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ANALYTIC CONTINUATION OF THE SECOND KIND EULER POLYNOMIALS AND THE EULER ZETA FUNCTION

C. S. RYOO

Department of Mathematics
Hannam University, Daejeon 306-791, Korea

ABSTRACT. In this paper we study that the second kind Euler numbers E_n and Euler polynomials $E_n(x)$ are analytic continued to $E(s)$ and $E(s,w)$. We investigate the new concept of dynamics of the zeros of analytic continued polynomials. Finally, we observe an interesting phenomenon of 'scattering' of the zeros of $E(s,w)$.

AMS (MOS) Subject Classification. 11B68, 11S40.

1. Introduction

Throughout this paper, we always make use of the following notations:$\mathbb{N} = \{1,2,3,\dots\}$ denotes the set of natural numbers, $\mathbb{N}_0 = \{0,1,2,3,\dots\}$ denotes the set of nonnegative integers, $\mathbb{Z}$ denotes the set of integers, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{C}$ denotes the set of complex numbers. Recently, many mathematicians have studies different kinds of the Euler, Bernoulli and Genocchi numbers and polynomials (see [1–20]). Using computer, a realistic study for the second kind Euler polynomials $E_n(x)$ is very interesting. It is the aim of this paper to observe an interesting phenomenon of 'scattering' of the zeros of the second kind Euler polynomials $E_n(x)$ in complex plane. First, we introduce the second kind Euler numbers and Euler polynomials. As well known definition, the second kind Euler numbers E_n are defined by

$$F(t) = \frac{2e^t}{e^{2t}+1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}, \quad |t| < \frac{\pi}{2}, \text{ see [14].} \tag{1}$$

For $n \in \mathbb{N}_0$, numbers E_n meet $E_{2n+1} = 0$. By using computer, the numbers E_n can be determined explicitly. The first few E_n are listed in Table 1.

Table 1. The first few numbers E_n

degree n	0	1	2	3	4	5	6	7	8	9	10
E_n	1	0	-1	0	5	0	-61	0	1385	0	-50521

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The second kind Euler polynomials $E_n(x)$ are defined by the generating function:

$$F(x,t)=\sum_{n=0}^{\infty}E_n(x)\frac{t^n}{n!}=\left(\frac{2e^t}{e^{2t}+1}\right)e^{xt},\quad |t|<\frac{\pi}{2},\ \text{see [14]},\tag{2}$$

where we use the technique method notation by replacing $E(x)^n$ by $E_n(x)$ symbolically.

By the above definition (2) and Cauchy product, we obtain

$$\begin{aligned}\sum_{l=0}^{\infty}E_l(x)\frac{t^l}{l!}&=\left(\frac{2e^t}{e^{2t}+1}\right)e^{xt}=\sum_{n=0}^{\infty}E_n\frac{t^n}{n!}\sum_{m=0}^{\infty}x^m\frac{t^m}{m!}\\&=\sum_{l=0}^{\infty}\left(\sum_{n=0}^{l}E_n\frac{t^n}{n!}x^{l-n}\frac{t^{l-n}}{(l-n)!}\right)=\sum_{l=0}^{\infty}\left(\sum_{n=0}^{l}\binom{l}{n}E_nx^{l-n}\right)\frac{t^l}{l!}.\end{aligned}$$

By using comparing coefficients $\frac{t^l}{l!}$, we have the following theorem.

Theorem 1. *For $n\in\mathbb{N}_0$, one has*

$$E_n(x)=\sum_{k=0}^{n}\binom{n}{k}E_kx^{n-k}.$$

By Theorem 1 and some elementary calculations, we have

$$\begin{aligned}\int_a^b E_n(x)dx&=\sum_{l=0}^{n}\binom{n}{l}E_l\int_a^b x^{n-l}dx\\&=\sum_{l=0}^{n}\binom{n}{l}E_l\left.\frac{x^{n-l+1}}{n-l+1}\right|_a^b\\&=\frac{1}{n+1}\sum_{l=0}^{n+1}\binom{n+1}{l}E_l\ x^{n-l+1}\Big|_a^b.\end{aligned}$$

By Theorem 1, we get

$$\int_a^b E_n(x)dx=\frac{E_{n+1}(b)-E_{n+1}(a)}{n+1}.\tag{3}$$

Since $E_n(0)=E_n$, by (3), we have the following theorem.

Theorem 2. *For $n\in\mathbb{N}$, one has*

$$E_n(x)=E_n+n\int_0^x E_{n-1}(t)dt.$$

Then, it is easy to deduce that $E_k(x)$ are polynomials of degree k. Here is the list of the first Euler's polynomials. By using computer, the second kind Euler polynomials $E_n(x)$ can be determined explicitly. A few of them are

$$
\begin{aligned}
&E_0(x) = 1,\\
&E_1(x) = x,\\
&E_2(x) = x^2 - 1,\\
&E_3(x) = x^3 - 3x,\\
&E_4(x) = x^4 - 6x^2 + 5,\\
&E_5(x) = x^5 - 10x^3 + 25x,\\
&E_6(x) = x^6 - 15x^4 + 75x^2 - 61,\\
&E_7(x) = x^7 - 21x^5 + 175x^3 - 427x,\\
&E_8(x) = x^8 - 28x^6 + 350x^4 - 1708x^2 + 1385,\\
&E_9(x) = x^9 - 36x^7 + 630x^5 - 5124x^3 + 12465x,\\
&E_{10}(x) = x^{10} - 45x^8 + 1050x^6 - 12810x^4 + 62325x^2 - 50521,\\
&\cdots
\end{aligned}
$$

2. Analytic Continuation of the second kind Euler numbers

By using the second kind Euler numbers and polynomials, the second kind Euler zeta function and Hurwitz Euler zeta functions are defined. From (1.1), we note that

$$\frac{d^k}{dt^k}F(t)\bigg|_{t=0} = 2\sum_{n=0}^{\infty}(-1)^n(2n+1)^k, \quad k \in \mathbb{N}.$$

By using the above equation, we are now ready to define the second kind Euler zeta functions.

Definition 3. For $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 0$, define the second kind Euler zeta function by

$$\zeta_E(s) = 2\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^s}.$$

Notice that the Euler zeta function can be analytically continued to the whole complex plane, and these zeta function have the values of the Euler numbers at negative integers. That is, the second kind Euler numbers are related to the second kind Euler zeta function as

$$\zeta_E(-k) = E_k.$$

By using (2), we note that

$$\frac{d^k}{dt^k}F(x,t)\bigg|_{t=0} = 2\sum_{n=0}^{\infty}(-1)^n(2n+x+1)^k, \quad k \in \mathbb{N}, \tag{4}$$

and

$$\left(\frac{d}{dt}\right)^k \left(\sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}\right)\Bigg|_{t=0} = E_k(x), \quad \text{for } k \in \mathbb{N}. \tag{5}$$

By (4) and (5), we are now ready to define the Hurwitz Euler zeta functions.

Definition 4. We define the Hurwitz zeta function $\zeta_E(s,x)$ for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 0$ by

$$\zeta_E(s,x) = 2\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+x+1)^s}.$$

Note that $\zeta_E(s,x)$ is a meromorphic function on $\mathbb{C}$. Relation between $\zeta_E(s,x)$ and $E_k(x)$ is given by the following theorem.

Theorem 5. *For $k \in \mathbb{N}$, we have*

$$\zeta_E(-k,x) = E_k(x). \tag{6}$$

We now consider the function $E(s)$ as the analytic continuation of the second kind Euler numbers. From the above analytic continuation of the second kind Euler numbers, we consider

$$\begin{aligned} &E_n \mapsto E(s), \\ &\zeta_E(-n) = E_n \mapsto \zeta_E(-s) = E(s). \end{aligned} \tag{7}$$

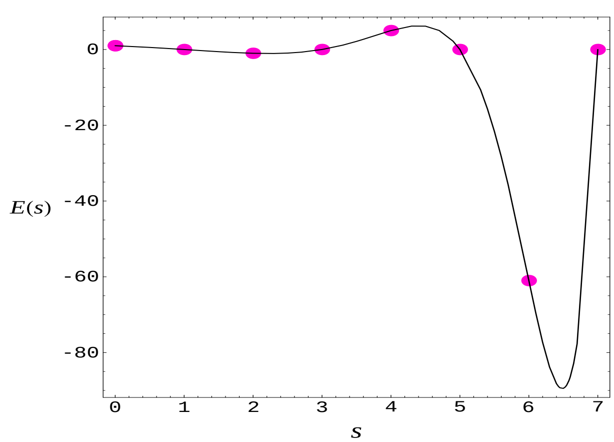

FIGURE 1. The curve $E(s)$ runs through the points of all E_n

All the Euler number E_n agree with $E(n)$, the analytic continuation of the second kind Euler numbers evaluated at n (see Figure 1),

$$E_n = E(n) \text{ for } n \geq 0. \tag{8}$$

In fact, we can express $E'(s)$ in terms of $\zeta_E'(s)$, the derivative of $\zeta_E(s)$.

$$\begin{aligned} E(s) &= \zeta_E(-s), \\ E'(s) &= -\zeta_E'(-s) \\ E'(2n+1) &= -\zeta_E'(-2n-1) \text{ for } n \in \mathbb{N}_0. \end{aligned} \tag{9}$$

From the relation (9), we can define the other analytic continued half of the second kind Euler numbers

$$\begin{aligned} E(s) = \zeta_E(-s), \quad E(-s) = \zeta_E(s) \\ \Rightarrow E(-n) = \zeta_E(n), n \in \mathbb{N}. \end{aligned} \tag{10}$$

By (10), we have

$$\lim_{n \to \infty} E_{-n} = \zeta_E(n) = 2.$$

The curve $E(s)$ runs through the points $E_{-n} = E(-n)$ and grows ~ 2 asymptotically as $-n \to \infty$ (see Figure 2).

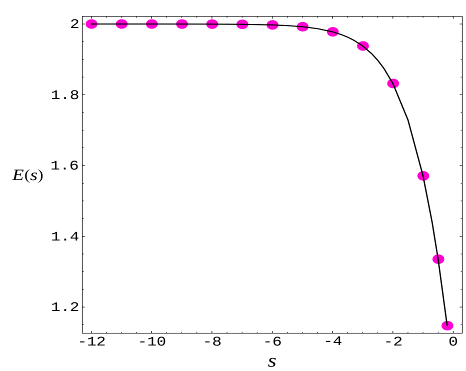

FIGURE 2. The curve $E(s)$ runs through the points E_{-n}

3. Zeros of analytic continued polynomials of the second kind Euler polynomials

Our main purpose in this section is to investigate the new concept of dynamics of the zeros of analytic continued polynomials. The analytic continuation can be then obtained as

$$\begin{aligned}
&n \mapsto s \in \mathbb{R}, x \mapsto w \in \mathbb{C},\\
&E_k \mapsto E(k+s-[s]) = \zeta_E(-(k+(s-[s]))),\\
&\binom{n}{k} \mapsto \frac{\Gamma(1+s)}{\Gamma(1+k+(s-[s]))\Gamma(1+[s]-k)}\\
&\Rightarrow E_n(x) \mapsto E(s,w) = \sum_{k=-1}^{[s]} \frac{\Gamma(1+s)E(k+s-[s])w^{[s]-k}}{\Gamma(1+k+(s-[s]))\Gamma(1+[s]-k)}\\
&= \sum_{k=0}^{[s]+1} \frac{\Gamma(1+s)E((k-1)+s-[s])w^{[s]+1-k}}{\Gamma(k+(s-[s]))\Gamma(2+[s]-k)},
\end{aligned} \tag{11}$$

where $[s]$ gives the integer part of s, and so $s-[s]$ gives the fractional part.

By (11), we obtain analytic continuation of the second kind Euler polynomials.

$$\begin{aligned}
&E_0(w) = 1,\\
&E_1(w) = E(1,w) = w,\\
&E_2(w) = E(2,w) = -1 + w^2,\\
&E(2.2,w) \approx -1.05935 - 0.51637w + 1.10088w^2 + 0.13087w^3,\\
&E(2.4,w) \approx -1.01747 - 1.11835w + 1.09099w^2 + 0.31130w^3,\\
&E(2.6,w) \approx -0.84541 - 1.772532w + 0.92996w^2 + 0.53121w^3,\\
&E(2.8,w) \approx -0.514468 - 2.42561w + 0.577882w^2 + 0.77106w^3,\\
&E_3(w) = E(3,w) = -3w + w^3.
\end{aligned} \tag{12}$$

By using (12), we plot the deformation of the curve $E(2,w)$ into the curve of $E(3,w)$ via the real analytic continuation $E(s,w), 2 \le s \le 3, w \in \mathbb{R}$ (see Figure 3).

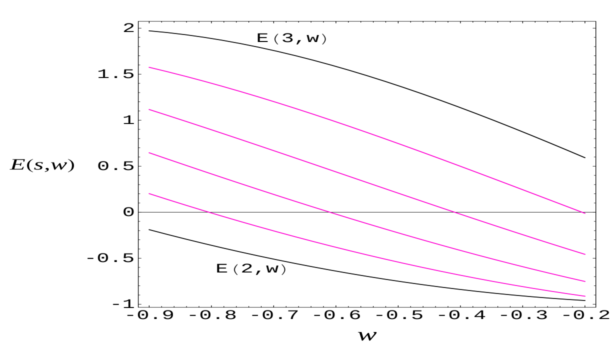

FIGURE 3. The curve of $E(s,w)$, $2 \le s \le 3$, $-0.8 \le w \le -0.2$

Stacks of zeros of $E(n,w)$ for $1 \le n \le 50$, forming a 3D structure are presented (Figure 4).

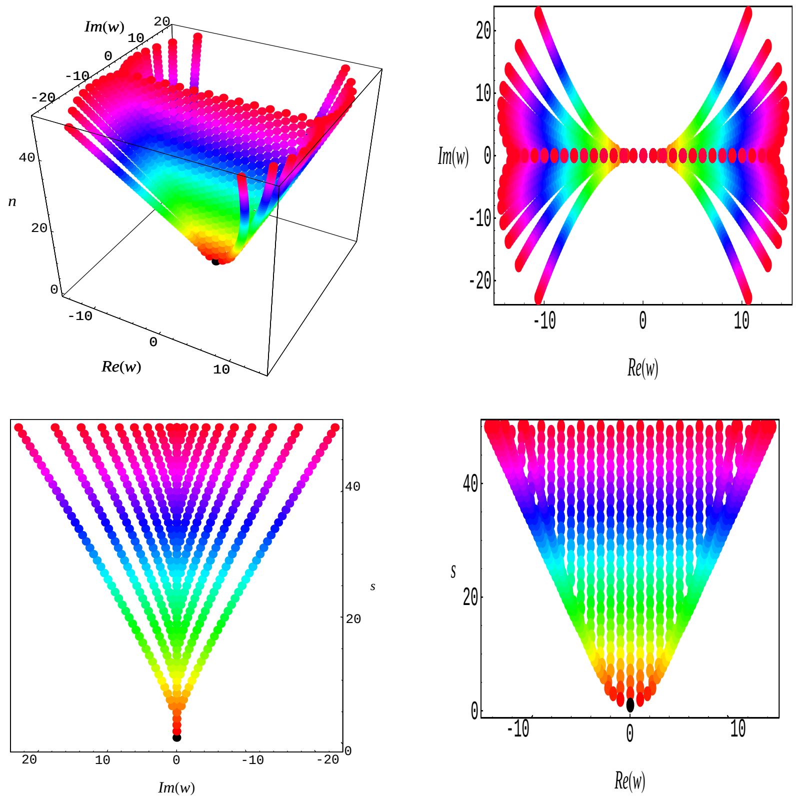

FIGURE 4. Stacks of zeros of $E(n, x)$ for $1 \le n \le 50$

In Figure 4 (top-right), we draw x and y axes but no z axis in three dimensions. In Figure 4 (bottom-left), we draw y and z axes but no x axis in three dimensions. In Figure 4 (bottom-right), we draw x and z axes but no y axis in three dimensions. In Figure 4, we observe that $E_n(x)$, $x \in \mathbb{C}$, has $Re(x) = 0$ reflection symmetry in addition to the usual $Im(x) = 0$ reflection symmetry analytic complex functions (see Figure 4). The obvious corollary is that the zeros of $E(n, w)$ will also inherit these symmetries.

$$\text{If } E(n, w_0) = 0, \text{ then } E(n, -w_0) = E(n, w_0^*) = E(n, -w_0^*) = 0,$$

where $*$ denotes complex conjugation.

For $n \in \mathbb{N}_0$, it is easy to deduce that the second kind Euler polynomials $E_n(x)$ satisfy

$$\begin{aligned} \sum_{n=0}^{\infty} E_n(-x)\frac{(-t)^n}{n!} &= \frac{2e^{-t}}{e^{-2t}+1}e^{(-x)(-t)} \\ &= \frac{2e^{t}}{e^{2t}+1}e^{xt} \\ &= \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}. \end{aligned}$$

By using comparing coefficients $\frac{t^n}{n!}$ in the above equation, we have the following theorem.

Theorem 6. *For $n \in \mathbb{N}_0$, we have*

$$E_n(x) = (-1)^n E_n(-x). \tag{13}$$

Hence we have the following theorem.

Theorem 7. *If $n \equiv 1 \pmod 2$, then $E_n(0) = 0$, for $n \in \mathbb{N}$.*

Next, we investigate the beautiful zeros of the $E(s, w)$ by using a computer. We plot the zeros of $E(s, w)$ for $s = 9$, 9.5, 9.8, 10 and $w \in \mathbb{C}$ (Figure 5).

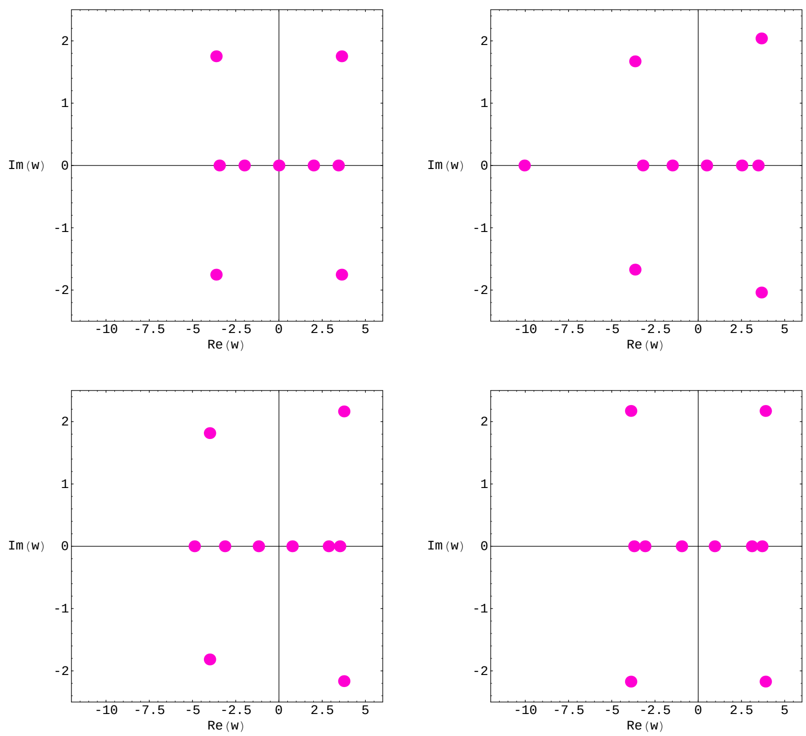

FIGURE 5. Zeros of $E(s, w)$ for $s = 9, 9.5, 9.8, 10$

In Figure 5 (top-left), we choose $s = 9$. In Figure 5 (top-right), we choose $s = 9.5$. In Figure 5 (bottom-left), we choose $s = 9.8$. In Figure 5 (bottom-right), we choose $s = 10$.

Stacks of zeros of $E_q(s, w)$ for $s = n + 1/2$, $1 \leq n \leq 50$, forming a 3D structure are presented (Figure 6).

In Figure 6 (top-right), we draw y and z axes but no x axis in three dimensions. In Figure 6 (bottom-left), we draw x and y axes but no z axis in three dimensions. In Figure 6 (bottom-right), we draw x and z axes but no y axis in three dimensions.

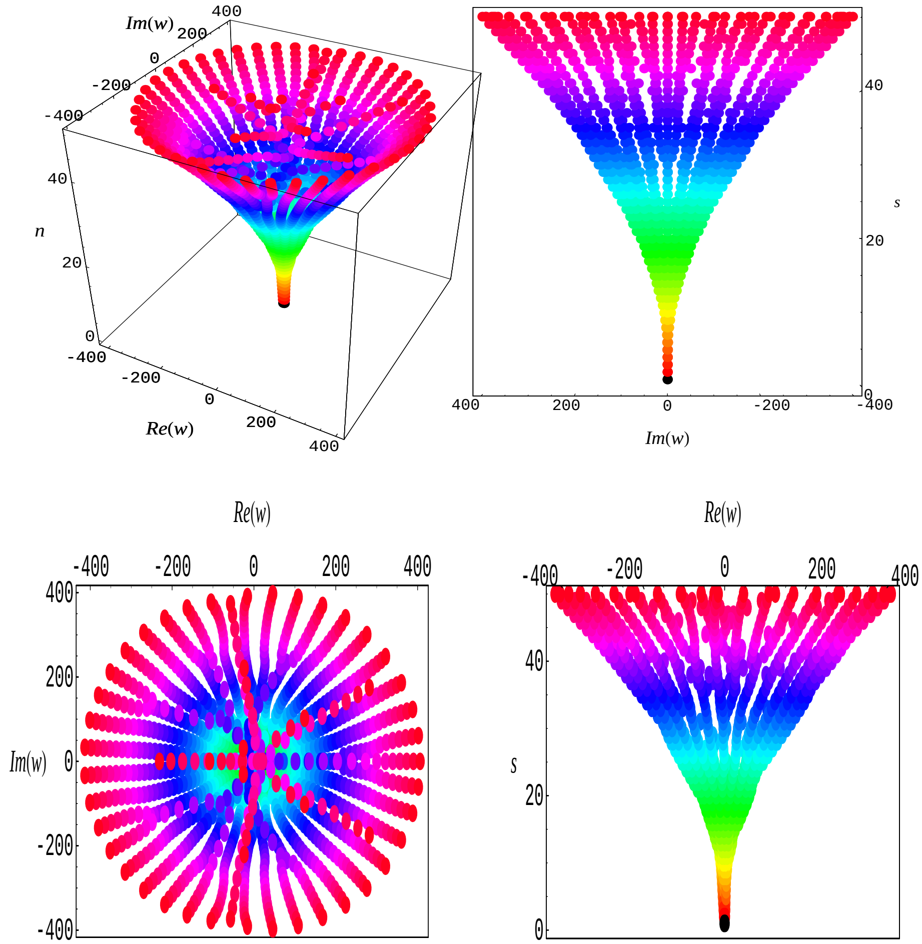

FIGURE 6. Stacks of zeros of $E_q(s,w)$ for $1 \leq n \leq 50$

However, we observe that $E(s,w), w \in \mathbb{C}$, has not $Re(w)=0$ reflection symmetry analytic complex functions (see Figure 5 and Figure 6).

Table 1. Numbers of real and complex zeros of $E(s,w)$

s	real zeros	complex zeros
4.5	5	0
5.5	4	2
6.5	3	4
7.5	4	4
8.5	5	4
9	5	4
9.3	6	4
9.5	6	4
9.8	6	4
10	6	4

Our numerical results for approximate solutions of real zeros of $E(s,w)$ are displayed. We observe a remarkably regular structure of the complex roots of Euler polynomials. We hope to verify a remarkably regular structure of the complex roots of Euler polynomials (Table 1). Next, we calculated an approximate solution satisfying $E(s,w), w \in \mathbb{R}$. The results are given in Table 2.

Table 2. Approximate solutions of $E(s,w) = 0, w \in \mathbb{R}$

s	w
6	−1.0000, 1.0000
6.5	−7.21395, −0.500166, 1.49742
7	−1.99547, 0.00000, 1.99546
7.5	−8.15937, −1.49886, 0.500067, 2.42679
8	−2.86463, −1.0000, 1.0000, 2.86461
8.5	−9.10419, −2.45462, −0.500361, 1.50051, 3.09031
9	−3.44019, −1.99828, −0.0020349, 2.00238, 3.43849
9.3	−18.4239, −3.26444, −1.69509, 0.294429, 2.30993, 3.46266
9.5	−10.0489, −3.19923, −1.49042, 0.489325, 2.52446, 3.47029
9.8	−4.88359, −3.12755, −1.17558, 0.772477, 2.87402, 3.52148
10	−3.70784, −3.07603, −0.95533, 0.949222, 3.10087, 3.69157

In Figure 7, we plot the real zeros of the the second kind Euler polynomials $E(s,w)$ for $s = n + \frac{1}{2}$, $1 \le n \le 50$ and $w \in \mathbb{C}$ (Figure 7). In Figure 7 (right), we choose $E(s,w)$ for $s = n + \frac{1}{2}$, $1 \le n \le 50$. In Figure 7 (left), we choose $E(n,w)$ for $1 \le n \le 50$.

The second kind Euler polynomials $E_n(w)$ is a polynomials of degree n. Thus, $E_n(w)$ has n zeros and $E_{n+1}(w)$ has $n+1$ zeros. When discrete n is analytic continued to continuous parameter s, it naturally leads to the question: How does $E(s,w)$, the analytic continuation of $E_n(w)$, pick up an additional zero as s increases continuously by one?

This introduces the exciting concept of the dynamics of the zeros of analytic continued polynomials – the idea of looking at how the zeros move about in the w complex plane as we vary the parameter s. More studies and results in this subject we may see references [9], [12]–[17].

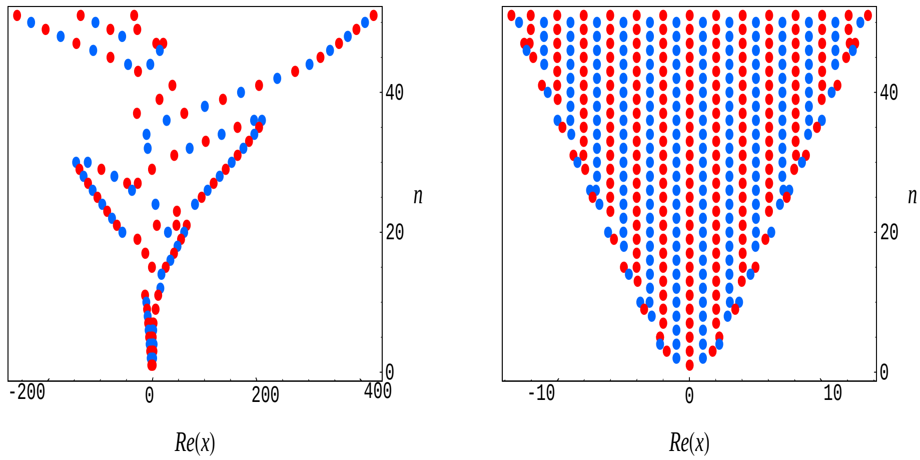

FIGURE 7. Real zeros of $E(s, w)$

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