Neural, Parallel, and Scientific Computations 22 (2014) 411-420

SELF-SIMILARITY, INTEGRABILITY, AND ACCORDIONS IN TRANSIENT STIMULATED RAMAN SCATTERING

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ABSTRACT. Accordions, like their better-known mathematical cousins — solitons — are similarity solutions of integrable partial differential equations. In the case of solitons, the similarity variable is $x - vt$, where x denotes position, t denotes time, and v denotes a velocity. In the case of accordions, the similarity variable is xt^{α}. It can be shown that the solutions to the transient stimulated Raman scattering equations will always tend toward one of a two-parameter set of accordion solutions for any initial condition. The early history of these equations and related experiments is reviewed. The recent observation of accordions by the Russell group at the Max-Planck Institute for Light in Erlangen, Germany has rewoken interest in this topic and has suggested new mathematical work. Related work at Princeton that is aimed at applications to high-energy laser pulse generation will also be described.

AMS (MOS) Subject Classification. 35Q60.

1. INTRODUCTION

Self-similarity is ubiquitous in nature [1], either exactly or approximately. Its presence allows us to understand complex phenomena that otherwise might be completely resistant to mathematical study.

The natural world in which we live contains three space and one time dimension. Sometimes, it is possible to analyze what is happening in these 3+1 dimensions using linear equations, but it is often not possible, since the behaviors that we observe are intrinsically nonlinear. Fluid systems that are described by the Navier-Stokes equation or electromagnetic systems that are described by Maxwell's equations with a nonlinear permittivity are two of the more important examples. It is often possible to simplify 3+1-dimensional systems to systems with one space and one time dimension by taking advantage of symmetries that exist in a system. Wave propagation in a single-mode optical fiber, or for that matter, any waveguide with a nonlinear dielectric response in which only a single transverse mode can propagate is one case in which this reduction occurs.

Received December 18, 2013

As Ya. B. Zeldovich eloquently states in the preface to [1]:

"Progress in numerical calculation brings great good... The problem of formulating rules and extending ideas from vast amounts of computational data or experimental results remains a matter for our brains, our minds."

However, even 1+1-dimensional systems can be frustratingly complex. Supercontinuum generation in optical fibers, in which many solitons emerge from a single pulse and in which the final outcome can depend sensitively on the initial conditions is one example of this complexity [2]. Chaotic conditions are often observed in 1+1-dimensional systems. So, further reduction of the complexity is often needed. Self-similarity, when present, allows us to reduce partial differential equations to ordinary differential equations.

As Ya. B. Zeldovich notes, there are two basic types of self-similarity that are observed in nature. In the first kind, we find $f(x,t) \to f(x - vt)$, where v is a constant. In this case, we have the well-known *soliton* self-similarity. In the second kind, we find $f(x,t) \to (t/t_0)^n f(xt^m/t_0^m)$, where m and n are constants. While often present in nature, this second self-similarity has attracted much less attention from the mathematical community. With the thought that a catchy name (like "soliton") could perhaps enhance the reputation of this somewhat neglected cousin of the soliton, I have proposed to call them "accordions," which mimics their observed behavior. When $m > 0$, the structure is observed to shrink in x as it expands in t; when $m < 0$, they both expand together.

However, even when we can find a self-similar solution, how do we know that it is important — that it will in fact appear in a system? As Ya. B. Zeldovich once again eloquently states:

"In nonlinear problems, exact special solutions sometimes appear to be useless; since there is no principle of superposition, one cannot immediately find a solution to the problem for arbitrary initial conditions."

Integrability or near-integrability is often the key to resolving this dilemma. The classic soliton systems like the Korteweg-de Vries equation or the nonlinear Schrödinger equation have prescribed initial data for all x at $t = 0$ that tend rapidly to zero as $x \to \pm\infty$. These systems have the property that the final asymptotic state will always consist of a dispersive continuum and a fixed number of solitons, whose parameters evolve linearly and thus can easily be determined at any time t from the initial data. Once they are known, one can determine the data as a function of x at any time t [3]. The transform that determines the parameters of the continuum and the solitons is somewhat messy, and the inverse transform is even messier, which limits their value as a computational technique. Nonetheless, these transforms are of great utility. Not only do they allow us to immediately infer the number of solitons that will exist at large times, but they also imply that chaos cannot exist.

How do we know that a system is integrable? The most accepted evidence of integrability is to find a Lax pair, which then becomes the basis for constructing a nonlinear transform that effectively linearizes the original nonlinear equation and for constructing its inverse transform. Lax pairs are known for a large number of nonlinear integrable systems.

Integrable systems are special, but nearly integrable systems are ubiquitous. Hence, integrable systems give us a window that allows us to peer into the inner workings of the more complex systems that typically appear in nature.

The equations that describe transient stimulated Raman scattering are [4]:

$$\frac{\partial A_L}{\partial z} = -i\frac{k_L}{k_S}\kappa_2 Q A_S, \quad \frac{\partial A_S}{\partial z} = -i\kappa_2 Q^* A_L, \quad \frac{\partial Q}{\partial t} + \Gamma Q = -i\kappa_1 A_S^* A_L, \tag{1.1}$$

where A_L and A_S are respectively the amplitudes of higher and lower amplitude waves that appear in Raman scattering, k_L and k_S are the corresponding wavenumbers, Q is the off-diagonal matrix element of the material excitation, κ_1 and κ_2 are parameters that indicate the strength of the nonlinearty, and $\Gamma \equiv 1/T_2$ is the decay rate for the material excitation.

These equations play an important role in nonlinear optics, and their study — both theoretical and experimental — by Bloembergen's group at Harvard in the 1970s contributed in part to his winning the Nobel prize in Physics. Chu and Scott [5] showed in 1975 that these equations have a Lax pair when $\Gamma = 0$. They then used the standard formalism that applies when the initial data goes to zero as $z \to \pm\infty$ to show that — as expected — the initial data would decompose into a dispersive component and a set of solitons as $t \to \infty$. However, no sign of solitons is seen in either experiments or simulations when pulses are short compared to T_2, which is the limit in which systems are integrable.

Why?

It is this mystery that I set out to address in the late-1980s when I was collaborating with experimentalists at the Naval Research Laboratory that were doing experiments on transient stimulated Raman scattering (TSRS) [6]. In these experiments, the pulse durations were 40 ps, while T_2 was 633 ps; so, the experiments definitely operated in a regime in which integrable behavior was expected. Moreover, simulations in which we set $\Gamma = 0$ showed almost exactly the same behavior as simulations in which we set $T_2 = 633$ ps. As the pulse propagates, rapid oscillations in both A_L and Q are observed, and the amplitudes of both slowly decay [7].

The source of the difficulty was quickly traced to the initial-boundary conditions. In the original work of Chu and Scott [5], it was assumed that at $t = 0$, all the dependent variables, A_L, A_s, and Q tend to zero as $z \to \pm\infty$. It was then possible to solve all variables in the half-plane $t > 0$ using the standard methods of scattering transform theory [3]. However, the physically relevant boundary conditions are to set $Q = 0$ for all $z \geq 0$ at $t = 0$, to set initial data for A_L and A_S for all $t \geq 0$ at $z = 0$ and then to find all the variables in the quarter plane ($z \geq 0$, $t \geq 0$).

In the end, I was able to completely resolve the problem with a group of distinguished collaborators. We were able to show that a unique solution exists to the Cauchy problem when properly framed. We were able to find all the similarity solutions to these equations and to show that only solutions with the similarity variable zt are consistent with the initial-boundary condition. Finally, we were able to show that with initial-boundary data in an appropriate function space, the solution will ultimately tend toward P_{III}, a Painlevé-3 function, and to relate the parameters of the solution to the initial-boundary conditions in a simple way. We will summarize this work in Section 2 of this contribution.

This problem has real mathematical significance, since it falls into the class of 1+1-dimensional problems in which the initial-boundary conditions are not imposed on a line, but instead on a more complicated boundary in a plane. This set of problems has been a topic of much recent work by Fokas and his collaborators, and this problem presented a non-linear — and hence non-trivial — application of this issue and his approaches.

At this time, there do not appear to be any engineering applications of this work. However, there may be applications of the related problem in which the pump wave (A_L) and the Stokes wave (A_S) are counter-propagating. It may be possible to use the Raman effect in either gases or plasmas to generate a highly intense pulse. In this case, the appropriate initial-boundary conditions must be specified on a strip in the z-t plane, rather than on the edges of a quarter plane. In Section 3, we review the experimental work to date, which began with the verification of the original theory on the quarter plane. The mathematical problems remain open.

2. SOLUTION OF THE THE QUARTER PLANE PROBLEM

An important first step in resolving this problem was to recognize that soliton solutions are necessary transient. This result was proved by Menyuk [8] in 1989. Perhaps even more important from a fundamental standpoint was to show that the Cauchy problem is well-posed, i.e. a unique solution problem exists with properly chosen data in the quarter plane ($t \geq 0$, $z \geq 0$). This demonstration was published by Menyuk and Seidman in 1992 [9]. It should be pointed out that these two results hold for any Γ and thus apply to non-integrable as well as integrable systems.

To make progress on the integrable problem, it is useful simplify Eq. 1.1. After setting $\Gamma = 0$, the first step is to remove the constant parameters by setting:

$A_1 = (k_S/k_L)^{1/2}A_L$, $A_2 = A_S$, $X = i(k_L/k_S)^{1/2}Q$, $\chi = \kappa_2 z$, and $\tau = \kappa_1 t$. This transformation produces the equations:

$$\frac{\partial A_1}{\partial \chi} = -A_2 X, \quad \frac{\partial A_2}{\partial \chi} = A_1 X^*, \quad \frac{\partial X}{\partial \tau} = A_1 A_2^*. \tag{2.1}$$

The next step is to normalize the dependent and independent variables. We first note that $K^2(\chi,\tau) = K^2(\tau) \equiv |A_1(\chi,\tau)|^2 + |A_2(\chi,\tau)|^2$ is independent of χ. Physically, this result corresponds to conservation of the photon number. As long as A_1 and A_2 are both in $\mathcal{L}^2$, we may define a total time, $T_\infty = \int_{-\infty}^{\infty} K^2(\tau)\, d\tau$. Using T_∞, we may define the normalized independent variables:

$$\bar{\tau} = \frac{1}{T_\infty}\int_{-\infty}^{\tau} K^2(\tau')\, d\tau', \quad \bar{\chi} = T_\infty \chi, \tag{2.2}$$

and the normalized dependent variables

$$\bar{A}_1 = \frac{A_1}{K(\tau)}, \quad \bar{A}_2 = \frac{A_2}{K(\tau)}, \quad \bar{X} = \frac{X}{T_\infty}. \tag{2.3}$$

With this set of transformations, the equations governing the barred quantities are the same as Eq. 2.1. However, the new set of variables satisfy $K^2(\bar{\tau}) = 1$ in the range [0,1] and $K^2(\bar{\tau}) = 0$ elsewhere. As a consequence, with no loss of generality, we may restrict our attention to problems that satisfy this condition.

Further simplification is possible. By making appropriate transformations, it is possible to choose $A_1(\chi = 0, \tau)$ and $A_2(\chi = 0, \tau = 0)$ (when it exists) to be real.

Given Eq. 2.1, the next step is to determine the similarity solutions. This task was carried out by Levi et al. [10], who demonstrated using Lie Algebraic techniques that this equation has three types of similarity solutions: phase waves, solitons (or more generally cnoidal wave functions), and accordions (P_{III} solutions). For the phase waves, the similarity variable is χ. For the cnoidal waves, the similarity variable is $\chi - \tau/\alpha^2$, where α is a constant parameter. Finally, for the accordions, the similarity variable is $\xi = \chi\tau$. Making the variable transformation,

$$\begin{gathered} B_1(\xi) = \exp\left(\frac{1}{2} ia \ln\tau\right) A_1(\chi,\tau), \quad B_2(\xi) = \exp\left(-\frac{1}{2} ia \ln\tau\right) A_2(\chi,\tau), \\ Y(\xi) = \exp(-ia\ln\tau) X(\chi,\tau), \end{gathered} \tag{2.4}$$

and also defining $B_1 = \cos(\beta_s/2)$, $B_2 = \sin(\beta_s/2)$, we find that β_s obeys the equation

$$\frac{d^2\beta_s}{d\xi^2} + \frac{1}{\xi}\frac{d\beta_s}{d\xi} - \frac{1}{\xi}\sin(\beta_s). \tag{2.5}$$

This equation is a standard form for the Painlevé-3 function, P_{III}, and it was first discovered in the context of transient stimulated Raman scattering by Elgin and O'Hare [11].

In order to make further analytical progress with this problem, it was necessary to find a way to make use of the Lax pair. Here, earlier work by Kaup [12] proved to be immensely valuable. Not only were the initial-boundary conditions in the formulation of Chu and Scott [5] not appropriate for the physical problem of interest here, but the usual application of the Lax pair would require all fields to tend to zero as $\tau \to \pm\infty$. That is not true for the field X. Kaup resolved both these issues by first reformulating the Lax pair and then by demonstrating that we would obtain the correct solution in the time range $\tau = [0, 1]$ by simply treating X as if it equaled zero at $\tau = 1$. In this case, however, the evolution of the scattering data becomes significantly more complicated than in the usual boundary-value problems. Nonetheless, Menyuk [13] was able to obtain a remarkably simple result. He proved that the solution tends asymptotically to a P_{III} solution whose parameters are completely governed by the behavior of the initial data at $\tau = 0$. Writing, $\beta_0 = \beta(\chi = 0, \tau = 0)$ and writing the similarity variable ξ as $\xi = (\chi - \chi_{\mathrm{off}})\tau$, Menyuk showed that the asymptotic solution corresponds to

$$\beta_s = \beta_0, \quad \chi_{\mathrm{off}} = \frac{1}{\sin\beta_0} \frac{d\beta}{d\tau}\bigg|_{\chi=0,\tau=0}. \tag{2.6}$$

Later, Fokas and Menyuk [14] revisited this problem using a powerful, general method that Fokas had developed with his collaborators. The method of solution was not only more elegant than the original approach that was used by Menyuk, but, additionally, Fokas and Menyuk extended the original solution, so that it could deal with logarithmically singular data at $\tau = 0$. This extension is physically important, because it corresponds to the situation in which there is a frequency mismatch between the material excitation and the frequency difference between the pump and Stokes waves. As noted in the Introduction, this problem has mathematical importance because it is an example of a solvable nonlinear problem with non-trivial initial-boundary conditions.

3. RECENT WORK

The original experiments of Duncan et al. [6] were no longer operating at the time that the mathematical questions were resolved, and there were no tests of the theoretical predictions until recently.

The key experimental advance that made the study of transient stimulated Raman scattering possible over long distances was the invention of photonic crystal fibers with air holes that run along the entire length of the fiber and the development of methods for introducing gases into the fibers [15]. By carefully designing the fibers, it is possible to avoid second-Stokes generation — a competing physical effect that invalidates Eq. 1.1 [16]. The development of these fibers not only made it possible to observe transient stimulated Raman scattering with co-propagating waves — which is the case covered by the mathematical treatment in Section 2 [17], but they also made it possible to study counter-propagating pump and Stokes waves, which makes it possible to amplify the Stokes wave to a high intensity [18].

This work may potentially have applications to the generation of highly intense light pulses. To date, the highest-energy pulses are generated in solid-state lasers, but, at some intensity, the nonlinear crystal that supplies the necessary saturable absorption is damaged. To address this issue, Malkin et al. [19] proposed to use backward-propagation stimulated Raman scattering in plasmas. A plasma cannot be damaged by high-intensity light. This proposal has been studied experimentally by Ren et al. [20, 21]. A difficulty with this experimental approach is that it is hard to hold the plasma in place sufficiently long with sufficiently high density to create a high-intensity output.

A potential solution to this problem is to use photonic crystal fibers. The fiber both guides the light and holds the gas in place. The total pulse energy that can be achieved in this way is small, but it may be possible to achieve very high peak intensities. It may be possible to use gases for this purpose, but, additionally, it has recently become possible to create plasmas inside photonic crystal fibers [22].

These experimental results correspond to initial-boundary conditions in which the pump wave A_L would be defined along the half-line ($z = 0$, $t \geq 0$), the Stokes wave A_S would be defined along the half-line ($z = Z$, $t \geq 0$), and the material excitation Q would be defined along the line segment ($0 \leq z \leq Z$, $t = 0$), where the pump wave is injected into the optical fiber at $z = 0$, and the Stokes wave is injected into the fiber at $z = Z$. To my knowledge, a careful mathematical study of the equations that govern the behavior in either gases or plasmas with this set of initial-boundary conditions has never been undertaken. Given the current interest and potential applications, as well as the mathematical interest in studying nonlinear systems with non-trivial initial-boundary conditions, this class of mathematical systems certainly merits careful study.

ACKNOWLEDGMENTS

The problem of transient stimulated Raman scattering was my first funded research problem in nonlinear optics. I am grateful for John Reintges for giving me the opportunity to work on this set of problems, as well as Michael Duncan and Rita Mahon, who carried out the experiments with him. I am grateful to my mathematics colleagues who participated with me in resolving the issues that I encountered while working on these problems or who encouraged me to keep working on them. These include Mark Ablowitz, Jerry Bona, Thanasis Fokas, Godehard Hilfer, David Kaup, Decio Levi, Tom Seidman, and Pavel Winternitz.

The recent renewal of interest in the subject has been led by the group of Philip Russell in Erlangen, Germany. I am grateful to John Dudley for making me aware of this work, to Amir Abdolvand who carefully explained his work and has hosted several visits to Erlangen, and to Philip Russell for making those visits possible.

In its early stages, this work was supported by the Naval Research Laboratory, and it was later supported by the National Science Foundation.

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