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HAAR WAVELET APPROACH FOR SOLVING NONLINEAR DIFFERENTIAL AND INTEGRAL EQUATIONS

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ABSTRACT. Haar wavelet function is considered to be a powerful tool for solving a number of problems in numerical analysis. We apply Haar wavelet quasi-linearization approach for solving nonlinear dynamical systems governed through ordinary differential equations such as initial and boundary value problems, oscillator equations, stiff problems. We also consider nonlinear integral equations through Haar wavelet approach. The Haar solutions so obtained have been compared with counter solutions which are available in literature.
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1. Introduction

A number of dynamical systems exhibit nonlinear phenomena in terms of ordinary differential equations and integral equations. Modeling and analysis of physical phenomena in applied sciences often generates nonlinear initial and boundary value ODE's and nonlinear integral equations. Integral equations are used as mathematical models for many physical situations and also occur as reformulations of other mathematical problems. Since many physical problems are modeled by integral equations, the numerical solutions of such differential and integral equations have been highly studied by many authors. The Bratu model[1] as a nonlinear differential equation appears in a number of applications such as the fuel ignition of the thermal combustion theory and in the Chandrasekhar model of the expansion of the universe. It simulates a thermal reaction process in a rigid material where the process depends on the balance between chemically generated heat and heat transfer by conduction. Several numerical methods such as the finite difference, finite element approximation and weighted residual methods have been implemented independently to handle the Bratu model and integral equations numerically.

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Fourier analysis provides the information of composition of a given function in terms of sinusoidal waves of different frequencies and amplitudes. Whereas though wavelet analysis we can know how a given function changes from one time period to the next. Chen and Wang have solved nonlinear stiff differential equation and time varying system by Haar wavelet approach[2,3]. Bujurke[4] has given the application of single-term Haar wavelet series in the solution of nonlinear oscillator equations. Wavelet analysis is also more flexible, in that we can chose a specific wavelet to match the type of function we are analyzing. We apply quasi-linearization Technique developed by us to find out solutions to nonlinear ODEs [5,6]. However, we restrict our efforts to Haar wavelet series approach to deal with nonlinear integral equations.

The article is organized as follows. In section 2, we describe the basic formulation of Haar wavelets and the numerical scheme. Section 3 describes the solution of Bratu type initial value problem and section 4 gives the numerical treatment and solution of integral equation using Haar wavelet approximation. Section 5 contains the conclusion of the presented work.

2. **Fundamentals of Haar Wavelets and Numerical Process**

In this section, we summarize the fundamentals of Haar wavelets. The structure of Haar wavelet family is based on multiresolution analysis[7]. A multiresolution analysis(MRA) $K = \{V_j \subset L_2 | j \in J \subset \mathbb{Z}\}$ of $\mathbb{X}$ consists of a sequence of nested spaces on $V_j \subseteq V_{j+1}$ at different levels j whose union is dense in $L_2(\mathbb{R})$. Let $L_2(\mathbb{X})$ be the space of functions with finite energy defined over a domain $\mathbb{X} \subseteq \mathbb{R}^n$ and $\langle .,\rangle$ be an inner product on $\mathbb{X}$. Bases of the spaces V_j are formed by the sets of scaling basis functions $\{\phi_{j,k} | k \in \kappa(j)\}$ in complete orthonormal system, where $\kappa(j)$ is an index set defined over all basis functions on level j. The strictly nested structure of the V_j implies the existence of difference spaces W_j such that $V_j \bigoplus W_j = V_{j+1}$. The W_j are spanned by sets of Haar wavelet basis functions $\{h_{j,k} | k \in K(j)\}$. For all levels j, V_j and W_j are subspaces of V_{j+1} implying the existence of refinement relationships.
The basic and simplest form of Haar wavelet is the Haar scaling function that appears in the form of a square wave over the interval $x \in [0, 1)$, denoted with $h_1(x)$ and generally written as [8]

$$h_1(x) = \begin{cases} 1, & x \in [0,1) \\ 0, & \text{elsewhere} \qquad 0 \leqslant x \leqslant 1 \end{cases} \tag{2.1}$$

The above expression, called Haar father wavelet, is the zeroth level wavelet which has no displacement and preserves dilation of unit magnitude.
According to the concept of MRA, as an example, the space V_j can be defined as

$$V_j = span\{h_{j,k}\}_{j=0,1,2,....,2^j-1} = W_{j-1} \bigoplus V_{j-1} \tag{2.2}$$
$$= W_{j-1} \bigoplus W_{j-2} \bigoplus V_{j-2} \bigoplus .... = \bigoplus_{j=1}^{J+1} W_j \bigoplus V_0$$

The Haar mother wavelet is the first level Haar wavelet that can be written as the linear combination of the Haar scaling function as

$$h_2(x) = h_1(2x) + h_1(2x+1) \tag{2.3}$$

Each Haar wavelet is composed of a couple of constant steps of opposite sign during its subinterval and is zero elsewhere. The term wavelet is used to refer to a set of orthonormal basis functions generated by dilation and translation of a compactly supported scaling function $h_1(x)$ (father wavelet) and a mother wavelet $h_2(x)$ associated with multiresolution analysis of $L_2(\mathbb{R})$. Thus, we can write the Haar wavelet family as

$$h_i(x) = h_i(2^j x - k) = \begin{cases} 1, & \frac{k}{2^j} \leq x < \frac{k+0.5}{2^j} \\ -1, & \frac{k+0.5}{2^j} \leq x < \frac{k+1}{2^j} \\ 0, & \text{elsewhere} \end{cases} \tag{2.4}$$

The Haar wavelet matrix of order 8×8 is given by

$$H = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$

where $0 \leqslant x \leqslant 1$ and for $i \geq 2, i = 2^j + k + 1, j \geqslant 0, 0 \leqslant k \leqslant 2^j - 1$ *and* $x_l = \frac{l-\frac{1}{2}}{2m}, l = 1, 2, ...2m.$ and $m = 2^j(j = 0, 1, ...J)$. Here J indicates the max. level of resolution. In case of minimal values $m = 1, k = 0$ then $i = 2$. For any fixed level m, there are m series of i to fill the interval corresponding to that level and for a provided J, the index number i can reach the maximum value $M = 2^{J+1}$, when including all levels of wavelets.

The operational matrix $P_{i,\alpha}(x)$ of order $2m \times 2m$ is derived from integration of Haar wavelet family with the aid of following formula:

$$P_{i,\alpha}(x) = \int_A^x \int_A^x \dots \int_A^x h_i(x)dx^\alpha = \frac{1}{(\alpha-1)!}\int_A^t (t-x)^{\alpha-1}h_i(x)dx \tag{2.5}$$

$$P_{i,2}(x) = \begin{cases} \frac{1}{2}(x-\frac{k}{2^j})^2, & x \in [\frac{k}{2^j}, \frac{k+0.5}{2^j}) \\ \frac{1}{4m^2} - \frac{1}{2}(x-\frac{k}{2^j})^2, & x \in [\frac{k+0.5}{2^j}, \frac{k+1}{2^j}) \\ \frac{1}{4m^2}, & x \in [\frac{k+0.5}{2^j}, 1) \\ 0, & \text{elsewhere} \end{cases} \tag{2.6}$$

The Haar wavelet function $y(x_l) \in L_2[0,1]$ may be expanded as

$$y(x_l) = \sum_{i=1}^{\infty} a_i h_i(x_l), \ i \in N \tag{2.7}$$

The orthogonality property puts a strong limitation on the construction of wavelets and allows us to transform any square integral function on the interval time $[0,1)$ into Haar wavelets series as

$$y(x_l) = a_1 h_1(x_l) + \sum_{i=2}^{\infty} a_i h_i(x_l), \ x_l \in [0,1] \tag{2.8}$$

Similarly the highest derivative can be written as wavelet series $\sum\limits_{i=1}^{\infty} a_i h_i(x_l)$. In applications, Haar series is always truncated to $2m$ terms[2], that is

$$\sum_{i=1}^{2m} a_i h_i(x_l) = a^T H_{2m}(x_l) \tag{2.9}$$

then we have used the quasi-linearization process. The quasi-linearization process is an application of the Newton Raphson Kantrovich approximation method in function space given by Bellman and Kalaba [9]. The idea and advantage of the method is based on the fact that linear equations can often be solved analytically or numerically while there are no useful techniques for obtaining the general solution of a nonlinear equation in terms of a finite set of particular solutions. Consider an n^{th}

order nonlinear ordinary differential equation

$$L^n y(x) = f(y(x), y^{(1)}(x), y^{(2)}(x), ..., y^{(n-1)}(x), x) \tag{2.10}$$

with the initial conditions

$$y(0) = \lambda_0, y^{(1)}(0) = \lambda_1, ..., y^{(n)}(0) = \lambda_n \tag{2.11}$$

Here L^n is the linear n^{th} order ordinary differential operator, f is nonlinear function of $y(x_l)$ and its derivatives are $y^{(s)}, s = 0, 1, 2, ..., n-1$.
The quasi-linearization prescription determines the $(r+1)^{th}$ iterative approximation to the solution of Eq.(2.10) and its linearized form is given by Eq.(2.12)

$$\begin{aligned} L^n y_{r+1}(x) &= f(y_r(x), y_r^{(1)}(x), y_r^{(2)}(x), ..., y_r^{(n-1)}(x), x) \\ &+ \sum_{s=0}^{n-1} (y_{r+1}^{(s)} - y_r^{(s)}) f_y^s(y_r(x), y_r^{(1)}(x), y_r^{(2)}(x), ..., y_r^{(n-1)}(x), x) \end{aligned} \tag{2.12}$$

where $y_r^0(x) = y_r(x)$. The functions $f_{y^{(s)}} = \frac{\partial f}{\partial y^{(s)}}$ are functional derivatives of the functions. The $zero^{th}$ approximation $y_0(x)$ is chosen from mathematical or physical considerations.

Now $y(x)$ at $(r+1)^{th}$ level can be written in the following form of Haar wavelet series as [10,11,12].

$$y_{r+1}^{(n)}(x_l) = \sum_{i=1}^{2m} a_i h_i(x_l) \tag{2.13}$$

Then using the concept of operational matrix and Haar wavelet technique, we can obtain the derivatives from following equation

$$y_{r+1}^{(0)}(x_l) = \sum_{i=1}^{2m} P_{i,n}(x_l) + x_l^n y_r^{(n-1)}(0) + x_l^{n-1} y_r^{(n-2)}(0) + ... + y_{r+1}^{(0)}(0) \tag{2.14}$$

3. Solutions of Nonlinear Differential Equations

Example 1. Consider the nonlinear boundary value problem

$$\frac{d^2y}{dx^2} = \frac{3}{2}y^2(x),\ y(0) = 1, y(1) = 4 \tag{3.1}$$

Numerical results for this problem are presented in Figure 1.

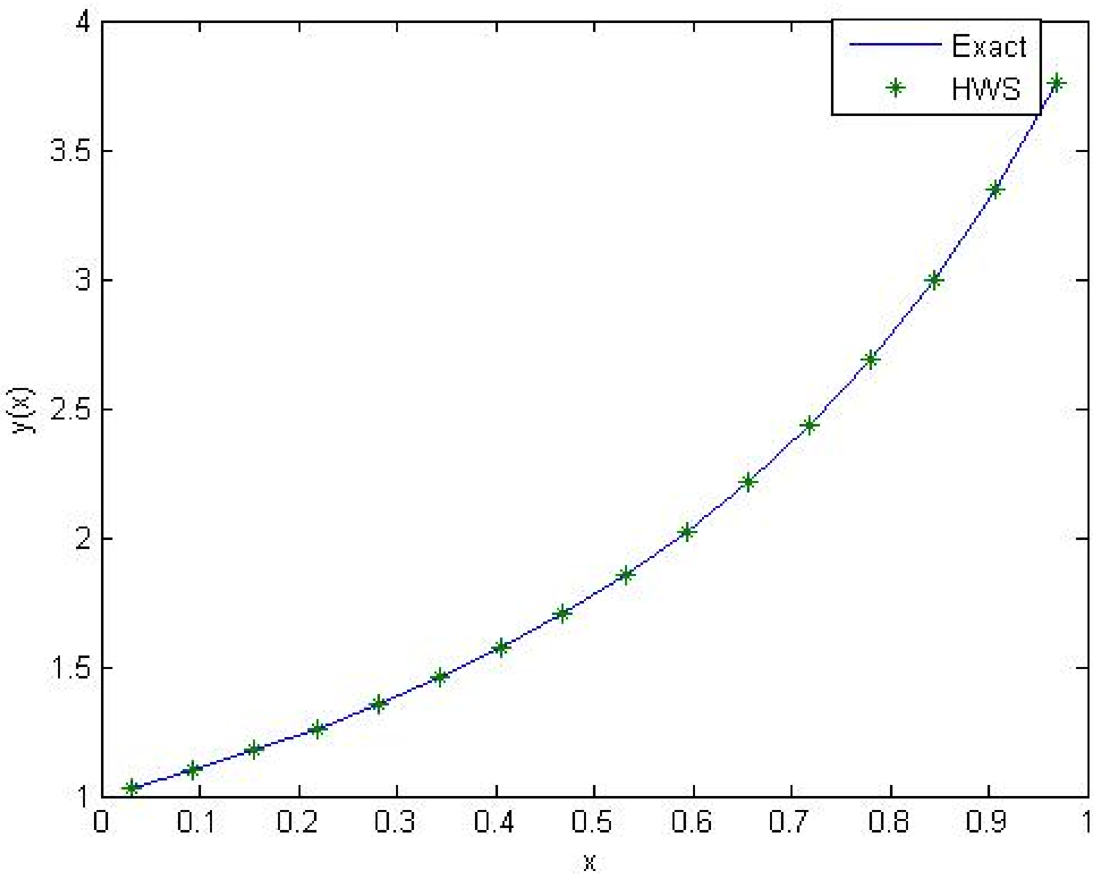

Figure 1. Plot of Example1 for $m = 16$

Example 2. Consider the well known Bratu type boundary value problem in one-dimensional planar coordinates of the following form [1].

$$\frac{d^2y}{dx^2} - 2e^y = 0, y(0) = y^{'}(0) = 0 \tag{3.2}$$

The standard Bratu problem (3.16) was used to model a combustion problem in a numerical slab. The Bratu model appears in a number of applications such as the fuel ignition of the thermal combustion theory. Exact solution of this problem is :$y(x) = -2\log(\cos(x))$ and Haar wavelet solution is shown in Figure 2.

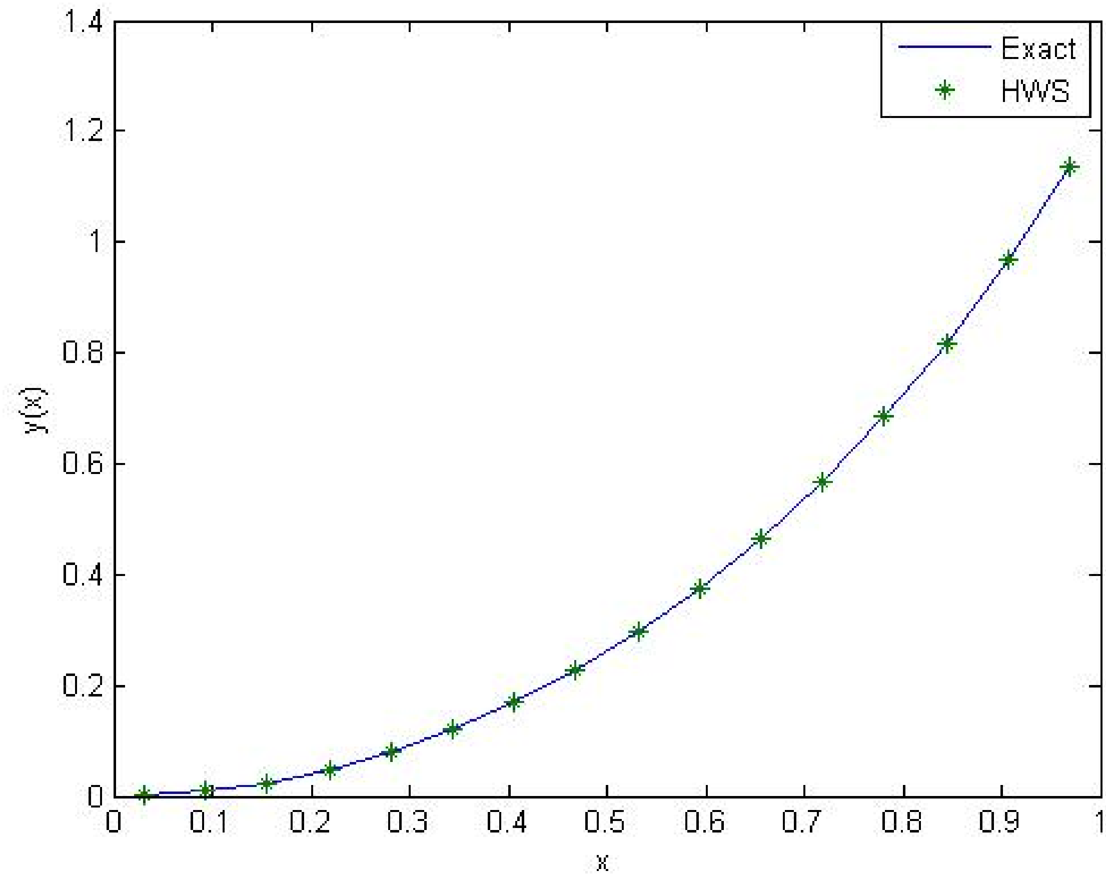

FIGURE 2. Plot of Example2 for $m = 16$

4. **Solutions of Nonlinear Integral Equations**

Consider Fredholm integral equation of the second kind

$$u(t) = \int_0^1 k(t,x)[u(x)]^p dx + f(t),\ \ p > 1, t \in [0,1) \tag{4.1}$$

The nonlinear term can be expressed as

$$[u(x)]^p = [h^T(x)c]^p = h^T(x)c_p \tag{4.2}$$

Here in c_p^T = Column vector whose elements are nonlinear combinations of elements of vector c. c_p may be termed as operational vector of the p^{th} power of function $u(x)$.

$$k(t,x) = h^T(t)Kh(x), K = K_{i,j_{0<i,j<2^{j-1}}},\ \ K_{i,j} =< h_i(t), < k(t,x), h_j(t) >> \tag{4.3}$$

Let$f(t) = h^T(t)d$ be known. From given equation

$$h^T(t)c = h^T(t)Kc_p + h^T(t)d \tag{4.4}$$

Taking inner product with $h(t)$ we find nonlinear system of equations

$$c - kc_p = d \tag{4.5}$$

which can be solved algebraically.

Example 3. Consider the nonlinear weakly singular Volterra-Hammerstein integral equation with algebraic nonlinearity and singular point $x = 0$.

$$u(x) = -\frac{x^4}{10} + \frac{5}{6}x^2 + \frac{3}{8} + \int_0^x \frac{1}{2x}u^2(t),\ \ x \in [0,1] \tag{4.6}$$

Exact solution of this equation is given in [13]. Haar wavelet solution is obtained by proposed method and is shown in Figure 3.

5. **Conclusion**

For nonlinear problems the Haar wavelet quasi-linearization approach is adopted. The cited examples clearly demonstrate that in solving nonlinear differential equations the Haar wavelet method can successfully compete with the exact solutions. The Haar wavelet method is also applicable for certain nonlinear integral equations. The main benefits of the Haar approach are simplicity (as a small number of grid points

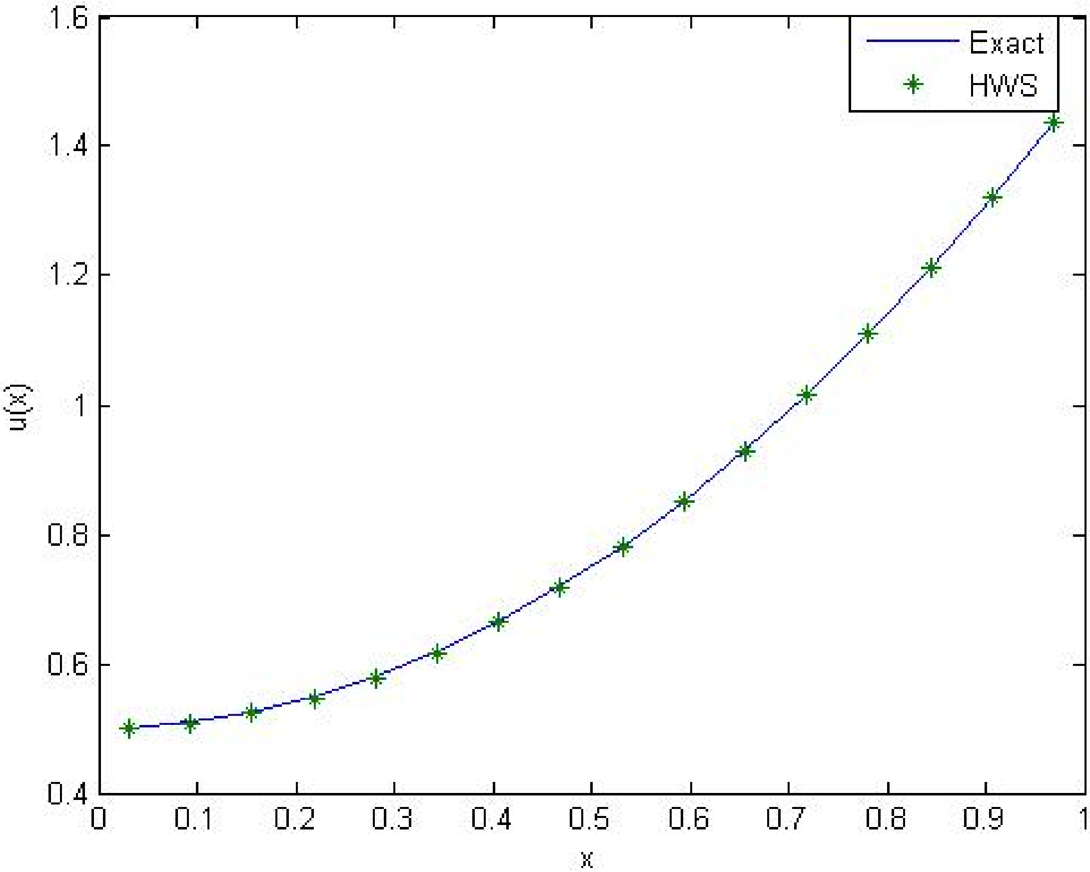

FIGURE 3. Plot of Example3 for $m = 16$

guarantee the necessary accuracy) and universality (as almost the same approach is applicable for a wide class of higher order differential and integral equations).

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