle \tilde{y}_i, x_k \rangle\} \\ &= \{\langle f, x_k \rangle\} \in ran(T^*_{\mathbb{X}}). \end{aligned}$$

∎

Lemma 2.6. *Let $\mathbb{X}$ and $\mathbb{Y}$ be two frames. Then $G_{\mathbb{X},\mathbb{Y}} = T^*_{\mathbb{X}} T_{\mathbb{Y}} : ran(T^*_{\mathbb{Y}}) \to ran(T^*_{\mathbb{X}})$ is bijective.*

Proof. Let $\{c_k\} \in ran(T^*_{\mathbb{Y}})$, and let $\tilde{\mathbb{X}}$ and $\tilde{\mathbb{Y}}$ be their dual frames. There exists an element $f \in \mathbb{H}$ such that $c_k = \langle f, y_k \rangle$. We note that $S_{\mathbb{X}}$ is self-adjoint. Let $S_{\mathbb{X}} f = g$. So $c_k = \langle f, y_k \rangle = \left\langle S^{-1}_{\mathbb{X}} g, y_k \right\rangle = \left\langle g, S^{-1}_{\mathbb{X}} y_k \right\rangle = \langle g, \tilde{y}_k \rangle$. Then,

$$\begin{aligned} T^*_{\mathbb{X}} T_{\mathbb{Y}}(\{c_k\}) &= T^*_{\mathbb{X}} \left(\textstyle\sum_i c_i y_i\right) \\ &= \{\langle \textstyle\sum_i c_i y_i, x_k \rangle\} \\ &= \{\langle \textstyle\sum_i \langle f, y_i \rangle y_i, x_k \rangle\} \\ &= \left\{\left\langle \textstyle\sum_i \left\langle S^{-1}_{\mathbb{X}} g, y_i \right\rangle y_i, x_k \right\rangle\right\} \\ &= \left\{\left\langle \textstyle\sum_i \left\langle g, S^{-1}_{\mathbb{X}} y_i \right\rangle y_i, x_k \right\rangle\right\} \\ &= \{\langle \textstyle\sum_i \langle g, \tilde{y}_i \rangle y_i, x_k \rangle\} \\ &= \{\langle g, x_k \rangle\} \in ran(T^*_{\mathbb{X}}). \end{aligned}$$

Thus if $T^*_{\mathbb{X}} T_{\mathbb{Y}}(\{c_k\}) = 0$, then $\{\langle g, x_i \rangle\} = 0$. So $g = S_{\mathbb{X}} f = 0$. Since $S_{\mathbb{X}}$ is injective, $f = 0$. So $c_k = \langle f, x_k \rangle = 0$. this proves that the operator is injective. Since $\mathbb{Y}$ is a frame, then $ran(T_{\mathbb{Y}}) = span\{y_k\}$. Any $f \in ran(T_{\mathbb{Y}})$ can be written as $f = T_{\mathbb{Y}}(\{c_k\})$ for some $\{c_k\} \in l_2(\mathbb{N})$. But then $T^*_{\mathbb{X}} f = T^*_{\mathbb{X}} T_{\mathbb{Y}}(\{c_k\})$. This implies that the range of $T^*_{\mathbb{X}}$ equals the range of $T^*_{\mathbb{X}} T_{\mathbb{Y}}$. This implies surjectivity. ∎

Remark 2.7. Note that this lemma remains true even if the frames $\mathbb{Y}$ and $\mathbb{X}$ are non Parseval.

Authors in [9] have provided conditions on a frame under which a frame can be decomposed into two frames. Their proof follows immediately from the lemma just proved. It's given here for its relevance.

Proposition 2.6. *Let $\mathbb{X} = \{x_k\}_k$ be a frame for $\mathbb{H}$. Let $\{m_k\}$ and $\{n_k\}$ be any two increasing sequences such that $\{m_k\} \bigcup \{n_k\} = \mathbb{N}$. Let $\mathbb{X}_1 = \{x_{m_k}\}$ be a frame for $\mathbb{H}$. Then $\mathbb{X}_2 = \{x_{n_k}\}$ is a frame for $\mathbb{H}$ iff there exists a bounded linear operator L from $l_2(\mathbb{N}) \to l_2(\mathbb{N})$ such that $L(\{\langle x_{m_k}, f\rangle\}) = \{\langle x_{n_k}, f\rangle\}$.*

Proof. If $\mathbb{X}_2$ is a frame, then the operator $L = T^*_{\mathbb{X}_2} T_{\mathbb{X}_1}$ with the help of the lemma 2.6 satisfies the condition. If there exists such a bounded operator L then the sequence $\mathbb{X}_2$ satisfies the frame inequalities as in [9]. Being a subsequence the upper inequality is obvious and the lower frame inequality follows from

$$\sum |\langle x_{n_k}, f\rangle|^2 = ||L(\{x_{m_k}, f\})||^2 \leq ||L||^2 \sum |\langle x_{m_k}, f\rangle|^2.$$

So we have

$$\sum |\langle x_{m_k}, f\rangle|^2 \geq \frac{\sum |\langle x_{n_k}, f\rangle|^2}{||L||^2} \geq \frac{A}{||L||^2}||f||^2.$$

This implies that $\mathbb{X}_2$ is a frame too. □

3. Main Results

New frames for a Hilbert space $\mathbb{H}$ can be designed from old ones with the help of the associated linear operators. Let $L : \mathbb{H} \to \mathbb{H}$ be bounded operator and let $\mathbb{X}$ be a frame for $\mathbb{H}$, with frame operator $S_{\mathbb{X}}$ and frame bounds $A \leq B$. Let $L(\mathbb{X}) := \{L(x_k)\}_k$. Obeidat et. el. in [10] have given conditions under which the sequence $L(\mathbb{X})$ is a frame for $\mathbb{H}$. Authors in [10, 11] have also studied conditions under which the sequences $\mathbb{X} + L(\mathbb{X})$ and $L_1(\mathbb{X}) + L_2(\mathbb{Y})$ are frames for the space $\mathbb{H}$, where L_1 and L_2 are bounded linear operators on $\mathbb{H}$. We include the orthogonality and state a condition under which such sums are frames. We study these with the aid of the analysis, synthesis and frame operators.

If $\mathbb{X}$ happens to be a frame, then its associated analysis, synthesis and frame operators are given by the following lemma.

Lemma 3.1. *The analysis, synthesis and frame operators for the frame $L(\mathbb{X})$ are given by $T^*_{\mathbb{X}} L^*, LT_{\mathbb{X}},$ and $LS_{\mathbb{X}}L$ respectively.*

Proof. The following simple calculation establishes this.

$$T^*_{L(\mathbb{X})} f \;=\; \{\langle f, L(x_i)\rangle\} \;=\; \{\langle L^* f, x_i\rangle\} \;=\; T^*_{\mathbb{X}} L^* f.$$

$$T_{L(\mathbb{X})}\{c_k\} = \sum_k c_k L(x_k) = \sum_k L(c_k x_k) = LT_{\mathbb{X}}\{c_k\}.$$

$$S_{L(\mathbb{X})}f = \sum_k \langle f, L(x_k)\rangle L(x_k) = L\sum_k \langle L^*f, x_k\rangle x_k = LS_{\mathbb{X}}L^*f.$$

□

The conditions under which $L(\mathbb{X})$ is a frame are given in [10, 11]. Since the range of analysis operator is closed, and the analysis operator is injective, $T_{\mathbb{X}}^*L^*$ is injective and $LT_{\mathbb{X}}$ is surjective. So it follows that $L(\mathbb{X})$ is a frame iff L is surjective as in [10]. It follows that L is surjective (L^* is injective). However, the sequences $L(\mathbb{X})$ and $L^*(\mathbb{X})$ are both frames iff the operator L is invertible. Another important result shown in [10, 11] is when the sequence $\mathbb{X} + L(\mathbb{X})$ is frame. A simple calculation proves the following lemma [10, 11].

Lemma 3.2. *The analysis, synthesis and frame operators for the frame* $\mathbb{X} + L(\mathbb{X})$ *are given by* $T_{\mathbb{X}}^*(I + L^*), (I + L)T_{\mathbb{X}},$ *and* $(I + L)S_{\mathbb{X}}(I + L^*)$ *respectively.*

It turns out that the sequence $\mathbb{X} + L(\mathbb{X})$ is a frame iff the operator $I + L$ is surjective. If we simply take $I + L > 0$, which is satisfied if we take $L > 0$, a positive operator, then the sequence $\mathbb{X}+L(\mathbb{X})$ provides frames for $\mathbb{H}$. An extra condition yields a Riesz bases for the Hilbert space $\mathbb{H}$. The following lemma is taken from [10].

Lemma 3.3. *Let* $L : \mathbb{H} \to \mathbb{H}$ *be a bounded operator and let* $\mathbb{X}$ *be a Riesz basis for* $\mathbb{H}$*, with operators,* $T_{\mathbb{X}}^*, T_{\mathbb{X}}, S_{\mathbb{X}}$ *as analysis, synthesis and frame operators with frame bounds* $A \leq B$*. Then* $L(\mathbb{X})$ *is a Riesz basis for* $\mathbb{H}$ *iff* L *is invertible on* $\mathbb{H}$*.*

Proof. Since $\mathbb{X}$ is a Riesz basis, $T_{\mathbb{X}}^*$ is an invertible operator. If L is invertible, then the operator $T_{\mathbb{X}}^*L^*$ is invertible too. But this is the analysis operator for the sequence $L(\mathbb{X})$. Hence the sequence $L(\mathbb{X})$ is a Riesz basis. If $L(\mathbb{X})$ is a Riesz basis, then the analysis operator $T_{\mathbb{X}}^*L^*$ is invertible. This implies that L^* is invertible. So L is invertible. □

For the sum $L_1(\mathbb{X}) + L_2(\mathbb{Y})$ to be a Riesz bases, we have the following result from [10].

Proposition 3.1. *Let* $\mathbb{X}$ *and* $\mathbb{Y}$ *be two Bessel sequences in* $\mathbb{H}$ *with analysis operators* $T_{\mathbb{X}}^*$*,* $T_{\mathbb{Y}}^*$ *and frame operators* $S_{\mathbb{X}}, S_{\mathbb{Y}}$ *respectively. Let* $L_1, L_2 : \mathbb{H} \to \mathbb{H}$ *be bounded linear operators. Then the following are equivalent.*
(1) $L_1(\mathbb{X}) + L_2(\mathbb{Y})$ *is a Riesz basis for* $\mathbb{H}$*.*
(2) $T_{\mathbb{X}}^*L_1^* + T_{\mathbb{Y}}^*L_2^*$ *is an invertible operator on* $\mathbb{H}$*.*

Remark 3.4. Authors in [11] (see Proposition 3.1 in [11]) mention that condition $S = L_1S_{\mathbb{X}}L_1 + L_2S_{\mathbb{Y}}L_2 + L_1T_1^*T_2L_2^* + L_2T_2^*T_1L_1^* > 0$ is equivalent to $L_1(\mathbb{X}) + L_2(\mathbb{Y})$ being a frame. Since the operator in condition (2) in the above Proposition 3.1 is an analysis operator for $L_1(\mathbb{X}) + L_2(\mathbb{Y})$, its invertibility implies that $L(\mathbb{X}) + L(\mathbb{Y})$ is a Riesz bases. This is given in [10] (Proposition 2.12 in [10]).

Finally, we provide a sufficient condition under which the sum $L_1(\mathbb{X}) + L_2(\mathbb{Y})$ is a frame.

Theorem 3.5. *Let $\mathbb{X}$ and $\mathbb{Y}$ be two Bessel sequences such that the frame operator $S_{\mathbb{X},\mathbb{Y}}$ is a zero operator. Let $L_1, L_2 : \mathbb{H} \to \mathbb{H}$ be bounded linear operators. Then the following are equivalent.*
(1) $L_1(\mathbb{X}) + L_2(\mathbb{Y})$ is a frame for $\mathbb{H}$.
(2) $L_1 S_{\mathbb{X}} L_1^* + L_2 S_{\mathbb{Y}} L_2^*$ is an invertible operator on $\mathbb{H}$.

Proof. Since the frame operators $S_{\mathbb{X},\mathbb{Y}} = 0$, i.e. the frames $\mathbb{X}$ and $\mathbb{Y}$ are orthogonal frames, we have $T_{\mathbb{Y}} T_{\mathbb{X}}^* = 0 = T_{\mathbb{X}} T_{\mathbb{Y}}^*$. The analysis operator for the sequence $L_1(\mathbb{X}) + L_2(\mathbb{Y})$ is $T_{\mathbb{X}}^* L_1^* + T_{\mathbb{Y}}^* L_2^*$ and the frame operator $S_{L_1(\mathbb{X})+L_2(\mathbb{Y})}$ is

$$\begin{aligned} S_{L_1(\mathbb{X})+L_2(\mathbb{Y})} &= (T_{\mathbb{X}}^* L_1^* + T_{\mathbb{Y}}^* L_2^*)^* (T_{\mathbb{X}}^* L_1^* + T_{\mathbb{Y}}^* L_2^*) \\ &= (L_1 T_{\mathbb{X}} + L_2 T_{\mathbb{Y}})(T_{\mathbb{X}}^* L_1^* + T_{\mathbb{Y}}^* L_2^*) \\ &= L_1 T_{\mathbb{X}} T_{\mathbb{X}}^* L_1^* + L_1 T_{\mathbb{X}} T_{\mathbb{Y}}^* L_2^* + L_2 T_{\mathbb{Y}} T_{\mathbb{X}}^* L_1^* + L_2 T_{\mathbb{Y}} T_{\mathbb{Y}}^* L_2^* \\ &= L_1 S_{\mathbb{X}} L_1^* + L_2 S_{\mathbb{Y}} L_2^*. \end{aligned}$$

(1) $\Rightarrow$ (2). Let $L_1(\mathbb{X}) + L_2(\mathbb{Y})$ be a frame. Then its frame operator in an invertible operator from Proposition 2.4. So (2) follows.
(2) $\Rightarrow$ (1). Let the operator in (2) be invertible. Above calculation shows that it is the frame operator for the sequence $L_1(\mathbb{X}) + L_2(\mathbb{Y})$. The invertibility of the frame operator implies that the sequence $L_1(\mathbb{X}) + L_2(\mathbb{Y})$ is a frame from Proposition 2.4. □

As an important consequence of Lemma 3.1, we have the following theorem for a pair of orthogonal frames.

Theorem 3.6. *Let $\mathbb{X}$ and $\mathbb{Y}$ be orthogonal frames for $\mathbb{H}$ and let L be a bounded linear operator on $\mathbb{H}$. Then the frames $L(\mathbb{X})$ and $L(\mathbb{Y})$ are orthogonal too.*

Remark 3.7. For the sequence $L(\mathbb{X})$ to be a frame, the operator L needs to be a surjective operator [11] and hence it is assumed to be a surjective operator in the Theorem 3.6.

Proof. From Lemma 3.1, the synthesis operators for $L(\mathbb{X})$ and $L(\mathbb{Y})$ are $LT_{\mathbb{X}}$ and $LT_{\mathbb{Y}}$ respectively. It follows that the frame operator $S_{L(\mathbb{X}),L(\mathbb{Y})}$ is given by

$$S_{L(\mathbb{X}),L(\mathbb{Y})} = (LT_{\mathbb{X}})(LT_{\mathbb{Y}})^* = LT_{\mathbb{X}} T_{\mathbb{Y}}^* L^* = L0L^* = 0.$$

□

Since the Gram matrix of $L(\mathbb{X})$ is $G_{L(\mathbb{X})} = (LT_{\mathbb{X}})^*(LT_{\mathbb{X}}) = T_{\mathbb{X}}^* L^* LT_{\mathbb{X}}$, the following corollary is immediate.

Corollary 3.8. *$G_{\mathbb{X}} = G_{L(\mathbb{X})}$ if L is unitary.*

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