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THE LAW OF THE ITERATED LOGARITHM IN VARIOUS CONTEXTS

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ABSTRACT. In this paper, we first discuss the origin of the law of the iterated logarithm and then focus on the law of the iterated logarithm in various contexts. Finally, we prove a law of the iterated logarithm for independent random variables.

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1. Origin and various contexts of the law of the iterated logarithm

The first law of the iterated logarithm (LIL) was introduced in probability theory in attempts to perfect Borel's theorem on normal numbers. Precisely, the first LIL was introduced to obtain the exact rate of convergence in Borel's theorem. We first define normal numbers:

Definition 1.1 (Normal Numbers)**.** For a real number $t \in [0, 1]$ consider its binary expansion given by $t = \sum_{i=1}^{\infty} 2^{-i} c_i$ where $c_i \in \{0, 1\}$. Let $N_n(t)$ denote the number of 1's in the first n places of the binary expansion of t. Then $\lim_{n\to\infty} \frac{N_n(t)}{n}$ is the relative frequency of the digit 1 in the binary expansion of t, and t is simply normal if $\displaystyle\lim_{n\to\infty} \frac{N_n(t)}{n} = \frac{1}{2}$.

We recall the classical theorem of Borel.

Theorem 1 (Borel)**.** *If $N_n(t)$ denote the number of occurrences of the digit 1 in the first $n-$places of the binary expansion of a number $t \in [0, 1)$, then $\displaystyle\lim_{n\to\infty} \frac{N_n(t)}{n} = \frac{1}{2}$ for a.e. t in Lebesgue measure.*

Hausdorff (1913) and Hardy-Littlewood (1914) attempted to obtain the size of the deviation $N_n(t) - n/2$ in the above theorem. But A. Khintchine (1924) obtained the definite answer to the size of the deviation as given by following theorem.

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Theorem 2 (Khintchine, 1924)**.** *If $N_n(t)$ denote the number of occurrences of the digit 1 in the first $n-$places of the binary expansion of a number $t \in [0,1)$, then for almost every $t \in [0,1)$, we have*

$$\limsup_{n\to\infty} \frac{N_n(t) - \frac{n}{2}}{\sqrt{\frac{1}{2} n \log\log n}} = 1.$$

This result is known as Khintchine's law of the iterated logarithm and this is the first law of the iterated logarithm. Because of the term log log, the result is popularly known as law of iteration. A few years later, the result of Khintchine was generalized by N. Kolmogorov to a wide class of sequences of independent random variables which in the words of K. L. Chung "is a crowning achievement in the classical probability theory". We recall the classical result of Kolmogorov.

Theorem 3 (N. Kolmogorov, 1929)**.** *Let $S_m = \sum_{k=1}^{m} X_k$ where $\{X_k\}$ is a sequence of real valued independent random variables. Let s_m be the variance of S_m. Suppose $s_m \to \infty$ and $|X_m|^2 \leq \dfrac{K_m s_m^2}{\log\log(e^e + s_m^2)}$ for some sequence of constants $K_m \to 0$. Then, almost surely,*

$$\limsup_{m\to\infty} \frac{S_m(t)}{\sqrt{2 s_m \log\log s_m^2}} = 1.$$

Kolmogorov's LIL provides the size of oscillation of partial sum of independent random variable from its expected mean and the size is approximated in terms of standard deviation. Over the years people have made many efforts to obtain an analogue of Kolmogorov's LIL in various settings in analysis. On the probability side, LIL of Kolmogorov has been extended in many directions with applications in different fields. Readers are refereed to survey article by N. Bingham [5] which has more than 400 references on the law of the iterated logarithm. We discuss various LIL results in analysis which were obtained to cope up with Kolmogorov's LIL. We mainly focus on the LIL of lacunary trigonometric series and dyadic martingales and then obtain a LIL for independent random variables. We add that the first LIL in analysis was obtained in the setting of lacunary trigonometric series.

Definition 1.2 (Lacunary Series)**.** A real trigonometric series with the partial sums $S_m(\theta) = \sum_{k=1}^{m}(a_k \cos n_k\theta + b_k \sin n_k\theta)$ which has $\frac{n_{k+1}}{n_k} > q > 1$ is called $q-$ lacunary series.

In the definition, the condition $\frac{n_{k+1}}{n_k} > q > 1$ is called gap condition (series is also termed as gap series) which states that the sequence $\{n_k\}$ increases at least as rapidly as a geometric progression whose common ratio is bigger than 1. Lacunary series exhibit many of the properties of partial sums of independent random variables. In the modern probability theory, lacunary series are called 'weakly dependent' random variables. The law of the iterated logarithm in the setting of lacunary series was first given by Salem and Zygmund. This result of Salem and Zygmund is the first law of the iterated logarithm in analysis [8].

Theorem 4 (R. Salem and A. Zygmund, 1950)**.** *Suppose that S_m is a $q-$ lacunary series and the n_k are positive integers. Set $B_m^2 = \frac{1}{2}\sum_{k=1}^{m}(|a_k|^2 + |b_k|^2)$ and $M_m = \max_{1\le k\le m}(|a_k|^2 + |b_k|^2)^{\frac{1}{2}}$. Suppose also that $B_m \longrightarrow \infty$ as $m \longrightarrow \infty$ and S_m satisfies the Kolmogorov-type condition: $M_m^2 \le K_m \dfrac{B_m^2}{\log\log(e^e + B_m^2)}$ for some sequence of numbers $K_m \downarrow 0$. Then*

$$\limsup_{m\to\infty} \frac{S_m(\theta)}{\sqrt{2B_m^2 \log\log B_m}} \le 1$$

for almost every $\theta \in T$, unit circle.

So the theorem gives us the upper bound for the size of oscillation of partial sums from its expected mean and the order of the size depends on the size of standard deviation. Salem and Zygmund assumed n_k to be positive integers and they only obtained the upper bound.

Later, M. Wiess gave the complete analogue of Kolmogorov's LIL in this setting. This result was the part of her Ph.D. thesis.

Theorem 5 (M.Weiss, 1959)**.** *Suppose $S_m(\theta) = \sum_{k=1}^{m}(a_k \cos n_k\theta + b_k \sin n_k\theta)$ is a $q-$ lacunary series. Set $B_m = \left(\frac{1}{2}\sum_{k=1}^{m}(|a_k|^2 + |b_k|^2)\right)^{\frac{1}{2}}$ and $M_m = \max_{1\le k\le m}(|a_k|^2 + |b_k|^2)^{\frac{1}{2}}$. Suppose also that $B_m \to \infty$ as $m \to \infty$ and S_m satisfies the Kolmogorov-type condition: $M_m^2 \le K_m \dfrac{B_m^2}{\log\log(e^e + B_m^2)}$ for some sequence of numbers $K_m \downarrow 0$. Then*

$$\limsup_{m\to\infty} \frac{S_m(\theta)}{\sqrt{2B_m^2 \log\log B_m}} = 1$$

for almost every θ in the unit circle.

Next, we define Rademacher functions:

Definition 1.3 (Rademacher functions)**.** Rademacher functions $r_j(t)$ are defined by

$$r_j(t) = \mathrm{sgn}(\sin(2^j\pi t)), j = 1, 2, 3, \cdots \quad \text{for} \quad t \in [0, 1]$$

where sgn is defined as

$$\mathrm{sgn(t)} = \begin{cases} 1, & \text{if } t \ge 0; \\ -1, & \text{if } t < 0. \end{cases}$$

Now we discuss another law of the iterated logarithm introduced by Salem and Zygmund. In this LIL, they considered tail sums of the lacunary series instead of n^{th} partial sums.

Theorem 6 (R. Salem and A. Zygmund, 1950)**.** *Suppose a lacunary series* $\tilde{S}_N(\theta) = \sum_{k=N}^{\infty}(a_k \cos n_k\theta + b_k \sin n_k\theta)$ *where* $c_k^2 = a_k^2 + b_k^2$ *satisfies* $\sum_{k=1}^{\infty} c_k^2 < \infty$*. Define* $\tilde{B}_N = \left(\frac{1}{2}\sum_{k=N}^{\infty} c_k^2\right)^{\frac{1}{2}}$ *and* $\tilde{M}_N = \max_{k \geq N} |c_k|$*. Suppose that* $\tilde{M}_N^2 \leq K_N \left(\dfrac{\tilde{B}_N^2}{\log\log \frac{1}{\tilde{B}_N}}\right)$ *for some sequence of numbers* $K_N \downarrow 0$ *as* $N \to \infty$*. Then*

$$\limsup_{N\to\infty} \frac{\tilde{S}_N(\theta)}{\sqrt{2\tilde{B}_N^2 \log\log \frac{1}{\tilde{B}_N}}} \leq 1$$

for almost every θ *in the unit circle.*

This result is popularly known as tail law of the iterated logarithm. We remark that the condition $\sum_{k=1}^{\infty} c_k^2 < \infty$ says that the given lacunary series converges a.e. and $\tilde{S}_N(\theta) = \sum_{k=1}^{\infty}(a_k \cos n_k\theta + b_k \sin n_k\theta) - \sum_{k=1}^{N-1}(a_k \cos n_k\theta + b_k \sin n_k\theta)$. This shows that the tail LIL gives the rate of convergence of partial sums of lacunary series to its limit function. Furthermore, the rate of convergence depends upon the standard deviation of the tail sums.
In 2012 S. Ghimire and C.N. Moore [10] obtained the lower bound of above tail law of the iterated logarithm.

Theorem 7 (Ghimire and Moore, 2012)**.** *Let* $S_m(x) = \sum_{k=1}^{m} a_k \cos(2\pi n_k x)$ *be a partial sum of a lacunary series where* $\frac{n_{k+1}}{n_k} \geq q > 1$ *and* $\sum_{k=1}^{\infty} a_k^2 < \infty$*. Assume that* $\max_{k\geq N} a_k^2 = o\left(\dfrac{\frac{1}{2}\sum_{k=N}^{\infty} a_k^2}{\log\log \frac{1}{\sqrt{\frac{1}{2}\sum_{k=N}^{\infty} a_k^2}}}\right)$*. Then for a.e.* x*,*

$$\limsup_{n\longrightarrow\infty} \frac{|\sum_{k=n}^{\infty} a_k \cos(2\pi n_k x)|}{\sqrt{2 \quad \frac{1}{2}\sum_{k=n}^{\infty} a_k^2 \quad \log\log \frac{1}{\sqrt{\frac{1}{2}\sum_{k=n}^{\infty} a_k^2}}}} \geq 1.$$

Next, we discuss LIL in the context of dyadic martingale. A dyadic subinterval of the unit interval $[0,1)$ is an interval of the form $[\frac{j}{2^n}, \frac{j+1}{2^n})$ where $n = 0, 1, 2\cdots$ and $j = 0, 1, \cdots 2^n - 1$.

Definition 1.4 (Dyadic martingale)**.** A dyadic martingale is a sequence of integrable functions, $\{f_n\}_{n=0}^{\infty}$ $f_n : [0,1) \to \mathbb{R}$ such that,

(i) for every n, f_n is $\mathfrak{F}_n-$ measurable where $\mathfrak{F}_n$ is the $\sigma-$algebra generated by dyadic intervals of the form $[\frac{j}{2^n}, \frac{j+1}{2^n})$, $j \in \{0, 1, 2, \cdots 2^n - 1\}$.
(ii) conditional expectation $E(f_{n+1}|\mathfrak{F}_n) = f_n$, where $E(f_{n+1}|\mathfrak{F}_n)(x) = \frac{1}{|Q_n|}\int_{Q_n} f_{n+1}(y)dy$, $|Q_n| = \frac{1}{2^n}, x \in Q_n$.

Definition 1.5. For a dyadic martingale, $\{f_n\}_{n=0}^{\infty}$, we define,

(i) increments: $d_k = f_k - f_{k-1}$. So $f_n(x) = \sum_{k=1}^{n} d_k(x) + f_0$.
(ii) quadratic characteristics or square function: $S_n^2 f(x) = \sum_{k=1}^{n} d_k^2(x)$.

(iii) limit function: $S^2f(x) = \lim\limits_{n\to\infty} S_n^2 f(x) = \sum_{k=1}^{\infty} d_k^2(x)$.

Burkholder and Gundy [2] proved

$$\{x : Sf(x) < \infty\} \overset{a.s}{=} \{x : \lim f_n \quad \text{exists}\}$$

where $\overset{a.s}{=}$ means the sets are equal upto a set of measure zero. From this result, we observe that dyadic martingales $\{f_n\}$ behave asymptotically well on the set $\{x : Sf(x) < \infty\}$. But what can be said about the asymptotic behavior of dyadic martingales on the complement of the given set? Its behavior is quit pathological on the set $\{x : Sf(x) = \infty\}$; in particular it is unbounded a.e. on this set. But it is possible to obtain the size of growth of $|f_n|$ on the set $\{x : Sf(x) = \infty\}$? The rate of growth of $|f_n|$ on $\{x : Sf(x) = \infty\}$ is precisely given by the martingale analogue of Kolmogorov's law of the iterated logarithm.

Theorem 8 (W. Stout, 1970)**.** *If $\{f_n\}_{n=0}^{\infty}$ is a dyadic martingale on $[0,1)$ then,*

$$\limsup_{n\to\infty} \frac{|f_n(x)|}{S_nf(x)\sqrt{2\log\log S_nf(x)}} \leq 1$$

almost everywhere on the set where $\{f_n\}$ is unbounded.

He also obtained the lower bound with some additional conditions.

Theorem 9 (W. Stout, 1970)**.** *With some control of the increments, i.e., $|d_n|^2 \leq K_n \dfrac{S_n^2 f}{\log\log(e^e + S_n^2 f)}$ almost everywhere on $S(f) = \infty$ for some sequence of constants $K_n \downarrow 0$, then $\limsup\limits_{n\to\infty} \dfrac{|f_n(x)|}{\sqrt{2S_n^2 f(x)\log\log S_nf(x)}} \geq 1$ almost everywhere on $\{x : Sf(x) = \infty\}$.*

2. **Law of the iterated logarithm for random variables**

Here, we obtain an upper bound in a law of the iterated logarithm for tail sums of weighted averages of independent random variables. This is not the first time that a tail LIL for independent random variables has been introduced, see [12], but our approach is different. We state our main result:

Theorem 10. *If $\{X_j; j \geq 1\}$ are independent and identically distributed symmetric random variables with $E(X_j^2) = 1$, $-1 \leq X_j \leq 1$, and $\{a_j; j \geq 1\}$ are real constants satisfying $\sum_{j=1}^{\infty} a_j^2 < \infty$, then for a.e. t,*

$$\limsup_{n\to\infty} \frac{\sum_{j=n}^{\infty} a_j X_j(t)}{\sqrt{2\sum_{j=n}^{\infty} a_j^2 \log\log \frac{1}{\sum_{j=n}^{\infty} a_j^2}}} \leq 1.$$

First we state some lemmas which will be used in the course of the proof.

Lemma 11 (Borel-Cantelli). *If $\{A_n\}$ is a sequence of events and $\sum_{n=1}^{\infty} P(A_n) < \infty$, then $P(\{A_n \ \ i.o.\}) = 0$.*

For the proof, see [3]

Lemma 12 (Borel-Cantelli, General version). *If $\{A_n\}$ is a sequence of independent events and $\sum_{n=1}^{\infty} P(A_n) = \infty$, then $P(\{A_n \ \ i.o.\}) = 1$.*

For the proof, see [3]

Lemma 13. *If X_i are independent random variables with the property $E(X_i) = 0$, then $S_n = \sum_{i=1}^{n} X_i$ is a martingale and S_n^2 is a submartingale.*

For the proof, see [3]

Lemma 14. *Let $\{(X_n, \mathfrak{F}_n)\}$ be a submartingale and let ϕ be an increasing convex function defined on $\mathbb{R}$. If $\phi(x)$ is integrable for every n, then $\{(\phi(X_n), \mathfrak{F}_n)\}$ is also a submartingale.*

For the proof, see [3]

Theorem 15 (Doob's Maximal Inequality). *If (X_n, β_n) is a submartingale, then for any $M > 0$,*

$$P\left(\max_{1\le k\le M} X_k \ge M\right) \le \frac{1}{M} E(X_n^+) \equiv \frac{1}{M} E(\max(X_n, 0)).$$

Theorem 16 (Hoeffding). *Let $Y_1, Y_2, \dots, Y_n$ be independent random variables with zero mean and bounded ranges: $a_i \le Y_i \le b_i, 1 \le i \le n$. Then for each $\eta > 0$,*

$$P\left(|\textstyle\sum_{i=1}^{n} Y_i| \ge \eta\right) \le 2\exp\left(\frac{-2\eta^2}{\sum_{i=1}^{n}(b_i - a_i)^2}\right).$$

Now we prove exponential estimates for partial and tail sums of random variables which will be used in our main results.

Lemma 17. *If $\{X_i; i \ge 1\}$ are independent and identically distributed symmetric random variables with $E(X_i) = 0$, $E(X_i^2) = 1$, $-1 \le X_i \le 1$, $Y_i = a_i X_i$ and $\{a_n; n \ge 1\}$ are real constants satisfying $\sum_{j=1}^{\infty} a_j^2 < \infty$, then $\forall\ \eta > 0$, $\forall\ \lambda > 0$*

$$P\left(\left\{t : \sup_{m\ge 1} |\textstyle\sum_{i=1}^{m} Y_i(t)| > \lambda\right\}\right) \le 4\sqrt{2\pi}\exp\left(\frac{(-1+\eta)\lambda^2}{2\sum_{k=1}^{\infty} a_k^2}\right),$$

.

Proof. For any $\gamma > 0$ and $\lambda > 0$, we have,

$$P\left(\left\{t : \sup_{1\le m\le n} |\textstyle\sum_{i=1}^{m} Y_i(t)| > \lambda\right\}\right)$$
$$\le P\left(\left\{t : \sup_{1\le m\le n} \exp\left(\textstyle\sum_{i=1}^{m} \gamma Y_i(t)\right) > e^{\gamma\lambda}\right\}\right) + P\left(\left\{t : \sup_{1\le m\le n} \exp\left(-\textstyle\sum_{i=1}^{m} \gamma Y_i(t)\right) > e^{\gamma\lambda}\right\}\right).$$

Now by Lemma 13, $\sum_{i=1}^m Y_i(w)$ is a martingale and clearly $\exp(\gamma x)$ is convex and increasing. Hence Lemma 14 proves $\exp\left(\gamma \sum_{i=1}^m Y_i(t)\right), \exp\left(-\gamma \sum_{i=1}^m Y_i(t)\right)$ are submartingales. Then Doob's maximal inequality gives,

$$P\left(\left\{t : \sup_{1\le m\le n} |\textstyle\sum_{i=1}^m Y_i(t)| > \lambda\right\}\right) = \frac{2}{e^{\gamma\lambda}} \int_t \exp\left(\gamma|\textstyle\sum_{i=1}^n Y_i(t)|\right) dP. \tag{2.1}$$

Using Hoeffding's Theorem, we get,

$$P\left(\{t : |\textstyle\sum_{i=1}^n Y_i(t)| \ge \lambda\}\right) \le 2\exp\left(\frac{-\lambda^2}{2\sum_{i=1}^n a_i^2}\right). \tag{2.2}$$

We note that $\forall \eta > 0$ we have,

$$P\left(\left\{t : \sup_{1\le m\le n} |\textstyle\sum_{i=1}^m Y_i(t)| > \lambda\right\}\right) \le 4\sqrt{2\pi}\exp\left(\frac{(-1+2\eta)\lambda^2}{2\sum_{i=1}^\infty a_i^2}\right). \tag{2.3}$$

This can be proved by using (2.2) in $\int_0^1 e^f dP = \int_{-\infty}^\infty e^\lambda P(\{f > \lambda\})d\lambda$ (follows easily from Fubini's theorem) followed by a simple substitution. Set $E_n := \left\{t : \sup_{1\le m\le n} |\sum_{i=1}^m Y_i(t)| > \lambda\right\}$, $E = \bigcup_{n=1}^\infty E_n$. Then using the elementary result $\lim_{n\to\infty} P(E_n) = P(E)$ together with (2.3) we get

$$P\left(\left\{t : \sup_{m\ge 1} |\textstyle\sum_{i=1}^m Y_i(t)| > \lambda\right\}\right) \le 4\sqrt{2\pi}\exp\left(\frac{(-1+2\eta)\lambda^2}{2\sum_{i=1}^\infty a_i^2}\right).$$

Now by the choice of η, we get the desired result. □

Lemma 18. *If $\{X_i; i \ge 1\}$ are independent and identically distributed symmetric random variables with $E(X_i) = 0$, $E(X_i^2) = 1$, $-1 \le X_i \le 1$, $Y_i = a_i X_i$ and $\{a_n; n \ge 1\}$ are real constants satisfying $\sum_{j=1}^\infty a_j^2 < \infty$, $\forall\ \eta > 0$, $\forall\ \lambda > 0$,*

$$P\left(\left\{t : \sup_{m\ge n} |\textstyle\sum_{k=1}^\infty Y_k(t) - \sum_{k=1}^m Y_k(t)| > \lambda\right\}\right) \le 4\sqrt{2\pi}\exp\left(\frac{(-1+\eta)\lambda^2}{2\sum_{k=n+1}^\infty a_k^2}\right)$$

.

Proof. Fix n. Define,

$$b_k = \begin{cases} 0, & \text{if } k \le n; \\ a_k, & \text{if } k > n. \end{cases}$$

Using the Lemma 17 for $\{\sum_{k=1}^m b_k X_k\}$, we get

$$P\left(\left\{t : \sup_{m\ge n} |\textstyle\sum_{k=1}^m Y_k(t) - \sum_{k=1}^n Y_k(t)| > \lambda\right\}\right) \le 4\sqrt{2\pi}\exp\left(\frac{(-1+\eta)\lambda^2}{2\sum_{k=n+1}^\infty a_k^2}\right). \tag{2.4}$$

Let $N >> n$ where n is fixed. Then using Levy's inequality we get,

(2.5)

$$P\left(\left\{t : \max_{n\le m\le N} |\textstyle\sum_{i=1}^N Y_i(t) - \sum_{i=1}^m Y_i(t)| > \lambda\right\}\right) \le 2P\left(\left\{t : |\textstyle\sum_{i=1}^N Y_k(t) - \sum_{i=1}^n Y_k(t)| > \lambda\right\}\right).$$

Since $N >> n$, we have from (2.4),

$$P\left(\left\{t : |\textstyle\sum_{k=1}^N Y_k(t) - \sum_{k=1}^n Y_k(t)| > \lambda\right\}\right) \le 4\sqrt{2\pi}M\exp\left(\frac{(-1+\eta)\lambda^2}{2\sum_{k=n+1}^\infty a_k^2}\right). \tag{2.6}$$

Hence from (2.5) and (2.6) we get,

(2.7)

$$P\left(\left\{t : \sup_{N\geq m\geq n} |\sum_{k=1}^{N} Y_k(t) - \sum_{k=1}^{m} Y_k(t)| > \lambda\right\}\right) \leq 4\sqrt{2\pi}M \exp\left(\frac{(-1+\eta)\lambda^2}{2\sum_{k=n+1}^{\infty} a_k^2}\right).$$

Set $E_N := \{t : \sup\limits_{N\geq m\geq n} |\sum_{k=1}^{N} Y_k(t) - \sum_{k=1}^{m} Y_k(t)| > \lambda\}$ and $E := \bigcup_{k=1}^{\infty} E_k$. As earlier using the result $\lim\limits_{n\to\infty} P(E_n) = P(E)$ together with (2.7) we get

$$P\left(\left\{t : \sup_{m\geq n} |\sum_{k=1}^{\infty} Y_k(t) - \sum_{k=1}^{m} Y_k(t)| > \lambda\right\}\right) \leq 4\sqrt{2\pi} \exp\left(\frac{(-1+\eta)\lambda^2}{2\sum_{k=n+1}^{\infty} a_k^2}\right). \qquad \square$$

Finally we prove our main result:

Proof. Let $\theta > 1$ and $\varepsilon > 0$. Assume that $\eta << 1$ such that $(1-\eta)(1+\varepsilon)^2 > 1$. Define $n_1 \leq n_2 \leq \cdots, n_k \to \infty$ by $n_k = \min\left(n : \sum_{j=n+1}^{\infty} a_j^2 < \frac{1}{\theta^k}\right)$. Then applying Lemma 18 for stopping time n_k, we have,

$$P\left(\left\{t : \sup_{n\geq n_k} |\sum_{i=1}^{\infty} Y_i(t) - \sum_{i=1}^{n} Y_i(t)| > \lambda\right\}\right) \leq 4\sqrt{2\pi}M \exp\left(\frac{(-1+\eta)\lambda^2}{2\sum_{i=n_k+1}^{\infty} a_i^2}\right).$$

We choose $\lambda = (1+\varepsilon)\sqrt{\frac{2}{\theta^k}\log\log\theta^k}$, $\varepsilon > 0$. With this λ and using $\sum_{i=n_k+1}^{\infty} a_i^2 < \frac{1}{\theta^k}$ above inequality becomes,

$$\begin{aligned} &P\left(\left\{t : \sup_{n\geq n_k} |\sum_{i=1}^{\infty} Y_i(t) - \sum_{i=1}^{n} Y_i(t)| > (1+\varepsilon)\sqrt{\frac{2}{\theta^k}\log\log\theta^k}\right\}\right) \\ &\leq \frac{4\sqrt{2\pi}}{k^{(1-\eta)(1+\varepsilon)^2} (\log\theta)^{(1-\eta)(1+\varepsilon)^2}}. \end{aligned}$$

So,

$$\begin{aligned} &\sum_{k=1}^{\infty} P\left(\left\{t : \sup_{n\geq n_k} |\sum_{i=1}^{\infty} Y_i(t) - \sum_{i=1}^{n} Y_i(t)| > (1+\varepsilon)\sqrt{\frac{2}{\theta^k}\log\log\theta^k}\right\}\right) \\ &\leq \frac{4\sqrt{2\pi}}{(\log\theta)^{(1-\eta)(1+\varepsilon)^2}} \sum_{k=1}^{\infty} \frac{1}{k^{(1-\eta)(1+\varepsilon)^2}} \\ &< \infty. \end{aligned}$$

So by Borel-Cantelli Lemma, for a.e. t, we have,

$$\sup_{n\geq n_k} |\sum_{i=1}^{\infty} Y_i(t) - \sum_{i=1}^{n} Y_i(t)| \leq (1+\varepsilon)\sqrt{\frac{2}{\theta^k}\log\log\theta^k}$$

for sufficiently large k, say $k \geq M$, such that M depends on t. Fix t. Choose $n \geq n_M$. Then there exists $k \geq M$, such that $n_k \leq n < n_{k+1}$. We note that

$$\frac{1}{\theta^{k+1}} \leq \sum_{j=n+1}^{\infty} a_j^2 < \frac{1}{\theta^k}.$$

Using this we have,

$$\left|\sum_{i=1}^{\infty} Y_i(t) - \sum_{i=1}^{n} Y_i(t)\right| \leq \sup_{m \geq n_k} \left|\sum_{i=1}^{\infty} Y_i(t) - \sum_{i=1}^{m} Y_i(t)\right|$$
$$< (1+\varepsilon)\sqrt{\theta}\sqrt{2\sum_{j=n+1}^{\infty} a_j^2 \log\log \frac{1}{\sum_{j=n+1}^{\infty} a_j^2}}.$$

Thus for a.e. t, we have,

$$\limsup_{n\to\infty} \frac{\left|\sum_{i=1}^{\infty} Y_i(t) - \sum_{i=1}^{n} Y_i(t)\right|}{\sqrt{2\sum_{j=n+1}^{\infty} a_j^2 \log\log \frac{1}{\sum_{j=n+1}^{\infty} a_j^2}}} \leq (1+\varepsilon)\sqrt{\theta}.$$

Letting $\theta \searrow 1$ and $\varepsilon \searrow 0$, we get,

$$\limsup_{n\to\infty} \frac{\left|\sum_{i=1}^{\infty} Y_i(t) - \sum_{i=1}^{n} Y_i(t)\right|}{\sqrt{2\sum_{j=n+1}^{\infty} a_j^2 \log\log \frac{1}{\sum_{j=n+1}^{\infty} a_j^2}}} \leq 1.$$

So,

$$\limsup_{n\to\infty} \frac{\sum_{j=n}^{\infty} a_j X_j(t)}{\sqrt{2\sum_{j=n}^{\infty} a_j^2 \log\log \frac{1}{\sum_{j=n}^{\infty} a_j^2}}} \leq 1.$$

This proves our upper bound result. □

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