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UPPER AND LOWER BOUNDS FOR THE TOTAL PRODUCT RATE VARIATION PROBLEM

SHREE RAM KHADKA[1] AND FRANK WERNER[2]

[1]Central Department of Mathematics, Institute of Science and Technology,
Tribhuvan University, P.O. Box 13143, Kathmandu, Nepal.
[2]Institute of Mathematical Optimization, Faculty of Mathematics,
Otto-von-Guericke University, P.O. Box 4120, Magdeburg, Germany.
Corresponding author: Shree Ram Khadka (shreeramkhadka@gmail.com)

ABSTRACT. The total product rate variation problem is a sequencing problem in mixed-model just-in-time production systems. In particular, this problem consists in the minimization of a function of the total deviations between the actual and the ideal cumulative productions of a variety of models of a common base product keeping the rate of usage of models as constant as possible. Several pseudo-polynomial exact algorithms and heuristics have been derived for this problem. In this paper, we propose an upper and a lower bound on the largest and smallest function values of a feasible solution of the problem when the m-th power of the total deviations between the actual and the ideal cumulative productions has to be minimized.

Key words: Bound, Product rate variation problem, Non-linear integer programming problem.

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1. Introduction

The total product rate variation problem (abbreviated as TPRVP) is a sequencing problem in mixed-model just-in-time production systems. Mixed-model just-in-time production systems with negligible change-over costs between the models have been used in order to respond to the customer demands for a variety of models of a common base product without holding large inventories or incurring large shortages. TPRVP is the problem of minimizing a function of the total deviations between the actual cumulative productions from the ideal one keeping the rate of usage of models as constant as possible, see Kubiak [6]. TPRVP has been widely investigated in the literature since it has a model with a strong mathematical base and wide real-world applications, see Dhamala and Khadka [2]. The existing exact solution approaches are pseudo-polynomial.

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In this paper, we propose an upper and a lower bound for TPRVP. We also derive an explicit value of a bound such that no instance has even a feasible solution for TPRVP with an objective function value smaller than this bound.

The remainder of the paper is as follows. In Section 2, we present a non-linear integer programming formulation of TPRVP. In Section 3, we derive an upper and a lower bound on the largest and smallest function values of a feasible solution of TPRVP, which is the major contribution of this paper. In Section 4, we derive the minimal possible function value of the m-th power of the total deviations between the actual and the ideal cumulative productions in the sense that for a smaller value, no instance has even a feasible solution for TPRVP. The last section concludes the paper.

2. Non-linear Integer Programming Formulation

Let D be the total demand of n different models with d_i copies of model $i, i = 1, 2, \ldots, n$, where $n \geq 2$ and $D = \sum_{i=1}^{n} d_i$. The time horizon is partitioned into D equal time units under the assumption that each copy of a model i, $i = 1, \ldots, n$, has the same processing time. A copy of a model is produced in a time unit k, which means that the copy of the model is produced during the time period from $k-1$ to $k, k = 2, 3, \ldots, D$. Let $r_i = \frac{d_i}{D}$ be the demand rate for model $i, i = 1, 2, \ldots, n$. Moreover, let x_{ik} and kr_i be the actual and the ideal cumulative productions, respectively, of model $i, i = 1, 2, \ldots, n$, produced during the time units 1 through k. An inventory holds if $x_{ik} - kr_i > 0$, and a shortage incurs if $kr_i - x_{ik} > 0$. We assign the same cost for both inventory and shortage. Miltenburg [8] and Kubiak and Sethi [7] gave an integer programming formulation for TPRVP as follows with m being a positive integer:

$$\text{minimize} \left[F_m = \sum_{k=1}^{D} \sum_{i=1}^{n} |x_{ik} - kr_i|^m \right]$$

subject to

$$\begin{aligned}
\sum_{i=1}^{n} x_{ik} &= k, \qquad k = 1, 2, \ldots, D \\
x_{i(k-1)} &\leq x_{ik}, \quad i = 1, 2, \ldots, n; \;\; k = 2, 3, \ldots, D \\
x_{iD} &= d_i, x_{i0} = 0, \quad i = 1, 2, \ldots, n \\
x_{ik} &\geq 0, \text{integer}, \quad i = 1, 2, \ldots, n; \;\; k = 1, 2, \ldots, D\ .
\end{aligned}$$

3. Bounds

3.1. **Upper Bound.** We set a horizontal line with a suitable value $B > 0$ intersecting the level curve for each copy $(i, j), i = 1, 2, \dots, n; j = 1, 2, \dots, d_i$, of the objective function of TPRVP on the planning horizon $[0, D]$. The horizontal line with the value B is called a bound for TPRVP. The intersecting points of the level curve of the objective function for each copy and the bound are important to determine the sequencing time units for all copies of all models. One seeks a smaller value of B so that the total deviations between actual and the ideal cumulative productions can be reduced with the sequencing time units not exceeding the planning horizon.

It is important to establish an upper and a lower bound so that one can minimize the total deviations in a reasonable time. A sequence corresponding to the minimum bound B_{min}, which satisfies the inequality

$$\sum_{k=1}^{D}\sum_{i=1}^{n} |x_{ik} - kr_i|^m \leq B_{min}, i = 1, 2, \dots, n; k = 1, 2, \dots, D \ ,$$

is optimal for TPRVP with the objective function F_m.

Let

$$\mathcal{X} = \{\mathbf{x} = (x_{ij}) | i = 1, 2, \dots, n; j = 1, 2, \dots, d_i\}$$

be the set of all feasible solutions for TPRVP.

A necessary and sufficient condition for the existence of a feasible sequence for the bottleneck product rate variation problem (abbreviated as BPRVP), which is the problem of minimizing a function of the maximum deviation between the actual cumulative productions and the ideal cumulative ones, with the objective of minimizing

$$\max_{i,k} |x_{ik} - kr_i| \,, i = 1, 2, \dots, n; k = 1, 2, \dots, D,$$

is that a bound with a value B_1 must satisfy the two inequalities

$$\sum_{i=1}^{n} (\lfloor k_2 r_i + B_1 \rfloor - \lceil (k_1 - 1) r_i - B_1 \rceil) \geq k_2 - k_1 + 1 \tag{3.1}$$

and

$$\sum_{i=1}^{n} (\lceil k_2 r_i - B_1 \rceil - \lfloor (k_1 - 1) r_i + B_1 \rfloor) \leq k_2 - k_1 + 1 \tag{3.2}$$

where $k_1, k_2 \in \{1, \dots, D\}$, $k_1 \leq k_2$. The interval $[k_1, k_2]$ overlaps with the time interval within which copy (i, j) can be sequenced, see Brauner and Crama [1]. The above considerations can be applied to the more general case of minimizing the m-th power of the maximum deviation between the actual cumulative productions and the ideal cumulative ones to derive an upper bound for TPRVP.

Theorem 3.1. *Let*

$$UB_m = nD\left(1 - \frac{1}{D}\right)^m .$$

Then UB_m is an upper bound on the largest value of the objective function F_m of a feasible solution for TPRVP.

Proof: If UB_m is an upper bound on the largest value of the objective function F_m of a feasible solution for TPRVP, then this bound UB_m satisfies the inequality

$$\sum_{k=1}^{D}\sum_{i=1}^{n} |x_{ik} - kr_i|^m \leq UB_m \tag{3.3}$$

for any feasible solution $\mathbf{x} \in \mathcal{X}$.

Let B_m be an upper bound on the largest function value of a feasible solution for BPRVP with the objective function

$$\max_{i,k} \ |x_{ik} - kr_i|^m, i = 1, 2, \ldots, n; k = 1, 2, \ldots, D.$$

Such a bound B_m has to satisfy the inequality

$$\max_{i,k} |x_{ik} - kr_i|^m \leq B_m, \ i = 1, 2, \ldots, n; \ k = 1, 2, \ldots, D .$$

Consider now

$$B_m = \left(1 - \frac{1}{D}\right)^m .$$

Then we can write

$$\left\lfloor k_2 r_i + \sqrt[m]{B_m} \right\rfloor = \left\lfloor k_2 r_i + 1 - \frac{1}{D} \right\rfloor .$$

If $k_2 r_i$ is an integer, we have

$$\left\lfloor k_2 r_i + 1 - \frac{1}{D} \right\rfloor = k_2 r_i, \ i = 1, 2, \ldots, n,$$

and if $k_2 r_i$ is not an integer, we have

$$k_2 r_i = \lfloor k_2 r_i \rfloor + \epsilon_i, \ i = 1, 2, \ldots, n,$$

where ϵ_i is the fractional part of $k_2 r_i$.
Since the inequalities

$$\frac{1}{D} \leq \epsilon_i \leq 1 - \frac{1}{D},$$

hold, we get

$$\begin{aligned} \left\lfloor k_2 r_i + 1 - \frac{1}{D} \right\rfloor &= \left\lfloor \lfloor k_2 r_i \rfloor + \epsilon_i + 1 - \frac{1}{D} \right\rfloor \\ &\geq \lfloor k_2 r_i \rfloor + 1 \\ &> k_2 r_i, \ i = 1, 2, \ldots, n . \end{aligned}$$

Therefore, we get

$$\left\lfloor k_2 r_i + \sqrt[m]{B_m} \right\rfloor \geq k_2 r_i, \ i = 1, 2, \ldots, n . \tag{3.4}$$

Thus, we can write

$$\left\lceil k_2 r_i - \sqrt[m]{B_m} \right\rceil = \left\lceil k_2 r_i - 1 + \frac{1}{D} \right\rceil .$$

If $k_2 r_i$ is an integer, we have

$$\left\lceil k_2 r_i - 1 + \frac{1}{D} \right\rceil = k_2 r_i,\ i = 1, 2, \ldots, n,$$

and if $k_2 r_i$ is not an integer, we have

$$\begin{aligned} \left\lceil k_2 r_i - 1 + \frac{1}{D} \right\rceil &= \left\lceil \lfloor k_2 r_i \rfloor + \epsilon_i - 1 + \frac{1}{D} \right\rceil \\ &\leq \lceil \lfloor k_2 r_i \rfloor \rceil \\ &= \lfloor k_2 r_i \rfloor \\ &< k_2 r_i,\ i = 1, 2, \ldots, n . \end{aligned}$$

Therefore, we obtain the inequality

$$\left\lceil k_2 r_i - \sqrt[m]{B_m} \right\rceil \leq k_2 r_i,\ i = 1, 2, \ldots, n . \tag{3.5}$$

Using the inequalities (3.4) and (3.5), a bound B_m satisfies the following inequalities:

$$\begin{aligned} \sum_{i=1}^{n} \left(\left\lfloor k_2 r_i + \sqrt[m]{B_m} \right\rfloor - \left\lceil (k_1 - 1) r_i - \sqrt[m]{B_m} \right\rceil \right) &\geq \sum_{i=1}^{n} k_2 r_i - \sum_{i=1}^{n} (k_1 - 1) r_i \\ &\geq k_2 - k_1 + 1 . \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} \sum_{i=1}^{n} \left(\left\lceil k_2 r_i - \sqrt[m]{B_m} \right\rceil - \left\lfloor (k_1 - 1) r_i + \sqrt[m]{B_m} \right\rfloor \right) &\leq \sum_{i=1}^{n} k_2 r_i - \sum_{i=1}^{n} (k_1 - 1) r_i \\ &\leq k_2 - k_1 + 1 . \end{aligned} \tag{3.7}$$

Similarly to the inequalities (3.1) and (3.2), the bound B_m satisfies the necessary and sufficient condition for the existence of a feasible sequence of any instance for BPRVP with the objective function

$$\max_{i,k} \ |x_{ik} - k r_i|^m , i = 1, 2, \ldots, n;\ k = 1, 2, \ldots, D .$$

Hence, the two inequalities (3.6) and (3.7) show that

$$B_m = \left(1 - \frac{1}{D} \right)^m$$

is an upper bound on the largest function value of a feasible solution for BPRVP with the objective function

$$\max_{i,k} \ |x_{ik} - k r_i|^m , i = 1, 2, \ldots, n;\ k = 1, 2, \ldots, D .$$

Now, we obtain

$$\begin{aligned}\sum_{k=1}^{D}\sum_{i=1}^{n}|x_{ik}-kr_i|^m &= \sum_{i=1}^{n}|x_{i1}-1r_i|^m+\cdots+\sum_{i=1}^{n}|x_{iD}-Dr_i|^m\\ &= |x_{11}-1r_1|^m+\cdots+|x_{n1}-1r_n|^m+\ldots\\ &+ |x_{1D}-Dr_1|^m+\cdots+|x_{nD}-Dr_n|^m\\ &\le nD\ \cdot\max_{i,k}\ |x_{ik}-kr_i|^m\\ &\le nD\left(1-\frac{1}{D}\right)^m .\end{aligned}$$

Hence, an upper bound UB_m on the largest value of the objective function F_m of a feasible solution for TPRVP is given by

$$UB_m = nD\left(1-\frac{1}{D}\right)^m . \square$$

3.2. **Lower Bound.** The importance of a lower bound LB_m on the optimal value of function F_m for TPRVP results from the fact that, if an instance with the demands $(d_1, d_2, \ldots, d_n)$ has a feasible sequence with an objective function value equal to the lower bound, this sequence is optimal. It is note-worthy that a lower bound

(3.8) $$B_1^* = 1 - r_{max}$$

on the absolute deviation objective function $|x_{ik} - kr_i|\,, i = 1,2,\ldots,n; k = 1,2,\ldots,D$, for BPRVP has been established by Steiner and Yeomans [9], and it has been generalized to

(3.9) $$B_m^* = (1 - r_{max})^m,$$

for this problem with the objective function $|x_{ik} - kr_i|^m\,, i = 1,2,\ldots,n; k = 1,2,\ldots,D$, (see [3, 4]).

Theorem 3.2. *Let*

$$LB_m = nD\,(1 - r_{max})^m\ ,$$

where $r_{max} = max\,\{r_i | i = 1,2,\ldots,n\}$. *Then* LB_m *is a lower bound on the optimal value of the objective function* F_m *for TPRVP.*

Proof: Let LB_m be a lower bound on the optimal value of function F_m for TPRVP. Such a bound LB_m has to satisfy the inequality

(3.10) $$LB_m \le \sum_{k=1}^{D}\sum_{i=1}^{n}|x_{ik}-kr_i|^m$$

for any feasible solution $\mathbf{x} \in \mathcal{X}$.

Now, we obtain

$$\begin{aligned}\sum_{k=1}^{D}\sum_{i=1}^{n}|x_{ik}-kr_i|^m &= \sum_{i=1}^{n}|x_{i1}-1r_i|^m+\dots+\sum_{i=1}^{n}|x_{iD}-Dr_i|^m \\ &= |x_{11}-1r_1|^m+\dots+|x_{n1}-1r_n|^m+\dots \\ &+ |x_{1D}-Dr_1|^m+\dots+|x_{nD}-Dr_n|^m \\ &\geq nD\cdot\min_{i,k}|x_{ik}-kr_i|^m .\end{aligned}$$

Using the lower bound (3.9) for BPRVP with the objective function

$$|x_{ik}-kr_i|^m, i=1,2,\dots,n; k=1,2,\dots,D,$$

a lower bound LB_m on the optimal value of function F_m for TPRVP can be established as

$$LB_m = nD\left(1-r_{max}\right)^m . \square$$

4. **Further Estimations**

In this section, we show that the minimal possible value of the objective function F_m is $\frac{nD}{3^m}$ in the sense that for a smaller value, no instance has even a feasible solution for TPRVP. For this derivation, the following lemma is useful.

Lemma 4.1. *Let*

$$UB_m = nD\left(1-\frac{1}{D}\right)^m$$

be an upper bound on the largest function value F_m of a feasible solution for TPRVP. Then for $i=1,2,\dots,n$; and $j=1,2,\dots,d_i$, the inequality

$$\frac{j-\sqrt[m]{\frac{UB_m}{nD}}}{r_i} \leq \frac{j-1+\sqrt[m]{\frac{UB_m}{nD}}}{r_i}+1$$

holds.

Proof: Let UB_m be an upper bound on the largest objective function value F_m of a feasible solution for TPRVP. Without loss of generality, we can consider

$$UB_m = nD\left(1-\frac{1}{D}\right)^m .$$

One can write

$$\begin{aligned}\frac{j-1+\sqrt[m]{\frac{UB_m}{nD}}}{r_i}+1-\frac{j-\sqrt[m]{\frac{UB_m}{nD}}}{r_i} &= \frac{j-1+1-\frac{1}{D}}{r_i}+1-\frac{j-1+\frac{1}{D}}{r_i} \\ &= \frac{1+r_i-\frac{2}{D}}{r_i} \\ &= \frac{D+d_i-2}{d_i} \\ &\geq 0\end{aligned}$$

for all $D \geq 2$.

Hence, the inequality

$$\frac{j - \sqrt[m]{\frac{UB_m}{nD}}}{r_i} \leq \frac{j - 1 + \sqrt[m]{\frac{UB_m}{nD}}}{r_i} + 1$$

holds for any feasible instance. □

Theorem 4.2. *Let a bound*

$$B < \frac{nD}{3^m}$$

be given. Then there is no instance that has a feasible sequence for TPRVP with the objective function F_m.

Proof: The lower bound

$$LB_m = nD(1 - r_{max})^m$$

implies the inequality

$$1 - r_{max} \leq \sqrt[m]{\frac{B}{nD}} \tag{4.1}$$

for any upper bound B on the largest objective function value F_m of a feasible solution for TPRVP.

Using Lemma 4.1, we have

$$\frac{j - \sqrt[m]{\frac{B}{nD}}}{r_i} \leq \frac{j - 1 + \sqrt[m]{\frac{B}{nD}}}{r_i} + 1$$

which can be written as

$$1 - r_i \leq 2\sqrt[m]{\frac{B}{nD}} \qquad \text{for } i = 1, \ldots, n.$$

Thus, we have

$$1 - r_{min} \leq 2\sqrt[m]{\frac{B}{nD}}$$

which yields

$$\sum_{i=1, r_{\acute{i}} \neq r_{min}}^{n} r_{\acute{i}} \leq 2\sqrt[m]{\frac{B}{nD}}$$

and

$$r_{max} \leq \sum_{i=1, r_{\acute{i}} \neq r_{min}}^{n} r_{\acute{i}} \leq 2\sqrt[m]{\frac{B}{nD}},$$

i.e., we have

$$1 - r_{max} \geq 1 - 2\sqrt[m]{\frac{B}{nD}}\ .$$

Then, using inequality (4.1), we obtain

$$1 - 2\sqrt[m]{\frac{B}{nD}} \leq \sqrt[m]{\frac{B}{nD}}\ .$$

Thus,

$$\frac{1}{3} \leq \sqrt[m]{\frac{B}{nD}} \cdot \square$$

5. Concluding Remarks

The total product rate variation problem is a sequencing problem in mixed-model just-in-time production systems. For this problem, several pseudo-polynomial exact solution algorithms and heuristics have been developed. An upper and a lower bound on the largest and smallest function values of a feasible solution of TPRVP are

$$nD\left(1-\frac{1}{D}\right)^m \qquad \text{and} \qquad nD(1-r_{max})^m,$$

respectively. These bounds can be used to develop an $O(D \log D)$ exact solution procedure recently given by Khadka and Werner [5] which improves the known exact algorithm by Kubiak from [6] with a complexity of $O(D^3)$. Moreover, it has been shown in this paper that the minimum value of the objective function F_m is $\frac{nD}{3^m}$ in the sense that for a smaller value, no instance has even a feasible solution for TPRVP.

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