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# SOBOLEV GRADIENT & APPLICATION TO NONLINEAR PSEUDO-DIFFERENTIAL EQUATIONS

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**ABSTRACT.** We seek to find critical points of a functional defined by

$$E_\alpha(u) = \int_{\mathbb{T}^d} \frac{1}{2}|A^{\alpha/2}u|^2 + F(x,u)\,dx, \quad \alpha \in (0,1],$$

on an infinite dimensional space (Sobolev Space), where $A$ is a self-adjoint, uniformly elliptic operator of order 2 with suitable symmetric and smoothness conditions on the coefficients and $F(x,y) = \int_0^y f(x,z)\,dz$ has the properties that the real-valued nonlinear functional $f(x,y)$ on $\mathbb{T}^d \times \mathbb{R}$ is continuous with respect to the spatial variable $x$ and Lipschitz continuous with respect to the functional component $y$.

We first consider Sobolev gradient of $E_\alpha$ as an element of a Sobolev space $H^{\alpha\beta}, \beta \in (0,1)$, then the steepest descent (Sobolev gradient descent) equation for $E_\alpha$. Under suitable initial and periodic boundary conditions, we prove existence and uniqueness of semi-flow (a strong solution) of this equation.

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## 1. Introduction

Consider a compact domain $\mathbb{T}^d = [0,1]^d$ (or the compact manifold $\mathbb{R}^d/\mathbb{Z}^d$). Let us consider a functional of type

$$E_\alpha(u) = \int_{\mathbb{T}^d} \frac{1}{2}|A^{\alpha/2}u|^2 + F(x,u)\,dx, \quad \alpha \in (0,1] \tag{1.1}$$

where $F(x,y) = \int_0^y f(x,z)\,dz$ has the properties that for each fixed $y \in \mathbb{R}$ the map $x \mapsto f(x,y)$ lies in $C(\mathbb{T}^d,\mathbb{R})$ and for each fixed $x \in \mathbb{T}^d$ the map $y \mapsto f(x,y)$ lies in $C^{0,1}(\mathbb{R},\mathbb{R})$, and $A$ is a linear, self-adjoint, and uniformly elliptic operator of order 2 on $L^2(\mathbb{T}^d)$ given by

$$(Au)(x) = -\sum_{i,j=1}^{d} ((a_{ij}(x)u_{x_i}(x))_{x_j}, \quad x \in \mathbb{T}^d \tag{1.2}$$

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with the properties that $a_{ij} \equiv a_{ji} \, \forall \, i, \; j = 1, \; 2 \ldots, d$ and $a_{ij} \in C^\infty(\mathbb{T}^d)$, and

$$\theta_1 |\xi|^2 \le \sum_{i,j=1}^{d} a_{ij}(x)\xi_i \xi_j \le \theta_2 |\xi|^2 \quad \forall \, x \in \mathbb{T}^d, \; \forall \, \xi \in \mathbb{R}^d. \tag{1.3}$$

for some positive real numbers $\theta_1, \; \theta_2$.

A reason why we are interested in the functional $E_\alpha$ is to find solutions of a pseudo-differential equations satisfied by critical points of $E_\alpha$. These critical points are those $u$ in Sobolev space $H^{\alpha\beta}, \; \beta > 0$ for which

$$\nabla_{\alpha\beta} E_\alpha(u) = 0,$$

where $\nabla_{\alpha\beta} E_\alpha(u)$, an element of $H^{\alpha\beta}$, is the Sobolev gradient of $E_\alpha$ at $u$, which we will discuss later in detail. Intuitively, we can think of such gradient as a counterpart of the gradient of a function defined on $\mathbb{R}^d$ in multi-variable calculus. It is defined in abstract sense on an infinite dimensional Hilbert space, namely on $H^{\alpha\beta}$ in our case.

As a method of finding critical points of $E_\alpha$, we consider the gradient descent equation

$$\partial_t u = -\nabla_{\alpha\beta} E_\alpha(u)$$

subject to an initial condition and periodic boundary condition. Then equilibrium solutions of this initial-boundary value problem are critical points of $E_\alpha$. Even though our future goal is to find such equilibrium solutions, in this paper we will prove existence and uniqueness of global semi-flow $u(t,x), \; t \ge 0$ of this initial-boundary value problem.

The results analogous to ours for the case $\alpha = 1$ and the nonlinear functional $F(x,y)$ satisfying stronger assumptions has been proved by T. Blass, R. de la lLave and E. Valdinoci in [1]. Further, T. Blass, R. de la lLave have used this method to compute numerical solution of perturbed nonlinear problem reducing to the one with periodic boundary condition for the case $\alpha = 1, \; A = -\Delta$, and $F(x,y)$ again satisfying stronger assumptions. Also analogous results for the case $A^\alpha$ replaced by $(-\Delta)^\alpha$ have been proved by R. de la lLave and E. Valdinoci in [2]. Our results generalize some of their results. The idea of the problem we mentioned here has been proposed in [1] with some outlines.

The proofs of the main results rely on abstract semigroup theory on a Banach space, spectral theory of linear unbounded self-adjoint operator on a Hilbert space, and many other concepts from functional analysis and theory of partial differential equations and pseudo-differential equations such as Riesz Representation Theorem, Banach Fixed Point Theorem, variation of constant formula, Sobolev Embedding Theorem, Inverse Operator Theorem etc.

Now we briefly discuss how we will develop a setting of our work by introducing some tools needed for establishing our main results and giving an overview of the proofs of these results. In section 1, we will introduce Euler-Lagrange Equation for $E_\alpha$, Sobolev spaces, Sobolev gradient, expressions for Sobolev gradient, and Sobolev gradient descent equation. We will begin section 2 with stating the main result. Then we will develop proof of this result throughout section 2. In section 2.1, we will characterize Sobolev spaces and define various fractional powers of operators involving $A$ using spectral theory of closed positive unbounded self-adjoint operators on a Hilbert space with compact resolvent. In section 2.2, we will construct a contraction $C_0$-semigroups generated by $-A$ and negative of fractional powers of operators involving $A$. More precisely, a closed densely defined unbounded linear operator $B$ on a Hilbert space $H$ is an infinitesimal generator of a contraction $C_0$-semigroup of bounded linear operators $T(t),\ t \geq 0$ on $H$ if $T(0) = I,\ T(t_1 + t_2) = T(t_1) \circ T(t_2)\ \forall\, t_1,\ t_2 \geq 0,\ \lim_{h\downarrow 0} ||T(h)u - u||_H = 0\ \forall\, u \in H$, and $||T(t)||_{\mathcal{L}(H)} \leq 1\ \forall\, t \geq 0$. Such semi-group has an important property, namely for each $u_0 \in D(B) = \{u \in H : \lim_{h\downarrow 0} ||\frac{T(h)-I}{h}u - Bu||_H = 0\}$

$$T(t)u_0 \in D(B) \text{ and } \frac{dT(t)}{dt}u_0 = BT(t)u_0\ \forall\, t \geq 0.$$

That is, $u(t) = T(t)u_0$ is a solution of abstract Cauchy problem

$$\begin{cases} \frac{du}{dt} = & Bu, \quad t > 0 \\ u(0) = & u_0 \end{cases}$$

Since $B$ can not generate more than one $C_0$-semigroup, $u = T(.)u_0$ is a unique solution of above abstract Cauchy problem that starts in $D(A)$ and always remains there (for more properties of $C_0$-semigroup, see [10, 15, 14]). In section 2.3, we will discuss some regularity and boundedness properties of a contraction $C_0$-semigroup. Finally in section 2.4, we will construct variation of constant formula (mild solution) for the Sobolev gradient descent equation and a contraction map on the space of mild solutions, which is a Banach space with suitable norm. We will apply Banach Fixed Point Theorem to find a unique fixed point, which will be a unique mild solution of the gradient descent equation. Finally we will use regularity and boundedness properties of semigroup and nonlinear operator appearing in the gradient descent equation to improve regularity of the fixed point. Hence we will obtain unique solution of the gradient descent equation.

1.1. **Euler-Lagrange Equation.** The main motivational to the problem of finding critical points of $E_\alpha$ is that these critical points are solutions of the Euler-Lagrange equation of $E_\alpha$

$$A^\alpha u + f(x, u) = 0, \quad x \in \mathbb{T}^d. \tag{1.4}$$

In other words, our interest is guided by a problem of solving the pseudo-differential equation (1.4). To see (1.4) as Euler-Lagrange equation of $E_\alpha$, if $\phi \in C^\infty(\mathbb{T}^d)$ then

$$\begin{aligned}\frac{d}{dt}E_\alpha(u+\tau\phi)|_{\tau=0} =&\frac{d}{d\tau}\{\frac{1}{2}(A^{\alpha/2}(u+\tau\phi), A^{\alpha/2}(u+\tau\phi))_{L^2} + \int_{\mathbb{T}^d} F(x, u+\tau\phi)\, dx\}|_{\tau=0} \\ =&(A^\alpha u + f(x,u), \phi)_{L^2} = 0 \quad \text{(because } A^{\alpha/2} \text{ is self-adjoint)}\end{aligned}$$

implies that $A^\alpha u + f(x,u) = 0$

We will easily see in section 1.4 that (1.4) is equivalent to $\nabla_{\alpha\beta}E_\alpha(u) = 0$ for each critical point $u$ of $E_\alpha$.

1.2. **Sobolev spaces.** Following [5, 8, 11, 14, 16], first we recall definitions of some fractional order Sobolev spaces and then introduce new Sobolev spaces.

For any $s \in \mathbb{R}$, $H^s(\mathbb{T}^d) = \{u \in \mathcal{D}'(\mathbb{T}^d) : \sum_{j\in\mathbb{Z}^d}(1+|j|^2)^{s/2}\hat{u}(j)e^{ij.x} \in L^2(\mathbb{T}^d)\} = D(\Lambda^s)$, where $\Lambda^s u(x) := \sum_{j\in\mathbb{Z}^d}(1+|j|^2)^{s/2}\hat{u}(j)e^{ij.x}$ for $u \in \mathcal{D}'(\mathbb{T}^d)$, is a Hilbert space with the inner product $(u,v)_{H^s(\mathbb{T}^d)} = (\Lambda^s u, \Lambda^s v)_{L^2(\mathbb{T}^d)} = \sum_{j\in\mathbb{Z}^d}(1+|j|^2)^s\hat{u}(j)\overline{\hat{v}(j)}$ which induces the norm $||u||_{H^s(\mathbb{T}^d)} = \{\sum_{j\in\mathbb{Z}^d}(1+|j|^2)^s|\hat{u}(j)|^2\}^{1/2}$.

In particular, if $s = 0$ then $H^0(\mathbb{T}^d) = \{u \in \mathcal{D}'(\mathbb{T}^d) : \sum_{j\in\mathbb{Z}^d}\hat{u}(j)e^{ij.x} \in L^2(\mathbb{T}^d)\} = L^2(\mathbb{T}^d)$ with the inner product $(u,v)_{H^0(\mathbb{T}^d)} = (\sum_{j\in\mathbb{Z}^d}\hat{u}(j)e^{ij.x}, \sum_{k\in\mathbb{Z}^d}\hat{v}(k)e^{ik.x})_{L^2(\mathbb{T}^d)} = \sum_{j\in\mathbb{Z}^d}\hat{u}(j)\overline{\hat{v}(j)} = (u,v)_{L^2(\mathbb{T}^d)}$ and the norm $||u||_{L^2(\mathbb{T}^d)} = \{\sum_{j\in\mathbb{Z}^d}|\hat{u}(j)|^2\}^{1/2}$. Moreover, if $s \geq 0$ then $\sum_{j\in\mathbb{Z}^d}|\hat{u}(j)|^2 \leq \sum_{j\in\mathbb{Z}^d}(1+|j|^2)^s|\hat{u}(j)|^2 \Rightarrow H^s(\mathbb{T}^d) \subset L^2(\mathbb{T}^d)$.

Notice that $\Lambda^\alpha = (I-\Delta)^{\alpha/2}$ on $\mathcal{D}'(\mathbb{T}^d)$ so that $D((I-\Delta)^{\alpha/2}) = H^\alpha(\mathbb{T}^d)$. For $\beta \in [0,1]$, the interpolation method (proposition 2.2 in [5]) yields $H^{\alpha\beta}(\mathbb{T}^d) = [L^2(\mathbb{T}^d), H^\alpha(\mathbb{T}^d)]_\beta = [L^2(\mathbb{T}^d), D(\Lambda^\alpha)]_\beta = D(\Lambda^{\alpha\beta}) = D(I-\Delta)^{\alpha\beta/2}$ with norm given by graph norm $||u||_{H^{\alpha\beta}(\mathbb{T}^d)} = ||(I-\Delta)^{\alpha\beta/2}u||_{L^2(\mathbb{T}^d)} = ||\Lambda^{\alpha\beta}u||_{L^2(\mathbb{T}^d)}$ and inner product $(u,v)_{H^{\alpha\beta}(\mathbb{T}^d)} = ((I-\Delta)^{\alpha\beta/2}u, (I-\Delta)^{\alpha\beta/2}v)_{L^2(\mathbb{T}^d)} = (\Lambda^{\alpha\beta}u, \Lambda^{\alpha\beta}v)_{L^2(\mathbb{T}^d)}$.

Next we define $H^{\alpha\beta}(\mathbb{T}^d) = \{u \in L^2(\mathbb{T}^d) : (I+A^\alpha)^{\beta/2}u \in L^2(\mathbb{T}^d)\}$ with inner product given by $(u,v)_{H^{\alpha\beta}(\mathbb{T}^d)} = ((I+A^\alpha)^{\beta/2}u, (I+A^\alpha)^{\beta/2}v)_{L^2(\mathbb{T}^d)}$ which induces the norm $||u||_{H^{\alpha\beta}(\mathbb{T}^d)} = ||(I+A^\alpha)^{\beta/2}u||_{L^2(\mathbb{T}^d)}$. In section 2.1, it will be implied that the operator $A_1^\alpha := I + A^\alpha$ satisfies the following properties: $\rho(A_1^\alpha)$ contains the sector $\mathcal{S}_{0,\phi} = \{0 \neq \lambda \in \mathbb{C} : \phi \leq |\arg(\lambda)| \leq \pi, \phi \in (0,\pi/2)\}$, and $||(\lambda - A_1^\alpha)^{-1}||_{\mathcal{L}(H^{\alpha\beta}(\mathbb{T}^d))} \leq M/|\lambda|$ for $\lambda \in \mathcal{S}_{0,\phi}$ for some $M > 0$. In other words, $A_1^\alpha$ is a sectorial positive operator so that fractional power of $A_1^\alpha$ on $H^{\alpha\beta}(\mathbb{T}^d)$ can be defined (for more detail, see section 1.3 in [18]).

In section 2.1, we will show that the spaces $H^{\alpha\beta}(\mathbb{T}^d)$ given in the last two paragraphs with their respective norms are the same.

1.3. **Sobolev gradients.** Sobolev gradient or $H^{\alpha\beta}(\mathbb{T}^d)$-gradient of $E_\alpha$ at $u \in H^{\alpha\beta}(\mathbb{T}^d)$ is a unique element $g \in H^{\alpha\beta}(\mathbb{T}^d)$ such that

$$DE_\alpha(u)\eta = (g,\eta)_{H^{\alpha\beta}(\mathbb{T}^d)} = ((I+A^\alpha)^{\beta/2}g, (I+A^\alpha)^{\beta/2}\eta)_{L^2(\mathbb{T}^d)}\ \forall\, \eta \in H^{\alpha\beta}(\mathbb{T}^d),$$

where $DE_\alpha(u)$ is Fréchet derivative of $E_\alpha$ at $u$. The Riesz Representation Theorem guarantees that such a $g \in H^{\alpha\beta}(\mathbb{T}^d)$ always exists, since $DE_\alpha(u)$ is a continuous linear functional on $H^{\alpha\beta}(\mathbb{T}^d)$. We write $g = \nabla_{H^{\alpha\beta}(\mathbb{T}^d)} E_\alpha(u)$.

In particular, if $\beta = 0$ then $DE_\alpha(u)\eta = (g,\eta)_{H^0(\mathbb{T}^d)} = (g,\eta)_{L^2(\mathbb{T}^d)}\, \forall\ \eta \in L^2(\mathbb{T}^d)$ and $g = \nabla_{H^0(\mathbb{T}^d)} E_\alpha(u)$. In this case, $g$ is $L^2(\mathbb{T}^d)$-gradient of $E_\alpha$ at $u$ and is also denoted by $\nabla_{L^2(\mathbb{T}^d)} E_\alpha(u)$.

Now we establish the formulas for $H^{\alpha\beta}(\mathbb{T}^d)$-gradient and $L^2(\mathbb{T}^d)$-gradient.

**Lemma 1.1.** *For every $\eta \in C^\infty(\mathbb{T}^d)$, we have*

*i)* $DE_\alpha(u)\eta = (A^\alpha u + f(x,u), \eta)_{L^2(\mathbb{T}^d)}$,

*ii)* $DE_\alpha(u)\eta = ((I+A^\alpha)^{1-\beta}u - (I+A^\alpha)^{-\beta}(u - V_y(x,u)), \eta)_{H^{\alpha\beta}(\mathbb{T}^d)}$

*Proof.* Since

$$\lim_{||\eta||_{L^2(\mathbb{T}^d)} \to 0} \frac{|E_\alpha(u+\eta) - E_\alpha(u) - DE_\alpha(u)\eta|}{||\eta||_{L^2(\mathbb{T}^d)}} = 0,$$

it follows that

$$E_\alpha(u+\eta) = E_\alpha(u) + DE_\alpha(u)\eta + o(||\eta||^2_{L^2(\mathbb{T}^d)})$$

On the other hand

$$\begin{aligned} E_\alpha(u+\eta) &= \int_{\mathbb{T}^d} \frac{1}{2}|A^{\alpha/2}(u+\eta)|^2 + F(x,u+\eta) \\ &= \int_{\mathbb{T}^d} \frac{1}{2}|A^{\alpha/2}u + A^{\alpha/2}\eta|^2 + F(x,u) + F_y(x,u)\eta + o(|\eta|^2) \\ &= E_\alpha(u) + (\eta, A^\alpha u)_{L^2(\mathbb{T}^d)} + \int_{\mathbb{T}^d} f(x,u)\eta + o(|\eta|^2) \\ &= E_\alpha(u) + (\eta, A^\alpha u + f(x,u))_{L^2(\mathbb{T}^d)} + o(|\eta|^2) \end{aligned}$$

Combining above two expressions for $E_\alpha(u+\eta)$, we obtain part(i) of lemma.

Since $(I+A^\alpha)^{\beta/2}$ is a self-adjoint operator, we have

$$\begin{aligned} DE_\alpha(u)\eta &= (A^\alpha u + f(x,u), \eta)_{L^2(\mathbb{T}^d)} \\ &= ((I+A^\alpha)^\beta (I+A^\alpha)^{-\beta}(A^\alpha u + u - u + f(x,u), \eta)_{L^2(\mathbb{T}^d)} \\ &= ((I+A^\alpha)^{-\beta}[(I+A^\alpha)u - \{u - f(x,u)\}], \eta)_{H^{\alpha\beta}(\mathbb{T}^d)} \\ &= ((I+A^\alpha)^{1-\beta}u - (I+A^\alpha)^{-\beta}\{u - f(x,u)\}, \eta)_{H^{\alpha\beta}(\mathbb{T}^d)} \end{aligned}$$

Thus part (ii) of lemma follows. □

Since $C^\infty(\mathbb{T}^d)$ is dense in each of $L^2(\mathbb{T}^d)$ and $H^{\alpha\beta}(\mathbb{T}^d)$, lemma 1.1 shows that

$$\nabla_{L^2(\mathbb{T}^d)} E_\alpha(u) = A^\alpha u + f(x,u) \tag{1.5}$$

$$\nabla_{H^{\alpha\beta}(\mathbb{T}^d)} E_\alpha(u) = (I+A^\alpha)^{1-\beta}u - (I+A^\alpha)^{-\beta}\{u - f(x,u)\} \tag{1.6}$$

1.4. **Sobolev gradient descent equations.** As we discussed early in the section, we want to solve nonlinear pseudo-differential equation (1.4) by finding the critical points of the functional $E_\alpha$. At these critical points, we expect to minimize (or maximize) $E_\alpha$ by considering $H^{\alpha\beta}(\mathbb{T}^d)$-gradient descent equation given by

$$\partial_t u = -\nabla_{H^{\alpha\beta}(\mathbb{T}^d)} E_\alpha(u)$$

which then by part (ii) of lemma 1.1 becomes

$$\partial_t u = -(I + A^\alpha)^{1-\beta} u + (I + A^\alpha)^{-\beta}(u - f(x, u))$$

We set linear operator $\mathbf{L} := -(I + A^\alpha)^{1-\beta}$ and nonlinear operator $\mathbf{N} := (I + A^\alpha)^{-\beta}(I(.) - f(x, .))$, impose initial condition $u(0, x) = u_0(x)$ and periodic boundary condition $u(t, x + e_j) = u(t, x) \quad (t \geq 0)$ for all $e_j = (0, \ \ldots, 1 \ \ldots, 0)$, $j = 1,\ 2 \ldots d,\ x \in \mathbb{T}^d$, and write $u(t, x) = u[t](x)$ by fixing $x$ to represent $t \mapsto u[t]$ as a map from time interval into a function space . Then we get the abstract Cauchy problem

$$\begin{cases} \frac{du[t]}{dt} = \mathbf{L}(u[t]) + \mathbf{N}(u[t]) \quad \text{if} \quad t > 0 \\ u[0] = u_0 \end{cases} \tag{1.7}$$

accompanied by above mentioned periodic boundary condition.

Because of the more complicated nature of $H^{\alpha\beta}(\mathbb{T}^d)$-gradient of $E_\alpha$ than its $L^2(\mathbb{T}^d)$-gradient, it seems more natural to consider the $L^2(\mathbb{T}^d)$-gradient. But one of main advantages of considering the former one is that numerical solution in higher Fourier frequency mode of (1.7) converges to its equilibrium solution much faster as compared to its $L^2(\mathbb{T}^d)$-gradient counterpart.

We notice that equilibrium solutions of (1.7) satisfy $\nabla_{H^{\alpha\beta}(\mathbb{T}^d)} E_\alpha(u) = 0$ which is equivalent to the nonlinear pseudo-differential equation (1.4) (Recall (1.6)). Therefore, existence of unique solution $u(t, x)$ to (1.7) is a key to study such equilibrium solutions.

## 2. The Existence and Uniqueness Theorem

In this section, we prove existence and uniqueness of global solution of the initial boundary-value problem (1.7) in a Sobolev space. Such a solution is a strong solution to the problem.

For the sake of simplicity, we will write $L^2$ for $L^2(\mathbb{T}^d)$, $L^\infty$ for $L^\infty(\mathbb{T}^d)$, $H^{\alpha\beta}$ for $H^{\alpha\beta}(\mathbb{T}^d)$, $(.,.)_{L^2}$ or $(.,.)_0$ for $(.,.)_{L^2(\mathbb{T}^d)}$, and $(.,.)_{\alpha\beta}$ for $(.,.)_{H^{\alpha\beta}(\mathbb{T}^d)}$ in the rest of the paper.

**Theorem 2.1** (Main). *Let $x \mapsto f(x, y) \in C(\mathbb{T}^d, \mathbb{R})$ for every $y \in \mathbb{R}$, $y \mapsto f(x, y) \in C^{0,1}(\mathbb{R}, \mathbb{R})$ for every $x \in \mathbb{T}^d$, and $u_0 \in L^\infty$. Then there exists a unique solution $u$ in*

$C([0,\infty), L^\infty \cap H^{2\alpha\beta}) \cap C^1((0,\infty), L^\infty \cap H^{2\alpha\beta})$ *of the initial-boundary value problem* (1.7).

2.1. **Characterization of Sobolev spaces.** In this section, we basically verify the observation we made at the end of section 1.2.

By inverse operator theorem (Theorem 8.2, [11]) $(A-\lambda I)^{-1}$ is compact in $L^2$ for $\lambda \notin \sigma(A)$. By theorem 8.3, [11], there is a complete orthonormal basis $\{w_j\}_{j=1}^\infty$ for $L^2$ of eigenfunctions of $A$ corresponding to the eigenvalues $\{\lambda_j\}_{j=1}^\infty$ with $|\lambda_j| \to \infty$ as $j \to \infty$. Moreover, each $w_j$ is smooth on $\mathbb{T}^d$ and $\sigma(A) = \sigma_p(A)$ (point spectrum).

Since $A$ is symmetric and positive definite, all eigenvalues of $A$ are real, positive and $0 < \lambda_1 \leq \lambda_2 \leq ..... \leq \lambda_j \leq ....$ (counting multiplicity) with $\lambda_j \to \infty$ as $j \to \infty$. So we can write $u \in L^2$ as Fourier series $u = \sum_j (u, w_j)_{L^2(\mathbb{T}^d)} w_j$.

Now onward we write $(u, w_j)_{L^2} = \hat{u}_j$. Then Parseval's identity ($L^2$-inner product) $(u,v)_{L^2} = \sum_j \hat{u}_j \overline{\hat{v}_j}$ yields $||u||_{L^2}^2 = (u,u)_{L^2} = \sum_j |\hat{u}_j|^2$.

Thus we can characterize $L^2$ as $L^2 = \{u \in \mathcal{D}'(\Omega) : \sum_j |\hat{u}_j|^2 < \infty\}$.

By proposition 10.3 in [11], for any complex $z$

$$A^z u = \sum_j \lambda_j^z \hat{u}_j w_j \tag{2.1}$$

(a spectral integral in discrete form). Moreover $w_j$ are eigenfunctions of $A^z$ with the corresponding eigenvalues $\lambda_j^z$.

We will discuss in section 2.2 that $-A$ generates a $C_0$-semigroup on $L^2$ and define $A^\alpha$ on $L^2$ in terms of $C_0$-semigroup generated by $-A$. Also we will prove that $-A^\alpha$ generates $C_0$-semigroup on $L^2$. Then $-(I + A^\alpha)$ will be infinitesimal generator of a $C_0$-semigroup on $L^2$ ([17]). So $(I + A^\alpha)$ which is negative of $-(I + A^\alpha)$ is a positive operator on $L^2$ from chapter 1 of [6]. In similar fashion as $A^\alpha$, we can define $(I+A^\alpha)^\beta$ on $L^2$ in terms of $C_0$-semigroup generated by $-(I + A^\alpha)$. But at this moment, we apply (2.1) to define an operator $(I + A^\alpha)^{1/2} : L^2 \to L^2$ by

$$D((I + A^\alpha)^{1/2}) = \{u \in L^2 : \sum_j (1 + \lambda_j^\alpha)|\hat{u}_j|^2 < \infty\},$$

$$(I + A^\alpha)^{1/2} u = \sum_j (1 + \lambda_j^\alpha)^{1/2} \hat{u}_j w_j$$

Here $D((I + A^\alpha)^{1/2})$ as a subspace of $L^2$ with inner product $(u,v)_{D((I+A^\alpha)^{1/2})} = ((I + A^\alpha)^{1/2}u, (I + A^\alpha)^{1/2}v)_{L^2} = \sum_j (1 + \lambda_j^\alpha)\hat{u}_j w_j$ that induces the graph norm $||u||_{D((I+A^\alpha)^{1/2})} = ||(I + A^\alpha)^{1/2}u||_{L^2} = \sum_j (1 + \lambda_j^\alpha)|\hat{u}_j|^2$ is equal to $D((I + (-\Delta)^\alpha)^{1/2})$ with analogous inner product and norm because two inner products induce the equivalent norms (see Chapter 1 in [11]). On the other hand, $\sum_{j\in\mathbb{Z}^d}(1 + |j|^{2\alpha})|\hat{u}(j)|^2 < \infty \Leftrightarrow \sum_{j\in\mathbb{Z}^d}(1 + |j|^2)^\alpha|\hat{u}(j)|^2 < \infty$ implies that $D((I + (-\Delta)^\alpha)^{1/2}) = D((I - \Delta)^{\alpha/2})$ with the respective inner products that induce the graph norms. Thus we have

$D((I+A^\alpha)^{1/2}) = H^\alpha$. For any $\beta \in [0,1]$, interpolation method yields $H^{\alpha\beta} = [H^0, H^\alpha]_\beta = H^{\alpha\beta+0(1-\beta)} = H^{\alpha\beta} = [L^2, D((I+A^\alpha)^{1/2})]_\beta = D((I+A^\alpha)^{\beta/2})$ with inner product $(u,v)_{D((I+A^\alpha)^{\beta/2})} = ((I+A^\alpha)^{\beta/2}u, (I+A^\alpha)^{\beta/2}v)_{L^2}$ that induces the graph norm $||u||_{D((I+A^\alpha)^{\beta/2})} = ||(I+A^\alpha)^{\beta/2}u||_{L^2} = \{\sum_{j\in\mathbb{Z}^d}(1+\lambda_j^\alpha)^\beta|\hat{u}_j|^2\}^{1/2}$.

Hence $H^{\alpha\beta}$ defined as $D((I+A^\alpha)^{\beta/2})$ with inner product that induces the graph norm is equal to $H^{\alpha\beta}$ defined as $D((I-(-\Delta))^{\alpha\beta/2})$ with the corresponding inner product that induces the graph norm. Finally, we summarize

$$H^{\alpha\beta} = \{u \in L^2 : (I+A^\alpha)^{\beta/2}u \in L^2\} \quad (\beta \in [0,1]),$$
$$(u,v)_{\alpha\beta} = ((I+A^\alpha)^{\beta/2}u, (I+A^\alpha)^{\beta/2}v)_{L^2} = \textstyle\sum_j (1+\lambda_j^\alpha)^\beta \hat{u}_j\overline{\hat{v}_j},$$
$$||u||_{\alpha\beta} = \{\textstyle\sum_j (1+\lambda_j^\alpha)^\beta|\hat{u}_j|^2\}^{1/2}$$

More interestingly, we can easily extend the definition of $H^{\alpha\beta}$ for any $\beta > 0$ via implementation of interpolation method.

We notice that $\{w_j\}$ is an orthogonal basis for $H^{\alpha\beta}$ but not necessarily orthonormal because $||w_j||^2_{H^{\alpha\beta}} = (1+\lambda_j^\alpha)^s \ \forall\, j$.

2.2. **Construction of some $C_0$-semigroups.** In order to prove theorem 2.1, we will first solve the abstract Cauchy problem obtained by dropping nonlinear term $\mathbf{N}(u)$ in the first equation of (1.7)

$$\begin{cases} \frac{du}{dt} = \mathbf{L}u \quad \text{if} \quad t > 0 \\ u(0) = u_0 \end{cases} \tag{2.2}$$

subject to the periodic boundary condition as mentioned in (1.7). Therefore, we need to construct $C_0$-semigroup $e^{t\mathbf{L}}u_0,\ t \geq 0$ generated by $\mathbf{L}$. This will be done by constructing various $C_0$-semigroups generated by operators involving fractional powers of $A$.

**Theorem 2.2.** *$-A$ generates a contraction $C_0$ semigroup on $L^2(\Omega)$.*

*Proof.* Define for each $t \geq 0$

$$e^{-tA}u := \sum_j e^{-t\lambda_j}\hat{u}_j w_j, \quad u \in L^2(\Omega).$$

Then for any real number $a$ and $u,\ v \in L^2(\Omega)$, we have $||e^{-tA}u||^2_{L^2} = \sum_j e^{-2t\lambda_j}|\hat{u}_j|^2 \leq \sum_j |\hat{u}_j|^2 = ||u||^2_{L^2}$ and

$$\begin{aligned} e^{-tA}(au+v) &= \sum_j e^{-t\lambda_j}(a\hat{u+v}) \\ &= a\sum_j e^{-t\lambda_j}\hat{u}_j w_j + \sum_j e^{-t\lambda_j}\hat{v}_j w_j \\ &= ae^{-tA}u + e^{tA}v. \end{aligned}$$

Thus $e^{-tA},\ t \geq 0$ is a one parameter family of bounded linear operators on $L^2$ with $||e^{-tA}||_{\mathcal{L}(L^2,L^2)} \leq 1\ \forall\ t \geq 0$. Further, it satisfies the following properties.

For every $u \in L^2$, it is not hard to show that $e^{-tA}u|_{t=0} = u$ and $(e^{-tA} \circ e^{-sA})u = e^{-(t+s)A}u\ \forall\ t \geq 0$, and for given any $\epsilon > 0$, to choose $N \in \mathbb{N}$ such that $\sum_{j=N+1}^{\infty} |\hat{u}_j|^2 < \epsilon^2/2$ since $\sum_j |\hat{u}_j|^2 < \infty$. With this $N$, we have

$$||e^{-tA}u - u||_{L^2}^2 \leq (1 - e^{-t\lambda_N})^2 \sum_{j=1}^{N} |\hat{u}_j|^2 + \sum_{j=N+1}^{\infty} |\hat{u}_j|^2$$

So we can choose $\delta(N) > 0$ small enough such that we can make the first term on the right side of above inequality less than $\epsilon^2/2$ for $0 < t < \delta(N)$ and hence $||e^{-tA}u - u||_{L^2} < \epsilon$ proving that $\lim_{t\downarrow 0} e^{-tA}u = u$ under $L^2$-norm.

At this point, we have proved that $e^{-tA},\ t \geq 0$ is a contraction $C_0$-semigruop of bounded linear operators on $L^2$.

Finally it remains to show that $-A$ is an infinitesimal generator of $e^{-tA},\ t \geq 0$. For $u \in D(A)$,

$$||\frac{e^{-tA}u - u}{t} - (-Au)||_{L^2}^2 = \sum_j (\frac{e^{-t\lambda_j} - 1}{t} + \lambda_j)^2 |\hat{u}_j|^2 = \sum_j t\lambda_j^2 (\frac{1}{2!} - \frac{t\lambda_j}{3!} + \dots)|\hat{u}_j|^2$$

and so $\lim_{t\downarrow 0} \frac{e^{-tA}u-u}{t}$ under $L^2$-convergence exists and equals to $-Au$. Hence $-A$ is an infinitesimal generator for $e^{-tA},\ t \geq 0$.

Thus the proof is complete. □

Since $-A$ is an infinitesimal generator of contraction $C_0$ semigroup $e^{-tA}$, $t \geq 0$ on $L^2$, so $-A$ is a closed operator and $D(A)$ is dense in $L^2$. From Chapter IX, section 11 in [12] (or [10, 18]), expressions for fractional power of a closed positive operator $A$ are given by

$$A^\alpha u = \frac{\sin \alpha\pi}{\pi} \int_0^\infty s^{\alpha-1}(sI + A)^{-1}Auds, \quad u \in D(A) \tag{2.3}$$

and

$$A^\alpha u = \frac{1}{\Gamma(-\alpha)} \int_0^\infty s^{-\alpha-1}(e^{-sA} - I)uds, \quad u \in D(A) \tag{2.4}$$

It is also proved that $-A^\alpha$ generates a contraction $C_0$-semigroup given by

$$e^{-tA^\alpha}u = \begin{cases} \int_0^\infty f_{t,\alpha}(s)e^{-sA}uds & \text{if } t > 0 \\ u & \text{if } t = 0 \end{cases} \tag{2.5}$$

where $f_{t,\alpha}$ in the integrand is defined by

$$f_{t,\alpha}(s) = \begin{cases} \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} e^{zs-tz^\alpha} dz & \text{if } s \geq 0 \\ 0 & \text{if } s < 0 \end{cases}$$

with $\sigma > 0, t > 0, 0 < \alpha < 1$ and shown that $f_{t,\alpha}(s) \geq 0$ for all $s > 0$

In proof of Theorem 1 in Chapter IX, section 11 in [12], it is established that

$$||e^{-tA^\alpha}u||_{L^2} \leq \sup_{t\geq 0} ||e^{-tA}u||_{L^2}$$

which follows that $||e^{-tA^\alpha}||_{\mathcal{L}(L^2)} \leq 1$ since $||e^{-tA}||_{\mathcal{L}(L^2)} \leq 1$.

We summarize the above discussions below:

**Theorem 2.3.** *$-A^\alpha$ generates a contraction $C_0$-semigroup $e^{-tA^\alpha}$, $t \geq 0$ given in 2.5 on $L^2$.*

Thus $(-I) + (-A^\alpha) = -(I + A^\alpha)$ also generates a contraction $C_0$-semigroup on $L^2$. But this is not what we want. However, we want to construct a contraction $C_0$-semigroup generated by $-(I + A^\alpha)$ on $H^{\alpha\beta}$. Then using this semigroup and formula 2.5, we will be able to construct a contraction $C_0$-semigroup generated by $\mathbf{L}$ on $H^{\alpha\beta}$. To this end, we use (2.1) and Sobolev spaces with corresponding inner products and norms discussed in section 2.1 to define a pseudo-differential operator $(I + A^\alpha) : D(I + A^\alpha) \subset H^{\alpha\beta} \to H^{\alpha\beta}$ of order $2\alpha$ by

$$D(I + A^\alpha) = \{u \in H^{\alpha\beta} : ||(I + A^\alpha)u||^2_{\alpha\beta} = \sum_j (1 + \lambda_j^\alpha)^{\beta+2} |\hat{u}_j|^2 < \infty\},$$

$$(I + A^\alpha)u = \sum_j (1 + \lambda_j^\alpha)\hat{u}_j w_j$$

**Theorem 2.4.** *$-(I+A^\alpha)$ generates a contraction $C_0$-semigroup, namely $e^{-t(I+A^\alpha)}$, $t \geq 0$, on $H^{\alpha\beta}$.*

*Proof.* Define

$$e^{-t(I+A^\alpha)}u := \sum_j e^{-t(1+\lambda_j^\alpha)}\hat{u}_j w_j, \quad t \geq 0,\ u \in H^{\alpha\beta}$$

Then rest of the proof is just analytic routine similar to we did in theorem 2.2, but this time we need to work with $H^{\alpha\beta}$-norm instead of $L^2$-norm. □

Either using the ideas of proof of theorem 2.3 (basically (2.3)-(2.5)) or defining a semigroup by $e^{t\mathbf{L}} := \sum_j e^{-t(1+\lambda_j^\alpha)^{1-\beta}} \hat{u}_j w_j$ on $H^{\alpha\beta}$ as in the proofs of theorem 2.2 and theorem 2.4, we can prove the following.

**Theorem 2.5.** *The linear operator $\mathbf{L}$ generates a contraction $C_0$-semigroup, namely $e^{t\mathbf{L}}u$, $t \geq 0$, on $H^{\alpha\beta}$, $\beta \in (0, 1)$.*

2.3. **Regularity and boundeness properties of $e^{tL}$.** The following technical result which states that $e^{t\mathbf{L}}$ increases regularity is very useful to prove regularity result of main theorem 2.1.

**Theorem 2.6.** *If $u \in L^2$, then $e^{t\mathbf{L}}u \in H^{\alpha\beta}$ for any $\beta > 0$. For instance, we have the following estimates:*

$$||e^{t\mathbf{L}}||_{\mathcal{L}(H^{\alpha\beta},H^{\alpha(\beta+2n\lambda)})} \leq \Big(\frac{n}{\sqrt{2}t}\Big)^n \quad (n \in \mathbb{N}) \tag{2.6}$$

*and for each $\mu \in (0,1)$, there exists a constant $C_{\mu,T}$ such that*

$$||e^{t\mathbf{L}}||_{\mathcal{L}(H^{\alpha\beta},H^{\alpha(\beta+2\mu\lambda)})} \leq C_{\mu,T}\Big(\frac{\mu}{t}\Big)^\mu \tag{2.7}$$

*Proof.* By [2], we have the following estimates on the semigroup and its generator:

If $B$ is a self adjoint and m-dissipative operator on a Hilbert space $H$, and if $u \in H$ then $e^{tB}u \in D(B^n)$ for all $n = 1, 2, ...$ and

$$||(-B)^n||_{\mathcal{L}(H)} = ||(B)^n||_{\mathcal{L}(H)} \leq \Big(\frac{n}{\sqrt{2}t}\Big)^n \tag{2.8}$$

and for any $\mu \in (0,1)$, there exists $C_{\mu,T}$ such that for any $t \in (0,T]$

$$||(-B)^\mu||_{\mathcal{L}(H)} \leq C_{\mu,T}\Big(\frac{\mu}{t}\Big)^\mu \tag{2.9}$$

Since $\mathbf{L}$ is self adjoint and generates a $C_0$-semigroup on $H^{\alpha\beta}$, by Lumer Phillips Theorem $\mathbf{L}$ is m-dissipative and therefore by (2.8)

$$||(-\mathbf{L})^n||_{\mathcal{L}(H^{\alpha\beta})} = ||(\mathbf{L})^n||_{\mathcal{L}(H^{\alpha\beta})} \leq \Big(\frac{n}{\sqrt{2}t}\Big)^n$$

and by (2.9), for any $\mu \in (0,1)$, there exists $C_{\mu,T}$ such that for any $t \in (0,T]$

$$||(-\mathbf{L})^\mu||_{\mathcal{L}(H^{\alpha\beta})} \leq C_{\mu,T}\Big(\frac{\mu}{t}\Big)^\mu$$

Since $(-\mathbf{L})^{\frac{1}{\delta}} = (\gamma + A^\alpha)$ where $\delta = 1 - \beta$, for $u \in H^{\alpha\beta}$, we have

$$||e^{t\mathbf{L}}u||_{\alpha(\beta+2n\delta)} = ||(-\mathbf{L})^n e^{t\mathbf{L}}u||_{\alpha\beta} \leq \Big(\frac{n}{\sqrt{2}t}\Big)^n ||u||_{\alpha\beta}, \tag{2.10}$$

$$||e^{t\mathbf{L}}u||_{\alpha(\beta+2\mu\delta)} = ||(-\mathbf{L})^\mu e^{t\mathbf{L}}u||_{\alpha\beta} \leq C_{\mu,T}\Big(\frac{\mu}{t}\Big)^\mu ||u||_{\alpha\beta} \tag{2.11}$$

By virtue of theorem 2.5, it suffices to prove that if $u \in L^2$ then $e^{t\mathbf{L}}u \in H^{\alpha\beta}$ for all $\beta \geq 1$ which follows from 2.6 and 2.7. □

We see from previous theorem 2.6 that if $u \in L^\infty \subset L^2$ then $e^{t\mathbf{L}}u \in H^{\alpha\beta},\ \beta \geq 0$. For large enough $\beta > 0$, $e^{t\mathbf{L}}u \in L^\infty$ by Sobolev Embedding Theorem. Thus we can assume that $e^{t\mathbf{L}}$ is an operator on $L^\infty$. Using the method analogous to proposition 3.6 in [1], we can establish bounds for $e^{t\mathbf{L}}$ and $\mathbf{N}$ as operators on $L^\infty$ into itself.

**Theorem 2.7.** *If $u \in L^\infty$, then (a) $||\mathbf{N}(u)||_\infty \leq ||u||_\infty + ||f(.,u)||_\infty$ and (b) for each $t \geq 0$, $||e^{t\mathbf{L}}u||_\infty \leq ||u||_\infty$.*

This theorem shows that $\mathbf{N} : L^\infty \to L^\infty$ is locally bounded and $e^{t\mathbf{L}} : L^\infty \to L^\infty$ is bounded with $||e^{t\mathbf{L}}||_{\mathcal{L}(L^\infty)} \leq 1$.

2.4. **Proof of main theorem 2.1.** Next we introduce an integral form of (1.7) and apply Banach Fixed Point Theorem to prove that a unique solution in this form always exists.

An intergral form of (1.7)

$$u(t,x) := e^{t\mathbf{L}}u_0(x) + \int_0^t e^{(t-s)\mathbf{L}}\mathbf{N}(u(s,x))\, ds$$

is called a *mild solution* of the equation.

We write $u(t,x) := u[t](x)$ to see that $t \mapsto u[t]$ defines a map from a time interval into a function space. Then for any $u_0 \in L^\infty$ and $T > 0$, we define a subspace

$$W_T = \{u \in C([0,T], L^\infty) : u[0] = u_0\}$$

of $C([0,T], L^\infty)$ with norm

$$||u||_{\infty,T} = \max_{0\le t\le T} ||u[t]||_\infty$$

Notice that if $0 < t \le T$ then $W_T \subseteq W_t$ and $||u[s]||_{\infty,t} \le ||u[s]||_{\infty,T}$

**Theorem 2.8.** *A map $\Psi : W_T \to W_T$ defined by*

$$\Psi(u[t]) := e^{t\mathbf{L}}u_0 + \int_0^t e^{(t-s)\mathbf{L}}\mathbf{N}(u[s])\, ds$$

*is a contraction map for some small $T > 0$ independent of $u_0$.*

*Proof.* We see that $\Psi(u[0]) = u_0$ and applying Theorem 2.7

$$\begin{aligned} ||\Psi(u[t])||_\infty &\le ||u_0||_\infty + \int_0^t [||u[s]||_\infty + ||f(.,u[s])||_\infty]\, ds \\ &\le ||u_0||_\infty + [||u||_{\infty,t} + \max_{0\le s\le t} ||f(.,u[s])||_\infty] \\ &< \infty \end{aligned}$$

implying that $\Psi$ defines a map on $W_T$.

Since $|f(x,y_1) - f(x,y_2)| \le L(f)|y_1 - y_2| \quad \forall\ x \in \mathbb{T}^d, \forall\ y_1,\ y_2 \in \mathbb{R}$ for some constant $L(f)$ depending on $f$, this yields

$$||f(.,u[s]) - f(.,u[s])||_\infty \le L(f)||u[s] - v[s]||_\infty \quad u,\ v \in W_t \tag{2.12}$$

If $u,\ v \in W_t$ then an argument analogous to the previous theorem 2.7 yields

$$\begin{aligned} ||\mathbf{N}(u[s]) - \mathbf{N}(v[s])||_\infty &\le ||u[s] - v[s]||_\infty + ||f(.,u[s]) - f(.,v[s])||_\infty \\ &\le [1 + L(f)]||u[s] - v[s]||_\infty \end{aligned}$$

and hence

$$\begin{aligned}||\Psi(u[t]) - \Psi(v[t])||_\infty &\leq \int_0^t ||e^{(t-s)\mathbf{L}}||_{\mathcal{L}(L^\infty)}||\mathbf{N}(u[s]) - \mathbf{N}(v[s])||_\infty \, ds \\ &\leq \int_0^t [1 + L(f)]||u[s] - v[s]||_\infty \, ds \\ &\leq t[1 + L(f)]||u - v||_{\infty,t}\end{aligned}$$

Choose $T = \frac{1}{2[1+L(f)]}$. Since $[1 + L(f)]$ is independent of $u_0$, $T$ is independent of $u_0$ and for every $t$ with $0 \leq t \leq T$

$$||\Psi(u[t]) - \Psi(v[t])||_\infty \leq \frac{1}{2}||u - v||_{\infty,T}$$

Taking supremum over all $t \in [0, T]$ on the left side, we get

$$||\Psi(u) - \Psi(v)||_{\infty,T} \leq \frac{1}{2}||u - v||_{\infty,T}$$

and hence $\Psi$ is a contraction map on $W_T$. □

*Proof of theorem 2.1:* As a closed subset of a Banach space, $W_T$ with the norm $||.||_{\infty,T}$ is complete. Applying Banach Fixed Point Theorem, theorem 2.8 now follows that the map $\Psi : W_T \to W_T$ has a unique fixed point $u_F \in W_T$ for a small enough $T$. That is,

$$u_F[t] = \Psi(u_F[t]) := e^{t\mathbf{L}}u_0 + \int_0^t e^{(t-s)\mathbf{L}}\mathbf{N}(u_F[s]) \, ds$$

which is, in fact, a mild solution of (1.7) in $C([0, T], L^\infty)$.

Since $T$ is independent of initial data $u_0$, we can follow the same method as above with starting solution at $t = T$ instead of starting at $t = 0$, we can obtain a unique mild solution in $C([T, 2T], L^\infty)$ of (1.7) with the initial data replaced by $u[T] = u_0$. Then we can apply concatenation (Gluing Lemma) to obtain a unique mild solution on $[0, 2T]$ of (1.7) with $u[0] = u_0$. Repeating this process indefinitely, we can obtain a unique mild solution on $[0, \infty)$ of (1.7) with $u[0] = u_0$. We again denote it by $u_F$. Then $u_F \in C([0, \infty), L^\infty)$ with $u_F[0] = u_0$.

Since $||(\gamma + A^\alpha)^{-\beta}u||_{\alpha(s+2\beta)} = ||u||_{\alpha s} \; \forall \; u \in H^{\alpha s}, \; s \geq 0$, the operator $(I + A^\alpha)^{-\beta}$ maps $H^{\alpha s}$ into $H^{\alpha(s+2\beta)}$, that is, it increases regularity by $2\alpha\beta$. In particular, $L^\infty \subset L^2 = H^0$, $(I + A^\alpha)^{-\beta}$ maps $L^\infty$ into $H^{2\alpha\beta}$. Also $||(I + A^\alpha)^{-\beta}||_{\mathcal{L}(H^{\alpha s}, H^{\alpha(s+2\beta)})} = 1$. Further, we notice that $\mathbf{N}$ increases regularity by $2\alpha\beta$ on regularity of $u - f(., u)$.

For instance, as $u_F[.] \in L^\infty$, by theorem 2.7 $\mathbf{N}(u_F[.]) \in L^\infty$ and so $\mathbf{N}(u_F[.]) \in H^{2\alpha\beta}$ with

$$||\mathbf{N}(u_F[.])||_{2\alpha\beta} = ||u_F[.] - f(., u_F[.]))||_0 \leq (||u_F[.]||_\infty + ||f(., u_F[.])||_\infty)$$

Regularity and contracting $C_0$-semigroup properties of $e^{t\mathbf{L}}$ (Theorems 2.5, 2.6) imply that $||e^{t\mathbf{L}}||_{\mathcal{L}(H^{2\alpha\beta})} \leq 1 \; \forall \; t \geq 0$. Applying this inequality and previous inequality, for each fixed $t \geq 0$ we get

$$||u_F[t]||_{2\alpha\beta} \leq ||e^{t\mathbf{L}}u_0||_{2\alpha\beta} + t(||u_F||_{\infty,t} + \max_{0\leq s\leq t} ||f(.,u_F[t])||_\infty) < \infty$$

This inequality together with (2.12) follows that $u_F \in C([0,\infty), H^{2\alpha\beta})$ and hence $u_F \in C([0,\infty), H^{2\alpha\beta} \cap L^\infty)$.

Till now we have only proved the unique existence of mild solution and improved its regularity. To complete the proof of theorem 2.1, we have to show that the map $t \mapsto u_F[t]$ is $C^1$ from $(0,\infty)$ into $H^{2\alpha\beta}$. In order to show this, $\frac{du}{dt} = \mathbf{L}(u_F) + \mathbf{N}(u_F)$ helps us anticipate what the first derivative of this map should be. So we define

$$v[t] := \mathbf{L}(u_F[t]) + \mathbf{N}(u_F[t]) = \mathbf{L}e^{t\mathbf{L}}u_0 + \int_0^t \mathbf{L}e^{(t-s)\mathbf{L}}\mathbf{N}(u_F[s])\, ds + \mathbf{N}(u_F[t]) \quad \text{for } t > 0$$

Then $\frac{1}{h}(u_F[t+h] - u_F[t]) - v[t]$

$$= \{\frac{1}{h}(e^{h\mathbf{L}} - I) - \mathbf{L}\}e^{t\mathbf{L}}u_0 + \int_0^\infty \{\frac{1}{h}(e^{h\mathbf{L}} - I) - \mathbf{L}\}e^{(t-s)\mathbf{L}}\mathbf{N}(u_F[s])\, ds$$
$$+\frac{1}{h}\int_t^{t+h} e^{(t+h-s)\mathbf{L}}\mathbf{N}(u_F[s])\, ds - \mathbf{N}(u_F[t])$$

Since $e^{t\mathbf{L}}u_0 \in D(\mathbf{L})$, by definition of infinitesimal generator $\lim_{h\downarrow 0} \frac{1}{h}(e^{h\mathbf{L}}-I)e^{t\mathbf{L}}u_0 = \mathbf{L}e^{t\mathbf{L}}u_0$ under $H^{2\alpha\beta}$-convergence. That is, $\lim_{h\downarrow 0} ||\{\frac{1}{h}(e^{h\mathbf{L}} - I) - \mathbf{L}\}e^{t\mathbf{L}}u_0||_{2\alpha\beta} = 0$

As we have $(e^{h\mathbf{L}} - I) = \int_0^h \ d(e^{\tau\mathbf{L}}) = \int_0^h \mathbf{L}e^{\tau\mathbf{L}}\ d\tau$ and $\mathbf{L}e^{(t-s)\mathbf{L}} = e^{(t-s)\mathbf{L}}\mathbf{L}$, so $|| \int_0^\infty \{\frac{1}{h}(e^{h\mathbf{L}}-I)-\mathbf{L}\}e^{(t-s)\mathbf{L}}\mathbf{N}(u_F[s])ds||_{2\alpha\beta} \leq \int_0^\infty ||\{\frac{1}{h}\int_0^h e^{\tau\mathbf{L}}\ d\tau - I\}e^{(t-s)\mathbf{L}}\mathbf{L}\mathbf{N}(u_F[s])||_{2\alpha\beta}ds$. Since for every $u \in H^{2\alpha\beta}$ $\lim_{h\downarrow 0} \frac{1}{h}\int_0^h e^{\tau\mathbf{L}}u\ d\tau = u$ under $H^{2\alpha\beta}$-convergence, the integrand on right side of above inequality approaches to 0 as $h \downarrow 0$. Thus $\lim_{h\downarrow 0} || \int_0^\infty \{\frac{1}{h}(e^{h\mathbf{L}} - I) - \mathbf{L}\}e^{(t-s)\mathbf{L}}\mathbf{N}(u_F[s])ds||_{2\alpha\beta} = 0$.

Substituting $s = t + \tau$, we get

$$\frac{1}{h}\int_t^{t+h} e^{(t+h-s)\mathbf{L}}\mathbf{N}(u_F[s])\, ds - \mathbf{N}(u_F[t]) = \frac{1}{h}\int_0^h e^{(h-\tau)\mathbf{L}}\mathbf{N}(u_F[\tau+t])\, d\tau - \mathbf{N}(u_F[t])$$
$$= \{\frac{1}{h}\int_0^h e^{(h-\tau)\mathbf{L}}\mathbf{N}(u_F[t])\, d\tau - \mathbf{N}(u_F[t])\} + \frac{1}{h}\int_0^h e^{(h-\tau)\mathbf{L}}\{\mathbf{N}(u_F[\tau+t]) - \mathbf{N}(u_F[t])\}\, d\tau$$

Because of the similar reason as in previous paragraph, the first term on the right side converges to 0 as $h \downarrow 0$ under $H^{2\alpha\beta}$-convergence. Applying $||e^{(t+h-s)\mathbf{L}}||_{2\alpha\beta} \leq 1$ and $||\mathbf{N}(u_F[\tau+t]) - \mathbf{N}(u_F[t])||_{2\alpha\beta} = ||(u_F[\tau+t] - f(.,u_F[\tau+t])) - (u_F[t] - f(.,u_F[t]))||_0 \leq [1 + L(f)]||u_F[\tau+t] - u_F[t]||_\infty$ with continuity of $\tau \mapsto u_F[\tau]$ from $[0,\infty)$ into $L^\infty$, the second term on the right side also converges to 0 as $h \downarrow 0$ under $H^{2\alpha\beta}$-convergence. Hence $\lim_{h\downarrow 0} ||\frac{1}{h}\int_t^{t+h} e^{(t+h-s)\mathbf{L}}\mathbf{N}(u_F[s])\, ds - \mathbf{N}(u_F[t])||_{2\alpha\beta} = 0$.

Thus $\lim_{h\downarrow 0} ||\frac{1}{h}(u_F[t+h] - u_F[t]) - v[t]||_{2\alpha\beta} = 0$. Therefore, $\frac{du_F[t]}{dt} = v[t] \in H^{2\alpha\beta}$. It is not hard to show that $v[t] \in L^\infty$. Since $t \mapsto v[t]$ is continuous, it is $C^1$.

Finally, we have $u_F[.] \in C^1((0,\infty), H^{2\alpha\beta} \cap L^\infty)$. Hence the proof of theorem 2.1 is complete.

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