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DCP PROPERTY OF CONVEX COMBINATIONS OF DE LA VALLÉE POUSSIN KERNELS

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ABSTRACT: Let $\mathcal{C}(\phi)$ denote the set of all univalent functions in the unit disk $\mathbb{D}$ which are convex in the direction $e^{i\,\phi}$. A function g analytic in the unit disk $\mathbb{D}$ is said to be in the class DCP (Directional Convexity Preserving) if it preserves the class $\mathcal{C}(\phi)$ under the Hadamard product, i.e. g belongs to the class DCP if $f * g \in \mathcal{C}(\phi)$ whenever $f \in \mathcal{C}(\phi)$. It has been proved in the literature that some well known and most applicable functions of a complex variable like exponential function e^{rz} for $0 < r \leq 1$ belongs to the class DCP. In this paper we further enlarge this class by establishing a criterion for closed convex hull of the de la Vallée Poussin kernels $V_\lambda(z) = \frac{\lambda z}{\lambda+1}\,{}_2F_1(1, 1-\lambda; 2+\lambda; -z),\ z \in \mathbb{D}$ to be in the class DCP.

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1. Introduction and Characterization of the class DCP

Definition 1.1. A domain $\Omega \subset \mathbb{C}$ is said to be a *starlike domain* with respect to a point $z_0 \in \Omega$ if the line segment joining z_0 to every other points $z \in \Omega$ lies entirely in Ω. Goemetrically the requirement is that every point of Ω be visible from z_0.

Definition 1.2. A domain $\Omega \subset \mathbb{C}$ is said to be a *convex domain* if it is starlike with respect to each of its point. In other words the domain Ω is convex if the line segments joining any two of its points lie entirely in Ω.

Definition 1.3. A domain $\Omega \subset \mathbb{C}$ is said to be *convex in the direction* $e^{i\phi}$, $\phi \in \mathbb{R}$, if and only if for every $a \in \mathbb{C}$ the set

$$\Omega \cap \left\{a + te^{i\phi}\ :\ t \in \mathbb{R}\right\}$$

is either connected or empty.

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Definition 1.4. Let $\mathcal{A}$ denote the set of all analytic functions in the unit disk $\mathbb{D} = \{z \; : |z| < 1\}$. Then we define some standars subclasses of $\mathcal{A}$ as follows:

$$\begin{aligned}
\mathcal{S} &= \{f \in \mathcal{A} \; : f \text{ is univalent and } f(0) = 0, \; f'(0) = 1\} \\
\mathcal{C} &= \{f \in \mathcal{S} \; : f(\mathbb{D}) \text{ is convex}\} \\
\mathcal{S}^* &= \{f \in \mathcal{S} \; : f(\mathbb{D}) \text{ is starlike with respect to the origin}\} \\
\mathcal{K}(\phi) &= \left\{f \in \mathcal{S} \; : f(\mathbb{D}) \text{ is convex in the direction } e^{i\phi}, \; \phi \in \mathbb{R}\right\}
\end{aligned}$$

Remark 1.5. It is well known that if $f \in \mathcal{C}$, then f maps each circle $|z| = r < 1$ on to a closed curve Γ which bounds a convex domain. In other words if $f \in \mathcal{C}$, then $g(z) = f(r\,z) \in \mathcal{C}$ for $0 < r < 1$. Similar result holds also for the class $\mathcal{S}^*$. That is if $f \in \mathcal{S}^*$, then f maps each circle $|z| = r < 1$ on to a closed curve Γ which bounds a starlike(with respect to the origin) domain, i.e. $g(z) = f(r\,z)$ is also starlike for $0 < r < 1$. But unlike the classes $\mathcal{C}$ and $\mathcal{S}^*$, the class $\mathcal{K}(\phi)$ does not entertain this property. In other words $f \in \mathcal{K}(\phi)$ does not necessarily imply $f(r\,z) \in \mathcal{K}(\phi)$ for $0 < r < 1$.

Remark 1.6. (Goodman - Saff Conjecture) In [3] and [5] it has been proved that for $r_0 := \sqrt{2} - 1 < r < 1$ generally $f \in \mathcal{K}(\phi)$ does not imply $f(r\,z) \in \mathcal{K}(\phi)$. But Goodman and Saff conjectured that such an implication may hold for $0 < r \leq r_0$. In 1987 J. Brown [2] proved that

$$f \in \mathcal{K}(\phi) \Rightarrow f(r\,z) \in \mathcal{K}(\psi), \; \psi \in \mathrm{I}(f),$$

where $I(f) \subset [0, 2\pi]$ is a set of positive measure. But unfortunately it was not shown that $\phi \in \mathrm{I}(f)$ and thus the Goodman - Saff conjecture was still open at that time.

In order to solve the Goodman-Saff conjecture Ruscheweyh and Salinas [9] introduced the class of functions which preserve the directional convexity under the Hadamard product.

Definition 1.7. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ be two functions in $\mathcal{A}$. Then the Hadamard product of $f(z)$ and $g(z)$, denoted by $(f * g)(z)$, is defined by

$$(f * g)(z) := \sum_{n=0}^{\infty} a_n b_n z^n \tag{1.1}$$

Definition 1.8. A function $g \in \mathcal{A}$ is called a Direction-Convexity-Preserving ($g \in DCP$) if and only if for every $\phi \in \mathbb{R}$ and for every $f \in \mathcal{K}(\phi)$, the function $g * f$ is convex in the direction $e^{i\,\phi}$.

Remark 1.9. Beside the members of $\mathcal{K}(\phi)$, we also use the term dirction convex or convex in the direction $e^{i\,\phi}$ also for not normalized functions of the class $\mathbb{A}$.

Remark 1.10. Another remark on the class $\mathcal{K}(\phi)$ is that unlike some other subclasses of $\mathcal{S}$ the class $\mathcal{K}(\phi)$ is not rotational invarient. That is $f \in \mathcal{K}(\phi)$ and $\alpha \neq 0, \pi (\mathrm{mod}\, 2\pi)$ do not always imply that $e^{i\alpha} f(e^{-i\,\alpha} z) \in \mathcal{K}(\phi)$. As a result $f \in DCP$ does not in general imply that the function $e^{i\,\alpha} g(z)$ is also in DCP.

In [9] one can finds a complete description of the members of DCP, namely

$$g \in DCP \quad \Longleftrightarrow \quad g(z) + i\,t\,z\,g'(z) \in \mathcal{K}(\frac{\pi}{2}) \text{ for all } t \in \mathbb{R}. \tag{1.2}$$

Further it is known that DCP functions are convex univalent. The following criterion for membership in DCP is a slight variant of [9, Theorem 4] (compare [11])

Lemma 1.11. *Let g be analytic in $\overline{\mathbb{D}}$, convex univalent and let $u(t) := Re\,(g(e^{it}))$, $t \in \mathbb{R}$. Then $g \in DCP$ if and only if*

$$\sigma_u := (u''(t))^2 - u'(t)\, u'''(t) \geq 0\,, \quad t \in \mathbb{R}\,. \tag{1.3}$$

Remark 1.12. The class DCP is not an isolated one but has a bearing on the geometric function theory and has been used to prove various results in this field of mathematics, for instance the preservation of convex harmonic functions in $\mathbb{D}$, and of Jordan curves in the plane with convex interior domain. It has been proved in the literature that some well known and most applicable functions of a complex variable like exponential function e^{rz} for $0 < r \leq 1$ belongs to the class DCP. We refer to [7], [9], [10] for more details.

2. DCP property of de la VallÉe Poussin Kernels

In this section we further enlarge the class DCP by establishing a criterion for closed convex hull of the de la Vallée Poussin kernels $V_\lambda(z) = \frac{\lambda z}{\lambda+1}\, {}_2F_1(1, 1-\lambda; 2+\lambda; -z)$, $z \in \mathbb{D}$ to be in the class DCP. The classical definition of the de la Vallée Poussin kernel of order $n \in \mathbb{N}$ is

$$\begin{aligned} \omega_n(t) &:= \frac{2^n (n!)^2}{(2n)!} (1 + \cos(t))^n \\ &= \frac{1}{\binom{2n}{n}} \sum_{k=-n}^{n} \binom{2n}{n+k} e^{ikt}. \end{aligned} \tag{2.1}$$

But here we are interested in the analytic version of the de la Vallée Poussin kernel

$$V_n(z) = \frac{1}{\binom{2n}{n}} \sum_{k=1}^{n} \binom{2n}{n+k} z^k, \quad z \in \mathbb{C}. \tag{2.2}$$

Note that

$$2\text{Re } V_n(e^{it}) = \omega_n(t) - 1, \quad n \in \mathbb{N}. \tag{2.3}$$

Ruscheweyh and Suffridge [12] extended the range of the parameter $n \in \mathbb{N}$ of the de la Vallée Poussin kernel to include all positive real numbers λ. We collect some of their results in the following lemma.

Lemma 2.1. (a) *For $\lambda > 0$, we have*

$$V_\lambda(z) = \frac{\lambda z}{\lambda + 1}\, {}_2F_1(1, 1-\lambda; 2+\lambda; -z), \quad z \in \mathbb{D}. \tag{2.4}$$

Furthermore, these functions extend continuously onto $\overline{\mathbb{D}}$, and we have

$$\begin{aligned} w_\lambda(t) &= \text{Re } V_\lambda(e^{it}) \\ &= -\frac{1}{2} + 2^{\lambda-1}\frac{(\Gamma(\lambda+1))^2}{\Gamma(2\lambda+1)}(1+\cos(t))^\lambda, \ t \in \mathbb{R}. \end{aligned} \tag{2.5}$$

(b) *For $\lambda > 0$, $V_\lambda(z)$ is analytic and convex univalent in $\mathbb{D}$.*

(c) *For $\lambda \geq \frac{1}{2}$, $w_\lambda(t)$ is strictly periodically monotone and three times continuously differentiable. Moreover,*

$$w''_\lambda(t)\, w''_\lambda(t) - w'''_\lambda(t)\, w'_\lambda(t) \geq 0, \quad t \in \mathbb{R}. \tag{2.6}$$

(d) *$V_\lambda(z)$ is in DCP for $\lambda \geq \frac{1}{2}$.*

In this section, we give a criterion for closed convex hulls of the de la Vallée Poussin kernels $V_\lambda(z)$ to be in DCP. But first we shall prove a result involving $\omega_\lambda(t)$, which we know from **Lemma 2.1** to be three times continuously differentiable for $\lambda \geq \frac{1}{2}$.

Lemma 2.2. *For $\lambda_k, \lambda_j \geq \frac{1}{2}$, set*

$$w_{jk}(t) = w''_{\lambda_k}(t)\, w''_{\lambda_j}(t) - \frac{1}{2}(w'''_{\lambda_k}(t)\, w'_{\lambda_j}(t) + w'_{\lambda_k}(t)\, w'''_{\lambda_j}(t)), \quad t \in \mathbb{R}. \tag{2.7}$$

Then the following hold:

(a) *Suppose $\lambda_k = \lambda_j \geq \frac{1}{2}$. Then*

$$w_{jk}(t) \geq 0, \quad t \in \mathbb{R}. \tag{2.8}$$

(b) *Suppose $\lambda_k \geq \lambda_j \geq \frac{1}{2}$. Then* (2.8) *holds if and only if*

$$\lambda_j \leq \lambda_k \leq \lambda_j + \frac{1}{4}\left(1 + \sqrt{16\lambda_j - 7}\right). \tag{2.9}$$

Proof: Part (a) is a particular case of part(b), and it follows directly from part(c) of **Lemma 2.1** if we take $\lambda = \lambda_k = \lambda_j$ in (2.6). To prove part (b), we first note that if we set $A = \lambda_k - \lambda_j$, then (2.9) holds if and only if A lies in the closed interval

$$\left[0,\ \frac{1}{4}\left(1 + \sqrt{16\lambda_j - 7}\right)\right]. \tag{2.10}$$

Hence, it is enough to prove that (2.8) holds if and only if A lies in the above closed interval. Now after simplification (using Mathematica), we get

$$\begin{aligned} w_{j\,k}(t) &= -K\{3\,\lambda_j^2 + 3\,\lambda_k^2 - 6\,\lambda_j\,\lambda_k - 2\,\lambda_J - 2\,\lambda_k \\ &\quad +2\,(-2\,\lambda_j^2 - 2\,\lambda_k^2 + 4\,\lambda_j\lambda_k + \lambda_j + \lambda_k - 2)\,\cos(t) \\ &\quad +(\lambda_j - \lambda_k)^2\,\cos(2\,t)\}, \end{aligned}$$

where

$$\begin{aligned} K &= \frac{1}{16}\frac{2^{\lambda_k+\lambda_j-2}\,\Gamma\,(\lambda_k+1)^2\,\Gamma\,(\lambda_j+1)^2}{\Gamma\,(2\,\lambda_k+1)\,\Gamma\,(2\,\lambda_j+1)} \\ &\quad \left(\lambda_k\,\lambda_j\,(1+\cos(t))^{\lambda_k+\lambda_j}\sec^4(\frac{t}{2})\right). \end{aligned}$$

Clearly K takes non-negative values for $t \in \mathbb{R}$. Hence $w_{j\,k}(t) \geq 0$ if and only if

$$\begin{aligned} &3\,\lambda_j^2 + 3\,\lambda_k^2 - 6\,\lambda_j\,\lambda_k - 2\,\lambda_j - 2\,\lambda_k \\ &\quad +2\,(-2\,\lambda_j^2 - 2\,\lambda_k^2 + 4\,\lambda_j\,\lambda_k + \lambda_j + \lambda_k - 2)\,\cos(t) \\ &\quad +(\lambda_j - \lambda_k)^2\cos(2t) \leq 0. \end{aligned}$$

If we now set $\lambda_k = A + \lambda_j$ and $y = \cos(t)$ in the above inequality, we get, after simplification,

$$-A + A^2 - 2\,\lambda_j + (A - 2 - 2\,A^2 + 2\,\lambda_j)\,y + A^2\,y^2 \leq 0.$$

The left side is a quadratic expression, say $F(y)$, in y, and $y = \cos(t)$ takes every value between -1 and 1. Let

$$\mathcal{X} = \{A \geq 0 : F(y) \leq 0 \text{ for } -1 \leq y \leq 1\}.$$

Then $w_{j\,k}(t) \geq 0$ for all real values of t if and only if $A = \lambda_k - \lambda_j$ lies in $\mathcal{X}$. Now $F(y)$ is linear in y if $A = 0$ and is quadratic in y with positive leading coefficient if $A > 0$. Hence

$$\mathcal{X} = \{A \geq 0 : F(1) \text{ and } F(-1) \leq 0\}.$$

It is easily seen that $F(1) = -2$. Hence

$$\mathcal{X} = \{A \geq 0 : F(-1) \leq 0\}.$$

Now

$$\begin{aligned} F(-1) &= 4\left(A^2 - \frac{1}{2}\,A + \frac{1}{2} - \lambda_j\right) \\ &= 4\left(A - \frac{1}{4}(1 - \sqrt{16\lambda_j - 7})\right) \\ &\quad \left(A - \frac{1}{4}(1 + \sqrt{(16\lambda_j - 7)})\right). \end{aligned}$$

By hypothesis, $\lambda_j \geq \frac{1}{2}$, and $A \geq 0$ by our assumption. If $A = 0$, then $F(-1) \leq 0$ and so $0 \in \mathcal{X}$. If $A > 0$, then $A - \frac{1}{4}(1 - \sqrt{16\lambda_j - 7}) > 0$ and hence

$$F(-1) \leq 0 \quad \Leftrightarrow \quad A - \frac{1}{4}(1 + \sqrt{16\lambda_j - 7}) \leq 0$$
$$\Leftrightarrow \quad A \leq \frac{1}{4}(1 + \sqrt{16\lambda_j - 7}).$$

We thus see that $\mathcal{X}$ is the closed interval represented by (2.10). Hence $w_{j\,k}(t) \geq 0$ for all $t \in \mathbb{R}$ if and only if A lies in the closed interval (2.10), that is if and only if (2.9) holds. This completes the proof of part (b).

Theorem 2.3. *For $\lambda \geq \frac{1}{2}$, we have*

$$\mathcal{F}_\lambda := \overline{\text{co}} \left\{ V_{\lambda_k}(z) \,:\, \lambda \leq \lambda_k \leq \lambda + \frac{1}{4}\left(1 + \sqrt{16\lambda - 7}\right) \right\} \subseteq DCP.$$

Proof: The set $\mathcal{F}_\lambda$ is by definition compact. Hence in order to prove the theorem, it suffices to show that the finite sums

$$V = \sum_{k=1}^{n} \mu_k V_{\lambda_k} \in DCP$$

for $\mu_k \geq 0$. Now V is analytic in $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$, because, by **Lemma 2.1**, the functions V_{λ_k} have the same properties. Set

$$u(t) := \text{Re}\; V(e^{it}) = \sum_{k=1}^{n} \mu_k \text{Re}\, V_{\lambda_k}(e^{it}) = \sum_{k=1}^{n} \mu_k\, w_{\lambda_k}(t).$$

Again by **Lemma 2.1**, $w_{\lambda_k}(t)$ is three times continuously differentiable and strictly periodically monotone, with the function decreasing on $(0,\ \pi)$ and increasing on $(\pi,\ 2\pi)$. Hence $u(t)$ is also three times continuously differentiable. Also, since $\mu_k \geq 0$, $u(t)$ is strictly periodically monotone. Therefore, by **Lemma 1.11** $V \in DCP$ if and only if

$$u''(t)\, u''(t) - u'''(t)\, u'(t) \geq 0, \quad t \in \mathbb{R}. \tag{2.11}$$

Now

$$\begin{aligned}(u''(t))^2 - u'''(t)\, u'(t) &= \left(\sum_{k=1}^{n} \mu_k\, w''_{\lambda_k}(t)\right)^2 \\ &\quad - \left(\sum_{k=1}^{n} \mu_k\, w'''_{\lambda_k}(t)\right)\left(\sum_{k=1}^{n} \mu_k\, w'_{\lambda_k}(t)\right) \\ &= \sum_{j,k=1}^{n} \mu_k\, \mu_j\, w_{j\,k}(t),\end{aligned} \tag{2.12}$$

where

$$w_{j\,k}(t) = w''_{\lambda_k}(t)\, w''_{\lambda_j}(t) - \frac{1}{2}(w'''_{\lambda_k}(t)\, w'_{\lambda_j}(t) + w'_{\lambda_k}(t)\, w'''_{\lambda_j}(t)). \tag{2.13}$$

Since $\lambda \le \lambda_k,\ \lambda_j \le \lambda + \frac{1}{4}\left(1+\sqrt{16\lambda-7}\right)$, we have $w_{jk} \ge 0$ by **Lemma 2.1**. Also $\mu_k, \mu_j \ge 0$. Hence (2.11) holds, thus completing the proof of the theorem.

Certainly the requirement

$$\sum_{j,\,k=1}^{n} \mu_k \mu_j\, w_{j\,k}(t) \ge 0 \text{ for } \mu_j,\ \mu_k \ge 0 \tag{2.14}$$

can hold even if not all $w_{j\,k}(t) \ge 0$ on $[0, 2\pi]$. One can expect a sharp bound (necessary condition) for inequality (2.14). Unfortunately, because of the very complicated nature of the functions $w_{j\,k}$, it is not easy to get a sharp condition. However, the following lemma will help to improve the previous result under certain conditions.

Lemma 2.4. *Let ϵ_k be numbers in $(0,1)$ such that $\sum_{k=1}^{n} \epsilon_k = 1$ and*

$$\frac{\epsilon_k}{1-\epsilon_j}\, \mu_j^2\, w_{j\,j}(t) + 2\,\mu_j\,\mu_k\,w_{j\,k}(t) + \frac{\epsilon_j}{1-\epsilon_k}\,\mu_k^2\,w_{k\,k}(t) \ge 0 \tag{2.15}$$

for $j \ne k$ and $t \in \mathbb{R}$. Then $\sum_{j,\,k=1}^{n} \mu_j\,\mu_k\,w_{j\,k} \ge 0$.

Proof: By (2.15), we have

$$\begin{aligned}
0 &\le \sum_{\substack{j,k=1\\ j\ne k}}^{n} \left(\frac{\epsilon_k}{1-\epsilon_j}\,\mu_j^2\,w_{jj} + 2\,\mu_j\,\mu_k\,w_{jk} + \frac{\epsilon_j}{1-\epsilon_k}\,\mu_k^2\,w_{kk} \right) \\
&= 2\sum_{\substack{j,k=1\\ j\ne k}}^{n} \left(\frac{\epsilon_k}{1-\epsilon_j}\right)\mu_j^2\,w_{jj} \;+\; 2\sum_{\substack{j,k=1\\ j\ne k}}^{n} \mu_j\,\mu_k\,w_{jk} \\
&= 2\sum_{j=1}^{n} \left(\sum_{\substack{k=1\\ j\ne k}}^{n} \epsilon_k \right) \left(\frac{\mu_j^2}{1-\epsilon_j} \right) w_{jj} + 2\sum_{\substack{j,k=1\\ j\ne k}}^{n} \mu_j\,\mu_k\,w_{jk} \\
&= 2\sum_{j=1}^{n} \mu_j^2\,w_{jj} + \;2\sum_{\substack{j,k=1\\ j\ne k}}^{n} \mu_j\,\mu_k\,w_{jk} \\
&= 2\left(\sum_{j=1}^{n} \mu_j^2\,w_{jj} + \sum_{\substack{j,k=1\\ j\ne k}}^{n} \mu_j\,\mu_k\,w_{jk} \right).
\end{aligned}$$

Hence

$$\sum_{j,k=1}^{n} \mu_j\,\mu_k w_{jk} = \sum_{j=1}^{n} \mu_j^2\,w_{jj} + \sum_{\substack{j,k=1\\ j\ne k}}^{n} \mu_j\,\mu_k\,w_{jj} \ge 0$$

and the lemma is proved.

Theorem 2.5. *Let $\lambda_k \geq \frac{1}{2}$, $k = 1, 2, \cdots, n$. Suppose there exists numbers $\epsilon_k \in (0,1)$ such that $\sum_{k=1}^{n} \epsilon_k = 1$ and*

$$\omega_{j\,k}(t) \geq -\sqrt{\frac{\epsilon_j\,\epsilon_k\,\omega_{j\,j}(t)\,\omega_{k\,k}(t)}{(1-\epsilon_j)(1-\epsilon_j)}} \tag{2.16}$$

for $t \in \mathbb{R}$ and $j \neq k$. Then

$$\sum_{k=1}^{n} \mu_k V_{\lambda_k}(z) \in DCP \tag{2.17}$$

for all $\mu_k \geq 0$.

Proof: As in Theorem 2.3, the function

$$\sum_{k=1}^{n} \mu_k\, V_{\lambda_k}(z) \in DCP$$

if and only if

$$\sum_{j,k=1}^{n} \mu_j\, \mu_k\, w_{j\,k}(t) \geq 0. \tag{2.18}$$

And by **Lemma 2.4**, the above inequality holds if

$$\frac{\epsilon_k}{1-\epsilon_j}\,\mu_j^2\, w_{j\,j} + 2\,\mu_j\,\mu_k\, w_{j\,k} + \frac{\epsilon_j}{1-\epsilon_k}\,\mu_k^2\, w_{k\,k} \geq 0 \tag{2.19}$$

for $j \neq k$. It was shown in **Lemma 2.2** that w_{jj}, $w_{kk} \geq 0$. Hence if $\mu_k = 0$ or if $w_{jk} \geq 0$, then (2.19) is obvious. If $w_{jj} = 0$, then $w_{jk} \geq 0$ by (2.16) and (2.19) again holds. So suppose $w_{jj} > 0$, $\mu_k > 0$, and $w_{jk} < 0$. Then by (2.16),

$$|w_{j\,k}(t)| \leq \sqrt{\frac{\epsilon_j\,\epsilon_k\, w_{j\,j}(t)\, w_{k\,k}(t)}{(1-\epsilon_j)(1-\epsilon_k)}}. \tag{2.20}$$

Also, (2.19) holds if and only if

$$\left(\frac{\mu_j}{\mu_k}\right)^2 + 2\,\frac{(1-\epsilon_j)\, w_{j\,k}}{\epsilon_k\, w_{j\,j}} \left(\frac{\mu_j}{\mu_k}\right) + \frac{\epsilon_j\,(1-\epsilon_j)\, w_{k\,k}}{\epsilon_k\,(1-\epsilon_k)\, w_{j\,j}} \geq 0. \tag{2.21}$$

Let Δ denote the discriminant of the quadratic equation

$$X^2 + 2\,\frac{(1-\epsilon_j)\, w_{j\,k}}{\epsilon_k\, w_{j\,j}} X + \frac{\epsilon_j\,(1-\epsilon_j)\, w_{k\,k}}{\epsilon_k\,(1-\epsilon_k)\, w_{j\,j}} = 0.$$

Then

$$\frac{1}{4}\,\Delta = \left(\frac{(1-\epsilon_j) w_{j\,k}}{\epsilon_k\, w_{j\,j}}\right)^2 - \frac{\epsilon_j (1-\epsilon_j) w_{k\,k}}{\epsilon_k (1-\epsilon_k) w_{j\,j}} \leq 0,$$

by (2.20). This proves (2.21) and with it the theorem.

If we take $n = 2$ in Theorem 2.5, then $1 - \epsilon_1 = \epsilon_2$ and $1 - \epsilon_2 = \epsilon_1$ and we obtain the following corollary.

Corollary 2.6. *Let* $\lambda_j, \lambda_k \geq \frac{1}{2}$. *If*

$$w_{j\,k}(t) \geq -\sqrt{w_{j\,j}(t)\, w_{k\,k}(t)}, \quad t \in \mathbb{R},$$

then

$$\mu_j\, V_{\lambda_j}(z) + \mu_k\, V_{\lambda_k}(z) \in DCP$$

for all $\mu_j, \mu_k \geq 0$.

REFERENCES

[1] M. Abramowitz and I.A. Stegun, editors, *Handbook of mathematical functions*, Dover, New York, 1970.

[2] J.E. Brown, *Level sets for functions convex in one direction*, Proc. Amer.Math.So.**100** (1997), 442–446.

[3] A.W. Goodman and E.B. Saff, *On univalent functions convex in one direction*, Proc. Amer. Math. Soc. **73** (1979), 183 - 187.

[4] K.P. Hadeler, *On copositive matrices*, Linear algebra and its applications **49** (1983), 79–89.

[5] W. Hengartner and G,Schober *A remarks on level curves for domains convex in one direction*, Applicable Anal.**3** (1973), 101–106.

[6] C.M. Pokhrel, *DCP property of a Certain Combination of de la Vallee Poussin Kernels*, The Nepali Mathematical Sciences Report, Vol. **29** No. **1** and **2**, (2009) 21-27.

[7] C.M. Pokhrel, St. Ruscheweyh and G.B.Thapa, *The DCP Radius of the Exponential Function*, Complex variable Vol. **50**, No. **7-11**, 10 June - 15 September (2005), 645 -651.

[8] G. Pólya and I.J. Schoenberg, Remarks on the de -la Vallée Poussin means and convex conformal maps of the circle, *Pacific J. Math.* **8**, (1958), 295–333.

[9] St. Ruscheweyh and L. Salinas, *On the preservation of direction convexity and the Goodman - Saff conjecture*, Ann. Acad. Sci. Fenn. **14** (1989), 63 - 73.

[10] St. Ruscheweyh and L. Salinas, *On the preservation of periodic Monotonicity*, Constr. Approx. **8** (1992), 129 - 140.

[11] St. Ruscheweyh and L. Salinas,On the DCP Property of Exponential Partial Sums, *J. Comp. Anal. Appl.*, (to appear).

[12] St. Ruscheweyh and T.J. Suffridge, *A continuous extension of the de-la Vallée Poussin means*, J. Anal. Math., **89**(2003),Pp. 155-167.

[13] I.J. Schoenberg, *On variation diminishing approximation method: On numerical approximation (R.E. Langer, ed.)*, Univ. Wisconsin Press (1959), 249 - 279.