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CRITICAL SOURCE AND CRITICAL WIDTH FOR A PARABOLIC QUENCHING PROBLEM IN AN INFINITE STRIP

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ABSTRACT. This article studies the quenching phenomena of a semilinear parabolic initial-boundary value problem in an infinite strip. It proves that there exists a unique number α^* (corresponding to the strength of the source) such that the solution u exists globally for $\alpha < \alpha^*$ and quenches in a finite time for $\alpha > \alpha^*$. A computational method is devised to find α^*. Also, a method to compute the critical width (corresponding to the number L^* such that u exists globally for $L < L^*$ and quenches in a finite time for $L > L^*$) of the infinite strip is given.

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1. Introduction

Let α, T and L be positive real numbers, $x = (x_1, x_2, \ldots, x_{N-1}, x_N)$ be a point in the N-dimensional Euclidean space $\mathbb{R}^N$, $S = (-L, L) \times \mathbb{R}^{N-1}$ be the infinite strip, and $\partial S = \{x : x_1 \in \{-L, L\}\}$. We would like to study the quenching phenomena of the following semilinear parabolic initial-boundary value problem in an infinite strip:

$$\begin{gathered} u_t - \triangle u = \alpha f(u) \text{ in } S \times (0, T], \\ u(x, 0) = 0 \text{ on } \bar{S},\ u(x, t) = 0 \text{ on } \partial S \times (0, T], \end{gathered} \tag{1.1}$$

where $\bar{S}$ is the closure of S, f is a given function such that $\lim_{u \to c^-} f(u) = \infty$ for some positive constant c, and $f(u)$ and its derivatives $f'(u)$ and $f''(u)$ are positive for $0 \leq u < c$. A similar problem with a concentrated nonlinear source was studied by Chan and Tragoonsirisak [1], [2], [3].

A solution u of (1.1) is said to quench in a finite time if there exists a number $t_q \in (0, \infty)$ such that

$$\sup\left\{u(x, t) : x \in \mathbb{R}^N\right\} \to c^- \text{ as } t \to t_q.$$

In Section 2, we prove that there exists a unique number α^* (which corresponds to the critical strength of the nonlinear source) such that u exists globally for $\alpha < \alpha^*$, and u quenches in a finite time for $\alpha > \alpha^*$. We also give a computational method for

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finding the critical value α^*. Dai and Gu [4] studied the above problem. They showed that u exists globally for $L < L^*$ and u quenches in a finite time for $L > L^*$; they did not compute L^*. In Section 3, we give a method to compute L^*.

2. The Critical Value $\boldsymbol{\alpha}^*$

From now on and for the rest of this paper, let $(x, \tilde{x}) = (x_1, x_2, \ldots, x_{N-1}, x_N)$ where x stands for x_1, and $S = (0, L)$. Due to symmetry, the problem (1.1) is equivalent to the following one-dimensional problem:

$$\begin{gathered} u_t - u_{xx} = \alpha f(u) \text{ in } S \times (0, T], \\ u(x,0) = 0 \text{ on } [0, L]\,, \; u_x\,(0,t) = 0 = u\,(L,t) \text{ for } 0 < t \leq T. \end{gathered} \tag{2.1}$$

By using Green's second identity, the problem (2.1) is converted into the nonlinear integral equation,

$$u\,(x,t) = \alpha \int_0^t \int_0^L g\,(x,t;\xi,\tau)\, f\,(u\,(\xi,\tau))\, d\xi d\tau, \tag{2.2}$$

where $g\,(x,t;\xi,\tau)$ is Green's function corresponding to the problem (2.1) and is given by

$$\begin{aligned} &g\,(x,t;\xi,\tau) \\ &= \frac{2}{L}\sum_{n=1}^{\infty}\left(\cos\frac{(2n-1)\,\pi x}{2L}\right)\left(\cos\frac{(2n-1)\,\pi\xi}{2L}\right)\exp\left(-\frac{(2n-1)^2\,\pi^2\,(t-\tau)}{4L^2}\right) \end{aligned} \tag{2.3}$$

(cf. Chan and Tragoonsirisak [1]).

Let

$$v(x,t) = \int_0^t \int_0^L g\,(x,t;\xi,\tau)\, d\xi d\tau. \tag{2.4}$$

Lemma 2.1. *For any $x \in D$, v is positive and is a strictly increasing function of t. Furthermore, $\lim_{t\to\infty} v\,(x,t)$ exists.*

Proof. For any $x \in D$, let

$$w(x,t) = v(x,t+h) - v(x,t),$$

where h is any positive number less than T. Then,

$$w(x,t) = \int_0^{t+h} \int_0^L g(x,t+h;\xi,\tau) d\xi d\tau - \int_0^t \int_0^L g(x,t;\xi,\tau) d\xi d\tau.$$

Let $\sigma = \tau - h$. Then,

$$\begin{aligned} &\int_0^{t+h} \int_0^L g(x,t+h;\xi,\tau) d\xi d\tau \\ &= \int_0^h \int_0^L g(x,t+h;\xi,\tau) d\xi d\tau + \int_h^{t+h} \int_0^L g(x,t+h;\xi,\tau) d\xi d\tau \end{aligned}$$

$$= \int_0^h \int_0^L g(x, t+h; \xi, \tau) d\xi d\tau + \int_0^t \int_0^L g(x, t+h; \xi, \sigma + h) d\xi d\sigma$$
$$= \int_0^h \int_0^L g(x, t+h; \xi, \tau) d\xi d\tau + \int_0^t \int_0^L g(x, t; \xi, \sigma) d\xi d\sigma$$

since $g(x, t+h; \xi, \sigma+h) = g(x, t; \xi, \sigma)$. Thus for $0 < t \leq T - h$,

$$w(x,t) = \int_0^h \int_0^L g(x, t+h; \xi, \tau) d\xi d\tau > 0$$

since for $x \in D$ and $t > 0$, $g(x, t; \xi, \tau)$ is positive (cf. Chan and Tragoonsirisak [1]). Thus for any $x \in D$, $v(x,t)$ is a strictly increasing function of t.

From (2.3) and (2.4),

$$v(x,t) \leq \frac{2}{L} \int_0^t \int_0^L \sum_{n=1}^{\infty} \exp\left(-\frac{(2n-1)^2 \pi^2 (t-\tau)}{4L^2}\right) d\xi d\tau$$
$$\leq \frac{2}{L} \int_0^t \int_0^L \sum_{n=1}^{\infty} \exp\left(-\frac{n^2 \pi^2 (t-\tau)}{4L^2}\right) d\xi d\tau$$
$$= 2 \int_0^t \sum_{n=1}^{\infty} \exp\left(-\frac{n^2 \pi^2 (t-\tau)}{4L^2}\right) d\tau.$$

Since $\sum_{n=1}^{\infty} \exp\left[-n^2\pi^2 (t-\tau) / (4L^2)\right]$ converges uniformly, we have

$$v(x,t) \leq 2 \sum_{n=1}^{\infty} \int_0^t \exp\left(-\frac{n^2 \pi^2 (t-\tau)}{4L^2}\right) d\tau$$
$$= 2 \sum_{n=1}^{\infty} \frac{4L^2}{n^2 \pi^2} \left(1 - \exp\left(-\frac{n^2 \pi^2 t}{4L^2}\right)\right)$$
$$\leq 2 \sum_{n=1}^{\infty} \frac{4L^2}{n^2 \pi^2}$$
$$= \frac{8L^2}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Since $\sum_{n=1}^{\infty} n^{-2} = \pi^2/6$ (cf. Stromberg [5, p. 518]), we have

$$v(x,t) \leq \frac{4L^2}{3}.$$

Thus, the lemma is proved. □

Theorem 2.2. *There exists a unique α^* such that u exists globally for $\alpha < \alpha^*$, and u quenches in a finite time for $\alpha > \alpha^*$.*

Proof. For $u(x,t) \leq c/2$, it follows from (2.2) and $f'(u) > 0$ that

$$u(x,t) \leq \alpha f\left(\frac{c}{2}\right) v(x,t).$$

From Lemma 2.1, $v(x,t)$ is bounded by some positive number M. Then,

$$u\left(x,t\right) \leq \alpha f\left(\frac{c}{2}\right) M.$$

By choosing

$$\alpha \leq \frac{c}{2f\left(\frac{c}{2}\right) M},$$

we have $u\left(x,t\right) \leq c/2$ for $t \in (0,\infty)$. Thus, the solution u of the problem (2.1) exists globally for α sufficiently small.

For any fixed $x \in D$, it follows from (2.2) that

$$u\left(x,t\right) \geq \alpha f\left(0\right) v\left(x,t\right).$$

From Lemma 2.1, there exists a positive number K such that $\lim_{t\to\infty} v\left(x,t\right) = K$. Thus, there exists $\tilde{t} \in (0,\infty)$ such that for $t \geq \tilde{t}$, $v\left(x,t\right) \geq K/2$. We have

$$u\left(x,t\right) \geq \frac{\alpha f\left(0\right) K}{2}.$$

By choosing

$$\alpha \geq \frac{2c}{f\left(0\right) K},$$

we have $u\left(x,t\right) \geq c$. Thus, the solution u of the problem (2.1) quenches in a finite time for α sufficiently large.

Let us consider the sequence $\{u_n\}$ given by $u_0\left(x,t\right) = 0$, and for $n = 0,1,2,\ldots$,

$$u_{n+1}(x,t) = \alpha \int_0^t \int_0^L g\left(x,t;\xi,\tau\right) f\left(u_n\left(\xi,\tau\right)\right) d\xi d\tau.$$

Using mathematical induction, we have $0 < u_1 < u_2 < u_3 < \cdots < u_n < u_{n+1}$ in $S \times (0,T]$. Since u_n is an increasing sequence as n increases, (2.2) follows from the Monotone Convergence Theorem (cf. Stromberg [5, pp. 266–268]) with $\lim_{n\to\infty} u_n(x,t) = u(x,t)$.

To show that the larger the α, the larger the solution u, let $\alpha > \beta$, and consider the sequence $\{v_n\}$ given by $v_0\left(x,t\right) = 0$, and for $n = 0,1,2,\ldots$,

$$v_{n+1}(x,t) = \beta \int_0^t \int_0^L g\left(x,t;\xi,\tau\right) f\left(v_n\left(\xi,\tau\right)\right) d\xi d\tau.$$

Similarly,

$$v(x,t) = \beta \int_0^t \int_0^L g\left(x,t;\xi,\tau\right) f\left(v\left(\xi,\tau\right)\right) d\xi d\tau,$$

where $v(x,t) = \lim_{n\to\infty} v_n(x,t)$. Since for $n = 1,2,3,\ldots$, $u_n > v_n$, we have $u \geq v$. Hence, the solution u is a nondecreasing function of α.

Since u exists globally for α sufficiently small, and quenches in a finite time for α sufficiently large, there exists a unique value α^* such that u exists globally for $\alpha < \alpha^*$, and u quenches in a finite time for $\alpha > \alpha^*$. □

To find the critical value α^*, we consider the steady state of the problem (2.1). We state without proof the following result since its proof is similar to that of Theorem 2.4 of Chan and Tragoonsirisak [2] for a problem with a concentrated nonlinear source.

Lemma 2.3. *If $u(x,t) \leq C$ for some constant $C \in (0,c)$, then $u(x,t)$ converges from below to a solution $U(x) = \lim_{t\to\infty} u(x,t)$ of the nonlinear two-point boundary value problem:*

$$-U''(x) = \alpha f(U) \ \ in \ \ (0,L), \quad U'(0) = 0, \ \ U(L) = 0. \tag{2.5}$$

Furthermore,

$$U(x) = \alpha \int_0^L G(x;\xi) f(U(\xi))\, d\xi \tag{2.6}$$

where

$$G(x;\xi) = \begin{cases} L-\xi, & x \leq \xi, \\ L-x, & x > \xi. \end{cases} \tag{2.7}$$

is Green's function corresponding to the problem (2.5)*.*

From (2.6) and (2.7),

$$U(x) = \alpha(L-x) \int_0^x f(U(\xi))\, d\xi + \alpha \int_x^L (L-\xi) f(U(\xi))\, d\xi. \tag{2.8}$$

Lemma 2.4. *The solution $U(x)$ of the problem* (2.5) *attains its maximum at $x = 0$.*

Proof. From (2.8),

$$\begin{aligned} U'(x) &= \alpha(L-x) f(U(x)) - \alpha \int_0^x f(U(\xi))\, d\xi - \alpha(L-x) f(U(x)) \\ &= -\alpha \int_0^x f(U(\xi))\, d\xi < 0. \end{aligned}$$

Thus, $U(x)$ is a strictly decreasing function of x, and the lemma is proved. □

To find computationally the critical value α^*, let us find its upper bound. From (2.8),

$$U(0) = \alpha \int_0^L (L-\xi) f(U(\xi))\, d\xi < c. \tag{2.9}$$

Since $f(u)$ and its derivative $f'(u)$ are positive for $0 \leq u < c$,

$$\alpha \int_0^L (L-\xi) f(U(\xi))\, d\xi \geq \alpha f(0) \int_0^L (L-\xi)\, d\xi = \frac{\alpha L^2 f(0)}{2}. \tag{2.10}$$

From (2.9) and (2.10),

$$\frac{\alpha L^2 f(0)}{2} < c. \tag{2.11}$$

Thus, an upper bound $\bar{\alpha}$ for α^* is given by

$$\bar{\alpha} = \frac{2c}{L^2 f(0)}. \tag{2.12}$$

From Lemma 2.4, it is sufficient to consider (2.8) at $x = 0$ in order to determine whether the solution quenches. From (2.8), we construct the sequence $\{U_k\}$ given by $U_0(x) = 0$, and for $k = 1, 2, 3, \ldots$,

$$U_k(x) = \alpha (L - x) \int_0^x f(U_{k-1}(\xi))\, d\xi + \alpha \int_x^L (L - \xi) f(U_{k-1}(\xi))\, d\xi. \tag{2.13}$$

We give below the steps to compute α^* by using Mathematica version 9:

Step 1: We input the value L, the function $f(u)$, and the value c.

Step 2: Let $\alpha\_low^{(n)}$ and $\alpha\_up^{(n)}$ be the $(n+1)$th estimates of lower and upper bounds of α^* respectively, and $\alpha = \left(\alpha\_low^{(n)} + \alpha\_up^{(n)}\right)/2$ be the $(n+1)$th approximation of α^*. Initially, we let a lower bound $\alpha\_low^{(0)}$ be zero, and compute an upper bound $\alpha\_up^{(0)}$ of α^* from (2.12).

Step 3: Let $h = L/m$, where m denotes the number of subdivisions with $j = 0, 1, 2, \ldots, m$. From (2.13),

$$U_k(jh) = \alpha (L - jh) \int_0^{jh} f(U_{k-1}(\xi))\, d\xi + \alpha \int_{jh}^L (L - \xi) f(U_{k-1}(\xi))\, d\xi \tag{2.14}$$

with $U_0(x) = 0$ for $0 \le x \le L$.

Step 4: At the kth iteration, if $U_k(0) \ge c$, then we let $\alpha\_low^{(n+1)} = \alpha\_low^{(n)}$, $\alpha\_up^{(n+1)} = \alpha$, and go to Steps 2 to 4; otherwise, we compute $U_k(jh)$ for $j = 1, 2, 3, \ldots, m$. If $\max_{j=0,1,2,\ldots,m}(U_k(jh) - U_{k-1}(jh)) < \delta$ for a given tolerance δ, then the sequence $\{U_k\}$ converges; we let $\alpha\_low^{(n+1)} = \alpha$, $\alpha\_up^{(n+1)} = \alpha\_up^{(n)}$, and go to Steps 2 to 4. However, if $\max_{j=0,1,2,\ldots,m}(U_k(jh) - U_{k-1}(jh)) \ge \delta$, then we use the interpolation to approximate $U_k(x)$ and continue the iterative process for the $(k+1)$th iteration.

Step 5: After n iterations, if $\left|\alpha\_up^{(n)} - \alpha\_low^{(n)}\right| < \epsilon$ for a given tolerance ϵ, then $\left(\alpha\_low^{(n)} + \alpha\_up^{(n)}\right)/2$ is accepted as the final estimate of α^*.

For illustrations of the above computational scheme, let $L = 2$ and $f(u) = 1/(c-u)^p$, where c and p are positive numbers such that $p \ge 1$. From (2.12),

$$\bar{\alpha} = \frac{c^{p+1}}{2}.$$

Using Steps 1 to 5 with $\epsilon = 10^{-4}$, $\delta = 10^{-6}$, and $m = 20$, we obtain the following tables for α^* (to four significant figures). For $c = 1$, we get the following table:

p	α^*
1	.1251
2	.07413
3	.05276

For $p = 1$, we get the following table:

c	α^*
2	.5004
3	1.126
4	2.002

3. The Critical L^*

The problem (1.1) was studied by Dai and Gu [4]. They showed that u exists globally for $L < L^*$ and u quenches in a finite time for $L > L^*$. In this section, we give a method to compute L^*. From (2.11),

$$L^2 < \frac{2c}{\alpha f(0)}.$$

Thus, an upper bound $\bar{L}$ for L^* is given by

$$\bar{L} = \sqrt{\frac{2c}{\alpha f(0)}}. \tag{3.1}$$

We give the following steps to compute L^* by using Mathematica version 9:

Step 1: We input the value α, the function $f(u)$, and the value c.

Step 2: Let $L\_low^{(n)}$ and $L\_up^{(n)}$ be the $(n+1)$th estimates of lower and upper bounds of L^* respectively, and $L = \left(L\_low^{(n)} + L\_up^{(n)}\right)/2$ be the $(n+1)$th approximation of L^*. Initially, we let a lower bound $L\_low^{(0)}$ be zero, and compute an upper bound $L\_up^{(0)}$ of L^* from (3.1).

Step 3: Let $h = L/m$, where m denotes the number of subdivisions with $j = 0, 1, 2, \ldots, m$. We use (2.14) with $U_0(x) = 0$ for $0 \le x \le L$.

Step 4: At the kth iteration, if $U_k(0) \ge c$, then we let $L\_low^{(n+1)} = L\_low^{(n)}$, $L\_up^{(n+1)} = L$, and go to Steps 2 to 4; otherwise, we compute $U_k(jh)$ for $j = 1, 2, 3, \ldots, m$. If $\max_{j=0,1,2,\ldots,m}(U_k(jh) - U_{k-1}(jh)) < \delta$ for a given tolerance δ, then the sequence $\{U_k\}$ converges; we let $L\_low^{(n+1)} = L$, $L\_up^{(n+1)} = L\_up^{(n)}$, and go to Steps 2 to 4. However, if $\max_{j=0,1,2,\ldots,m}(U_k(jh) - U_{k-1}(jh)) \ge \delta$, then we use the interpolation to approximate $U_k(x)$ and continue the iterative process for the $(k+1)$th iteration.

Step 5: After n iterations, if $\left|L\_up^{(n)} - L\_low^{(n)}\right| < \epsilon$ for a given tolerance ϵ, then $\left(L\_low^{(n)} + L\_up^{(n)}\right)/2$ is accepted as the final estimate of L^*.

For illustrations of the above computational scheme, let $\alpha = 1$ and $f(u) = 1/(c-u)^p$, where c and p are positive numbers such that $p \ge 1$. From (3.1),

$$\bar{L} = \sqrt{2c^{p+1}}.$$

Using Steps 1 to 5 with $\epsilon = 10^{-4}$, $\delta = 10^{-6}$, and $m = 20$, we obtain the following tables for L^* (to four significant figures). For $c = 1$, we get the following table:

p	L^*
1	.7077
2	.5448
3	.4597

For $p = 1$, we get the following table:

c	L^*
2	1.415
3	2.122
4	2.830

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