Neural, Parallel, and Scientific Computations 22 (2014) 547-562

# ACCURATE VARIABLE STEP EXPONENTIAL FINITE DIFFERENCE METHOD FOR NUMERICAL SOLUTION OF TWO POINT BOUNDARY VALUE PROBLEMS

P. K. PANDEY

Department of Mathematics,
Dyal Singh College (University of Delhi)
Lodhi Road, New Delhi-110003, India
pramod_10p@hotmail.com

**ABSTRACT.** In this article, we presented an exponential finite difference scheme, a numerical method for solution of two point boundary value problems with variable mesh and Dirichlet boundary conditions. The idea in exponential difference schemes is to discretize the differential equation using an exponential function. Under appropriate condition, we have discussed the local truncation error and the convergence of the proposed method. The accuracy of the proposed method has been tested through the numerical experiments and the numerical result for considered model problems demonstrate computational efficiency of the method. We conclude from numerical experiments that method is convergent and has at least second order accuracy which is in good agreement with the theoretically established order of the method.

**Key words.** Exponential finite difference method, Inverse Problem, Quadratic order, Two-point boundary value problem, Variable Mesh Nonlinear Method

**AMS (MOS) Subject Classification.** 65L10, 65L12

## 1. Introduction

Two point boundary value problems are of common occurrence in many areas of sciences and engineering. These problems are characterised by the necessity to find a solution from an equation with given coefficients and given boundary conditions. This class of problems has gained importance in the literature for the variety of their applications. In most cases it is impossible to obtain solutions of these problems using analytical methods which satisfy the given specified boundary conditions. In these cases we resort to approximate solution of the problems and is presently well studied matter. In the literature, there are many different methods and approaches in solving these boundary value problems. The last few decades have seen substantial progress in the development of one of the widely used method known as finite difference method in solving these boundary value problems [1, 2, 3].

Present address: Department of Natural Resource, Engineering and Management, University of Kurdistan Hewler, Erbil, Kurdistan Region, Iraq

Received May 15, 2014 1061-5369 $15.00 

In this article we proposed a method for the numerical solution of the boundary value problems of the form

$$y''(x) = f(x, y), \quad a < x < b, \tag{1}$$

subject to the boundary conditions

$$y(a) = \eta \quad and \quad y(b) = \delta,$$

where $\eta$ and $\delta$ are real constants and $f$ is continuous on $(x, y)$ for all $x \in [a, b]$, $y \in \Re$.

The existence and uniqueness of the solution to problem (1) is assumed. Further we assumed that problem (1) is well posed with continuous derivatives and that the solution depends differentially on the boundary conditions. The specific assumption on $f(x, y)$ to ensure existence and uniqueness will not be considered [4, 5, 6].

Over the last few decades, finite difference methods [7, 8, 9] have generated renewed interest and in recent years, variety of specialized techniques [10, 11, 12] for the numerical solution of boundary value problems in ODEs have been reported in the literature. Recently, an exponential finite difference method with uniform step size was proposed in [13] for the numerical solution of linear two point boundary value problem. This method generated impressive numerical results for the problem (1). Hence, the purpose of this article is to propose an exponential finite difference method with variable step length for problem (1). In this case we have difference equations with variable coefficients. The development of this accurate numerical method for two-point boundary-value problems plays a paramount role in the approximate solution of boundary value problems with a small parameter affecting highest derivative of the differential equation. The behavior of the solution changes very rapidly near to this coefficient. The occurrence of this coefficient creates difficulty for most standard numerical schemes with uniform mesh in solving these problems. A variable mesh method concept overcomes this difficulty because mesh size changes rapidly and so this method well suit for solving these problems[9, 14].

In other words if we know that there is some cause which affect solution of the problem and approximate solution of the problem is known. If permissible error in solution is known in advance. Consider a problem to determine this cause which changes the behavior of this approximate solution with the given boundary conditions and permissible error in solution. Such problems are occurred in many real-world situations and this may be considered as an inverse problem [1]. The propose variable exponential finite difference method takes this fact into account. We hope that others may find the proposed method as an improvement and accurate to those existing finite difference method for two-point boundary value problems.

Our idea is to apply the exponential finite difference method to discretize equation (1) in order to get a system of algebraic equations. In addition, if we apply a linearization technique, the method results in a tridiagonal matrix for the nodal values. A method of at least quadratic order for the numerical solution of problem (1) is proposed. To the best of our knowledge, no similar method for the numerical solution of problem (1) has been discussed in literature so far.

We have presented our work in this article as follows. In the next section we derived our exponential finite difference method. In section 3, local truncation error and convergence of the method are discussed in Section 4. The application of the developed method to the problems (1) has been presented and illustrative numerical results have been produced to show the efficiency of the new method in Section 5. Discussion and conclusion on the performance of the method are presented in Section 6.

## 2. **The Exponential Difference Method**

We defined $N$ finite numbers of mesh points of the domain $[a,b]$, in which the solution of the problem (1) is desired, as $a = x_0 < x_1 < x_2 < \cdots < x_N < x_{N+1} = b$, using nonuniform step length $h$ such that $x_{i+1} = x_i + h_{i+1}$, $i = 0, 1, 2, \ldots, N$ and $d_i = h_{i+1} - h_i$, $i = 1, 2, \ldots, N$. Suppose we wish to determine numerical approximation of the theoretical solution $y(x)$ of the problem (1) at the nodal point $x_i$, $i = 1, 2, \ldots, N$. We denote numerical approximation of $y(x)$ at node $x = x_i$ as $y_i$ . Let us denote $f_i$ the approximation of the theoretical value of the source function $f(x, y(x))$ at mesh $x = x_i$, $i = 0, 1, 2, \ldots, N + 1$. We can define other notations $f_{i\pm1}$, $y_{i\pm1}$, in the similar way used in this article. Following the ideas in [12, 13], we propose an approximation to the theoretical solution $y(x_i)$ of the problem (1) by the exponential difference scheme as,

$$a_2 y_{i+1} + a_0 y_i + a_1 y_{i-1} = b_0 h_i^2 f_i \exp(\phi(x_i)), \quad i = 1, 2, \ldots, N \tag{2}$$

where $a_0, a_1, a_2$ and $b_0$ are unknown function and its argument is $r_i$ and $\phi(x_i)$, is an unknown sufficiently differentiable function of $x$. Let us define a function $F_i(h, y)$ and associate it with (2) as,

$$F_i(h, y) \equiv a_2 y_{i+1} + a_0 y_i + a_1 y_{i-1} - b_0 h_i^2 f_i \exp(\phi(x_i)) = 0, \tag{3}$$

Assume that $\phi(x_i)$ can be expand in Taylor series about point $x = x_{i-1}$. Hence we write $\phi(x_i)$ in Taylor series,

$$\phi(x_i) = \phi(x_{i-1}) + h_i \phi'(x_{i-1}) + O(h_i^2), \tag{4}$$

The application of (4) in the expansion of $\exp(\phi(x_i))$ will provide an $O(h_i^2)$ approximation of the form as,

$$\exp(\phi(x_i)) = \exp(\phi(x_{i-1}))(1 + h_i \phi'(x_{i-1})) + O(h_i^2) \tag{5}$$

Expand $F_i(h, y)$ in Taylor series about mesh point $x = x_i$ and using (5) in it, we have

(6)
$$F_i(h,y) \equiv \Bigg\{(a_0{+}a_1{+}a_2)y_i{+}h_i\left(\left(1+\frac{d_i}{h_i}\right)a_2-a_0\right)y_i'{+}\frac{h_i^2}{2}\left(\left(1+\frac{d_i}{h_i}\right)^2a_2+a_0\right)y_i'' + \frac{h_i^3}{6}\left(\left(1+\frac{d_i}{h_i}\right)^3a_2-a_0\right)y_i^{(3)}\Bigg\} - b_0h_i^2f_i\exp(\phi(x_{i-1}))(1+h_i\phi'(x_{i-1})) = 0$$

On comparing the coefficients of $h_i^p$, $p = 0, 1, 2, 3$ both side in (6), we get the following system of nonlinear equations

$$a_0 + a_1 + a_2 = 0,$$
$$\left(1+\frac{d_i}{h_i}\right)a_2 - a_0 = 0,$$
$$\left(\left(1+\frac{d_i}{h_i}\right)^2a_2+a_0\right)y_i'' - 2b_0f_i\exp(\phi(x_{i-1})) = 0,$$

(7)
$$\left(\left(1+\frac{d_i}{h_i}\right)^3a_2-a_1\right)y_i^{(3)} - 6b_0f_i\exp(\phi(x_{i-1}))\phi'(x_{i-1})) = 0,$$

To determine the unknown $a_0$, $b_0$, $\phi(x_{i-1})$ and $\phi'(x_{i-1})$ in (7), we have to assign arbitrary value to some unknown. To simplify the system of equations in (7), we have considered the following assumption:

(8)
$$\phi(x_{i-1}) = 0.$$

Using (8) in (7) and solved the reduced system of equations, we obtained

$$a_0 = \left(1+\frac{d_i}{h_i}\right)a_2,$$

$$b_0 = \frac{(1+\frac{d_i}{h_i})(2+\frac{d_i}{h_i})a_2}{2},$$

(9)
$$\phi'(x_{i-1}) = \frac{d_iy_i^{(3)}}{3h_if_i}.$$

Write $f'$ for $y^{(3)}$ in (9) and substituting the values of $\phi(x_i)$ and $\phi'(x_i)$ from (8) and (9) in (4), we have

(10)
$$\phi(x_i) = \frac{d_if_i'}{3f_i}.$$

Finally substitute the values of $a_0$, $b_0$, $\phi(x_i)$ from (9) and (10) in (2), we obtain our proposed exponential difference method as

(11)
$$y_{i+1} - \left(2+\frac{d_i}{h_i}\right)y_i + \left(1+\frac{d_i}{h_i}\right)y_{i-1} = h_i^2\left(1+\frac{d_i}{h_i}\right)\left(2+\frac{d_i}{h_i}\right)f_i\exp\left(\frac{d_if_i'}{3f_i}\right).$$

For each nodal point, we will obtain the nonlinear system of equations given by (11) or a linear system of equations if the source function is $f(x)$. In the derived numerical method (11) exponential function $\exp(\frac{d_i f_i'}{3f_i})$ has argument $\frac{d_i f_i'}{3f_i}$. If $f_i$ in denominator of the fraction become zero in the domain of the solution, we take exponential series expansion of the function $\exp(\frac{d_i f_i'}{3f_i})$ and neglecting the term of second and higher order. So we have following method.

$$y_{i+1} - \left(2+\frac{d_i}{h_i}\right) y_i + \left(1+\frac{d_i}{h_i}\right) y_{i-1} = h_i^2 \left(1+\frac{d_i}{h_i}\right)\left(2+\frac{d_i}{h_i}\right)\left(f_i + \frac{d_i}{3} f_i'\right).$$

For computational purpose reported in Section 4, we have used following second order finite difference approximation in place of $f_i'$ in (11):

$$h_i f_i' = \frac{f_{i+1} + \frac{d_i}{h_i}(2+\frac{d_i}{h_i}) f_i - (1+\frac{d_i}{h_i})^2 f_{i-1}}{(1+\frac{d_i}{h_i})(2+\frac{d_i}{h_i})}. \tag{12}$$

## 3. **Local Truncation Error**

We can write following expression for the term in (11) with the help of (12):

$$\exp\left(\frac{d_i f_i'}{3f_i}\right) = \exp\left(\frac{\frac{d_i}{h_i}(f_{i+1} + \frac{d_i}{h_i}(2+\frac{d_i}{h_i}) f_i - (1+\frac{d_i}{h_i})^2 f_{i-1})}{3(1+\frac{d_i}{h_i})(2+\frac{d_i}{h_i}) f_i}\right)$$

Write the expansion of exponential function by neglecting the second and higher order terms, so we will obtain,

$$\exp\left(\frac{d_i f_i'}{3f_i}\right) \equiv 1 + \frac{\frac{d_i}{h_i}(f_{i+1} + \frac{d_i}{h_i}(2+\frac{d_i}{h_i}) f_i - (1+\frac{d_i}{h_i})^2 f_{i-1})}{3(1+\frac{d_i}{h_i})(2+\frac{d_i}{h_i}) f_i} \tag{13}$$

From (11) and (13), the truncation error $T_i$ at the nodal point $x = x_i$ may be written as [9, 15, 16],

$$\begin{aligned} T_i = {} & y_{i+1} - \left(2+\frac{d_i}{h_i}\right) y_i + \left(1+\frac{d_i}{h_i}\right) y_{i-1} \\ & - \frac{h_i^2}{2}\left(2+\frac{d_i}{h_i}\right)\left(2+\frac{d_i}{h_i}\right) f_i \left(1 + \frac{\frac{d_i}{h_i}(f_{i+1} + (2+\frac{d_i}{h_i})\frac{d_i}{h_i} f_i - (1+\frac{d_i}{h_i})^2 f_{i-1})}{3(2+\frac{d_i}{h_i})(1+\frac{d_i}{h_i}) f_i}\right). \end{aligned}$$

By the Taylor series expansion of $y$ at nodal point $x = x_i$ and using $y_i'' = f_i$, $y_i^{(3)} = f_i'$, etc. we have

$$\begin{aligned} T_i &= \left(\frac{h_{i+1}^4}{24} + \frac{(1+\frac{d_i}{h_i})h_i^4}{24}\right) y_i^{(4)} - \frac{(1+\frac{d_i}{h_i})(2+\frac{d_i}{h_i})}{36} \frac{(d_i h_i y_i^{(3)})^2}{f_i} + O(h_i^5) \\ &= \frac{(1+\frac{d_i}{h_i})(2+\frac{d_i}{h_i})h_i^4}{72} \left\{ 3\left(\left(\frac{d_i}{h_i}\right)^2 + \frac{d_i}{h_i} + 1\right) y_i^{(4)} - 2\left(\frac{d_i y_i^{(3)}}{h_i}\right)^2 \right\} + O(h_i^5). \end{aligned} \tag{14}$$

Thus we have obtained a truncation error at each node of $O(h_i^4)$.

## 4. Convergence of the Method

Let us substitute the value of $f_i'$ from (12) in the second order expansion of the exponential function in (11) then simplify expression, we have

$$y_{i+1}-\left(2+\frac{d_i}{h_i}\right)y_i+\left(1+\frac{d_i}{h_i}\right)y_{i-1}$$
$$=\frac{h_i^2f_i}{2}\left(2+\frac{d_i}{h_i}\right)\left(1+\frac{d_i}{h_i}\right)\left(1+\frac{\frac{d_i}{h_i}(f_{i+1}+(2+\frac{d_i}{h_i})\frac{d_i}{h_i}f_i-(1+\frac{d_i}{h_i})_i^2f_{i-1})}{3(2+\frac{d_i}{h_i})(1+\frac{d_i}{h_i})f_i}\right)$$
$$=\frac{h_i^2}{6}\left\{3\left(1+\frac{d_i}{h_i}\right)\left(2+\frac{d_i}{h_i}\right)f_i+\frac{d_i}{h_i}\left(f_{i+1}+\left(2+\frac{d_i}{h_i}\right)\frac{d_i}{h_i}f_i-\left(1+\frac{d_i}{h_i}\right)^2f_{i-1}\right)\right\}$$

Thus

$$-y_{i+1}+\left(2+\frac{d_i}{h_i}\right)y_i-\left(1+\frac{d_i}{h_i}\right)y_{i-1}+\frac{h_i^2}{6}(\alpha_if_i+\gamma_if_{i+1}+\beta_if_{i-1})=0 \tag{15}$$

where $\alpha_i=(2+\frac{d_i}{h_i})((\frac{d_i}{h_i})^2+3\frac{d_i}{h_i}+3)$, $\beta_i=-(1+\frac{d_i}{h_i})^2\frac{d_i}{h_i}$ and $\gamma_i=\frac{d_i}{h_i}$.

Let us define

$$\phi_1=\frac{h_1^2}{6}(\alpha_1f(x_1,y_1)+\gamma_1f(x_2,y_2))+\frac{h_1^2}{6}\beta_1f(x_0,y_0)+\left(1+\frac{d_1}{h_1}\right)y_0,\quad i=1$$

$$\phi_i=\frac{h_i^2}{6}(\alpha_if(x_i,y_i)+\gamma_if(x_{i+1},y_{i+1})+\beta_if(x_{i-1},y_{i-1})),\quad 2\le i\le N-1$$

$$\phi_N=\frac{h_N^2}{6}(\alpha_Nf(x_N,y_N)+\beta_Nf(x_{N-1},y_{N-1}))+\frac{h_N^2}{6}\gamma_Nf(x_{N+1},y_{N+1}))+y_{N+1},\quad i=N$$

Let us define column matrix $\boldsymbol{\phi}_{N\times1}$ and $\mathbf{y}_{N\times1}$as

$$\boldsymbol{\phi}=[\phi_1,\phi_2,\dots,\phi_N]'_{1\times N},\quad \mathbf{y}=[y_1,y_2,\dots,y_N]'_{1\times N}$$

where $[\cdots]'$ is transpose of column matrix. The difference method (11) represents a system of nonlinear equations in unknown $y_i$, $i=1,2,\dots,N$. Let us write (11) in matrix form as,

$$\mathbf{Dy}+\boldsymbol{\phi}(\mathbf{y})=\mathbf{0}, \tag{16}$$

where

$$\mathbf{D}=\begin{pmatrix} (2+\frac{d_1}{h_1}) & -1 & & & 0\\ -(1+\frac{d_2}{h_2}) & (2+\frac{d_2}{h_2}) & -1 & & \\ & -(1+\frac{d_3}{h_3}) & (2+\frac{d_3}{h_3}) & -1 & \\ \cdots & \cdots & \cdots & \cdots & \cdots\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & & & -(1+\frac{d_N}{h_N}) & (2+\frac{d_N}{h_N}) \end{pmatrix}_{N\times N}$$

is tridiagonal matrix. Let $\mathbf{Y}$ be the exact solution of (11), so it will satisfy matrix equation

$$\mathbf{DY}+\boldsymbol{\phi}(\mathbf{Y})+\mathbf{T}=\mathbf{0}, \tag{17}$$

where $\mathbf{Y}$ is column matrix of order $N \times 1$ which can be obtained replacing $y$ by $Y$ in matrix $\mathbf{y}$ and $\mathbf{T}$ is truncation error matrix in which each element has $O(h_i^4)$. Let us define

$$F_{i+1} = f(x_{i+1}, Y_{i+1}), \quad f_{i+1} = f(x_{i+1}, y_{i+1}), \quad F_{i-1} = f(x_{i-1}, Y_{i-1}),$$

$$f_{i-1} = f(x_{i-1}, y_{i-1}), \quad F_i = f(x_i, Y_i), \quad f_i = f(x_i, y_i),$$

After linearization of $f_{i+1}$, we have

$$f_{i+1} = F_{i+1} + (y_{i+1} - Y_{i+1})G_{i+1},$$

where $G_{i+1} = (\frac{\partial f}{\partial Y})_{i+1}$ . Thus

$$f_{i+1} - F_{i+1} = (y_{i+1} - Y_{i+1})G_{i+1}. \tag{18}$$

Similarly, we can linearize $f_{i-1}$, $f_i$, and obtained the following results :

$$f_{i-1} - F_{i-1} = (y_{i-1} - Y_{i-1})G_{i-1}, \tag{19}$$

$$f_i - F_i = (y_i - Y_i)G_i, \tag{20}$$

By Taylor series expansion of $G_{i\pm1}$ about $x = x_i$, and from (16)–(17), we can write

$$\boldsymbol{\phi}(\mathbf{y}) - \boldsymbol{\phi}(\mathbf{Y}) = \mathbf{PE}, \tag{21}$$

where $\mathbf{P} = (P_{lm})_{N\times N}$ is a tri-diagonal matrix defined as

$$P_{lm} = \frac{h_i^2}{6}(\alpha_i G_i), \quad i = l = m, \quad l = 1, 2, \ldots, N,$$

$$P_{lm} = \frac{h_i^2}{6}\gamma_i(G_i + h_{i+1}(\frac{\partial G}{\partial x})_i), \quad m = l + 1, \quad i = l = 1, 2, \ldots, N-1,$$

$$P_{lm} = \frac{h_i^2}{6}\beta_i(G_i - h_i(\frac{\partial G}{\partial x})_i), \quad i = l = m + 1, \quad m = 1, 2, \ldots, N-2,$$

and $\mathbf{E} = [E_1, E_2, \ldots, E_N]'_{1\times N}$, where $E_i = (y_i - Y_i)$, $i = 1, 2, \ldots, N$.

Let assume that the solution of difference equation (15) has no roundoff error. So from (16),(17) and (21) we have

$$(\mathbf{D} + \mathbf{P})\mathbf{E} = \mathbf{JE} = \mathbf{T}. \tag{22}$$

Let us define $G^0 = \{G_i : i = 1, 2, \ldots, N\}$

$$G_* = \min_{x\in[a,b]} \frac{\partial f}{\partial Y}, \quad G^* = \max_{x\in[a,b]} \frac{\partial f}{\partial Y},$$

then

$$0 \le G_* \le t \le G^*, \quad \forall t \in G^0.$$

We further define $H^0 = \{(\frac{\partial G}{\partial x})_i,\ i = 1,2,\ldots,N\}$. Let there exist some positive constant $W$ such that $|t^0| \leq W$, $\forall\ t^0 \in H^0$. So it is possible for very small $h_i$, $\forall\, i = 1,2,\ldots,N$,

$$|P_{lm}| \leq \left(2 + \frac{d_i}{h_i}\right), \quad \forall \quad i = l = m \quad l = 1,2,\ldots,N,$$

$$|P_{lm}| \leq 1, \quad \forall \quad m = l+1, \quad i = l = 1,2,\ldots,N-1,$$

$$|P_{lm}| \leq \left(1 + \frac{d_i}{h_i}\right), \quad \forall \quad i = l = m+1, \quad m = 1,2,\ldots,N-2.$$

Let $\mathbf{R} = [R_1, R_2, \ldots, R_N]'_{1\times N}$, denotes the row sum of the matrix $\mathbf{J} = (J_{lm})_{N\times N}$ where

$$R_1 = \left(1 + \frac{d_1}{h_1}\right) + \frac{h_1^2}{6}(\alpha_1 + \gamma_1)G_1 + \left(1 + \frac{d_1}{h_1}\right)\gamma_1\frac{h_1^3}{6}\left(\frac{\partial G}{\partial x}\right)_i, \quad l = i = 1,$$

$$R_l = \frac{h_i^2}{6}(\alpha_i + \gamma_i + \beta_i)G_i + \frac{h_i^3}{6}\left(\left(1 + \frac{d_i}{h_i}\right)\gamma_i - \beta_i\right)\left(\frac{\partial G}{\partial x}\right)_i, \quad 2 \leq l = i \leq N-1,$$

$$R_N = 1 + \frac{h_N^2}{6}(\alpha_N + \beta_N)G_N - \frac{h_N^3}{6}\beta_N\left(\frac{\partial G}{\partial x}\right)_N, \quad l = i = N.$$

Neglecting the higher order terms i.e. $O(h_i^3)$ in $R_i$ then it is easy to see that $\mathbf{J}$ is irreducible [15]. By row sum criterion and for sufficiently small $h_i$, $\forall\, i = 1,2,\ldots,N$, $\mathbf{J}$ is monotone [17]. Thus $\mathbf{J}^{-1}$ exist and $\mathbf{J}^{-1} \geq 0$. For the bound of $\mathbf{J}$, we define [18, 19]

$$d_l(\mathbf{J}) = |J_{ll}| - \sum_{l\neq m}^{N} |J_{lm}|, \quad l = 1,2,\ldots,N,$$

where

$$d_1(\mathbf{J}) = \left(1 + \frac{d_1}{h_1}\right) + \frac{h_1^2}{6}\left(1 + \frac{d_1}{h_1}\right)\left(\left(\frac{d_1}{h_1}\right)^2 + 4\frac{d_1}{h_1} + 6\right)G_1 - \frac{h_1^3}{6}\left(1 + \frac{d_1}{h_1}\right)\frac{d_1}{h_1}\left(\frac{\partial G}{\partial x}\right)_1,$$

$$d_l(\mathbf{J}) = \frac{h_i^2}{6}\left(1 + \frac{d_i}{h_i}\right)\left(2(\frac{d_i}{h_i})^2 + 5\frac{d_i}{h_i} + 5\right)G_i - \frac{h_i^3}{6}\left(1 + \frac{d_i}{h_i}\right)\left(\frac{d_i}{h_i}\right)^2\left(\frac{\partial G}{\partial x}\right)_i, \quad 2 \leq l = i \leq N-1,$$

$$d_N(\mathbf{J}) = 1 + \frac{h_N^2}{6}\left(3\left(\frac{d_N}{h_N}\right)^2 + 8\frac{d_N}{h_N} + 6\right)G_N + \frac{h_N^3}{6}\left(1 + \frac{d_N}{h_N}\right)^2\frac{d_N}{h_N}\left(\frac{\partial G}{\partial x}\right)_N, \quad l = i = N.$$

Neglecting the higher order terms i.e. $O(h_i^3)$ in above expressions. Let $d_l(\mathbf{J}) \geq 0$, $\forall\, l$ and

$$d_*(\mathbf{J}) = \min_{1\leq l\leq N} d_l(\mathbf{J}).$$

Then

$$\|\mathbf{J}^{-1}\| \leq \frac{1}{d_*(\mathbf{J})} \tag{23}$$

Thus from (22) and (23), we have

$$\|\mathbf{E}\| \leq \frac{1}{d_*(\mathbf{J})}\|\mathbf{T}\|. \tag{24}$$

It follows from (14) and (24) that $\|\mathbf{E}\| \to 0$ as $h_i \to 0$. Thus we conclude that method (11) converges and the order of the convergence of method (11) is at least quadratic.

## 5. Numerical Results

To illustrate our method and demonstrate its computationally efficiency, we consider some model problems. In each case, we took non uniform step size $h$. In Table 1–Table 5, we have shown the maximum absolute error (MAY) and root mean square error (RMS), computed for different values of $N$ and is defined as

$$MAY = \max_{1\leq i\leq N} |y(x_i) - y_i|.$$

$$RMS = \sqrt{\frac{(y(x_i) - y_i)^2}{N}}.$$

The given interval $[a, b]$ is divided into $(N + 1)$ parts such that $a = x_0 < x_1 < x_2 \cdots < x_N < x_{N+1} = b$ where $h_{i+1} = x_{i+1} - x_i$, $i = 0, 1, \ldots, N$ and $d_i = h_{i+1} - h_i$, $i = 1, 2, \ldots, N$. We can write

$$b - a = x_{N+1} - x_0 = (x_{N+1} - x_N) + (x_N - x_{N-1}) + \cdots + (x_1 - x_0)$$

$$b - a = h_{N+1} + h_N + h_{N-1} + \cdots + h_2 + h_1$$

$$b - a = (d_N + d_{N-1} + \cdots + d_2 + d_1 + h_1) + (d_{N-1} + \cdots + h_1) + \cdots + (d_1 + h_1) + h_1$$

$$b - a = (N + 1)h_1 + Nd_1 + (N - 1)d_2 + \cdots + 2d_{N-1} + d_N.$$

Thus, from this we can determine first step length $h_1$ and subsequent step lengths are given as $h_1 + d_1$, $h_1 + d_1 + d_2, \ldots$ and so on. For simplicity we assumed that $d = d_i, \forall\, i = 1, 2, \ldots, N$ and so we have obtained following formula to determine $h_1$

$$h_1 = \frac{b - a}{N + 1} - \frac{Nd}{2}.$$

In case of uniform mesh $d = 0$, so above formula for computation of step length becomes $h_1 = \frac{b-a}{N+1}$. Also the upper bound in selection of $d$ is given by

$$d \leq \frac{2(b - a)}{N(N + 1)}$$

The order of the convergence $(O_N)$ of the method (11) estimated by the formula

$$(O_N) = \log_m(\frac{MAY_N}{MAY_{mN}}).$$

We have used Newton-Raphson iteration method to solve the system of nonlinear equations arised from equation (23). All computations were performed on a MS Window 2007 professional operating system in the GNU FORTRAN environment version 99 compiler (2.95 of gcc) on Intel Duo Core 2.20 Ghz PC. The solutions are computed on $N$ nodes and iteration is continued until either the maximum difference between two successive iterates is less than $10^{(-10)}$ or the number of iteration reached $10^3$.

**Problem 1.** The first model problem is a linear problem [20] given by

$$y''(x) = \frac{-3\epsilon}{(\epsilon+x)^2}y, \quad y(-0.1) = \frac{-0.1}{\sqrt{(\epsilon+0.01)}}, \quad y(0.1) = \frac{0.1}{\sqrt{(\epsilon+0.01)}}, \quad x \in [-0.1, 0.1].$$

The analytical solution is $y(x) = \frac{x}{\sqrt{(\epsilon+x^2)}}$. The MAY computed by method (11) for different values of $N$ and $\epsilon$ are presented in Table 1-3.

**Problem 2.** The second model problem is a nonlinear problem

$$\epsilon y''(x) = \frac{3}{2}y^2, \quad y(0) = 4, \quad y(1) = 1 \quad x \in [0, 1].$$

The analytical solution is $y(x) = \frac{4}{(1+\frac{x}{\sqrt{(\varepsilon)}})^2}$. The MAY computed by method (11) for different values of $N$ are presented in Table 4.

**Problem 3.** The third model problem is a linear problem [21] given by

$$\epsilon y''(x) = \frac{4}{(x+1)^4}(1+\sqrt{\epsilon}(x+1))y - f(x), \quad y(0) = 2, \quad y(1) = -1, \quad x \in [0, 1].$$

where $f(x)$ is calculated so that $y(x) = -\cos(\frac{4\pi x}{x+1}) + \frac{3[\exp(\frac{-2\epsilon}{\sqrt{\epsilon}(x+1)}) - \exp(\frac{-1}{\sqrt{\epsilon}})]}{1-\exp(\frac{-1}{\sqrt{\epsilon}})}$ is analytical solution. The *MAY* and *RMS* computed by method (11)for different values of $N$, $\epsilon$ and $d$ are presented in Table 5.

**Problem 4.** The fourth model problem is a linear problem [22] given by

$$\epsilon y''(x) = (1 + x(1-x))y - f(x), \quad y(0) = 0, \quad y(1) = 0, \quad x \in [0, 1].$$

where $f(x)$ is calculated so that $y(x) = 1+(x-1)\exp(\frac{-x}{\sqrt{\epsilon}}) - x\exp(\frac{-(1-x)}{\sqrt{\epsilon}})$ is analytical solution. The *MAY* and *RMS* computed by method (11) for different values of $N$, $\epsilon$ and $d$ are presented in Tables 6 & 7.

**Problem 5.** The fifth model problem is a linear problem [23] given by

$$\epsilon y''(x) = y - f(x), \quad y(0) = 0, \quad y(1) = 1, \quad x \in [0, 1].$$

where $f(x)$ is calculated so that $y(x) = \exp(x) + \exp(\frac{-x}{\sqrt{\epsilon}}) - x(\exp(1) + \exp(\frac{-1}{\sqrt{\epsilon}})) - 2(1-x)$ is analytical solution. The *MAY* and *RMS* computed by method (11)for different values of $N$, $\epsilon$ as there in [24] and $d$ are presented in Table 8.

TABLE 1. Maximum absolute errors in $y(x) = \frac{x}{\sqrt{(\epsilon+x^2)}}$ for problem 1.

| $N$ | $h$ | $d$ | $\epsilon = 10^{-5}$ <br> $MAY$ |
|---|---|---|---|
| 200 | .50000002(-3) | .50251256(-5) | .12177229(-3) |
| 400 | .25000001(-3) | .12531328(-5) | .30279160(-4) |
| 800 | .12500000(-3) | .31289113(-6) | .11801720(-4) |
| 1600 | .62500000(-4) | .78173862(-7) | .79870224(-5) |

TABLE 2.

| $N$ | $h$ | $d$ | $\epsilon = 10^{-7}$ <br> $MAY$ |
|---|---|---|---|
| 200 | .47500004(-3) | .52763817(-5) | .83470345(-3) |
| 400 | .23750002(-3) | .13157894(-5) | .21815300(-4) |
| 800 | .11875001(-3) | .32853566(-6) | .23245811(-5) |
| 1600 | .59375003(-4) | .82082551(-7) | .10192394(-4) |

TABLE 3.

| $N$ | $h$ | $d$ | $\epsilon = 2 \times 10^{-9}$ <br> $MAY$ |
|---|---|---|---|
| 200 | .27500005(-3) | .72864318(-5) | .15497208(-5) |
| 400 | .12625001(-3) | .18734336(-5) | .21457672(-5) |
| | .12650002(-3) | .18721804(-5) | .75101852(-5) |
| | .12637500(-3) | .18728070(-5) | .58412552(-5) |
| 800 | .68750006(-4) | .45369211(-6) | .10728836(-4) |
| 1600 | .34375003(-4) | .11335209(-6) | .21040440(-4) |

TABLE 4. Maximum absolute errors in $y(x) = \frac{4}{(1+\frac{x}{\sqrt{(\varepsilon)}})^2}$ for problem 2.

| $N$ | $h$ | $d$ | Maximum absolute error $\epsilon = 1.0$ | $\epsilon = 10.0$ | $\epsilon = 100.0$ |
|---|---|---|---|---|---|
| 4 | .23500 | .10(-1) | .25690079(-1) | .12538433(-2) | .23126608(-4) |
| 8 | .10750 | .50(-2) | .65517426(-2) | .31304359(-3) | .52452087(-5) |
| 16 | .04375 | .25(-2) | .15363693(-2) | .77962875(-4) | .47683716(-6) |
| 32 | .01188 | .125(-2) | .40102005(-3) | .13351440(-4) | .23841858(-6) |

TABLE 5. Maximum absolute and root mean square errors in $y(x) = -\cos(\frac{4\pi x}{x+1}) + \frac{3[\exp(\frac{-2\epsilon}{\sqrt{\epsilon}(x+1)})-\exp(\frac{-1}{\sqrt{\epsilon}})]}{1-\exp(\frac{-1}{\sqrt{\epsilon}})}$ for problem 3.

| $N$ | $h$ | $d$ | $\epsilon = 10^{-2}$ | | $\epsilon = 1.0$ | |
|---|---|---|---|---|---|---|
| | | | $MAY$ | $RMS$ | $MAY$ | $RMS$ |
| 4 | .24250 | .5000(-2) | .30282277(0) | .17568053(0) | .93566823(0) | .59396017(0) |
| 8 | .11625 | .2500(-2) | .19129242(0) | .79953931(-1) | .20843291(0) | .12492458(0) |
| 16 | .5313(-1) | .1250(-2) | .66353500(-1) | .24727164(-1) | .45094490(-1) | .25948962(-1) |
| 32 | .2156(-1) | .6250(-3) | .12697011(-1) | .52606389(-2) | .91903210(-2) | .51786867(-2) |
| 64 | .5780(-2) | .3125(-3) | .16708672(-2) | .84878172(-3) | .18413067(-2) | .10121076(-2) |

TABLE 6. Maximum absolute and root mean square errors in $y(x) = 1 + (x-1)\exp(\frac{-x}{\sqrt{\epsilon}}) - x\exp(\frac{-(1-x)}{\sqrt{\epsilon}})$ for Problem 4.

| $N$ | $h$ | $d$ | $\epsilon = 10^{-4}$ | | $\epsilon = 10^{-6}$ | |
|---|---|---|---|---|---|---|
| | | | $MAY$ | $RMS$ | $MAY$ | $RMS$ |
| 4 | .24250 | .5000(-2) | .13921738(-1) | .10614859(-1) | .36870301(-1) | .22434477(-1) |
| 8 | .11625 | .2500(-2) | .35239458(-2) | .25695723(-2) | .12767732(-1) | .67855748(-2) |
| 16 | .5313(-1) | .1250(-2) | .83976984(-3) | .60213194(-3) | .29334426(-2) | .15995282(-2) |
| 32 | .2156(-1) | .6250(-3) | .19204617(-3) | .13964981(-3) | .60218573(-3) | .34017002(-3) |
| 64 | .5780(-2) | .3125(-3) | .46372414(-4) | .32933302(-4) | .11366606(-3) | .69415473(-4) |

TABLE 7.

| $N$ | $h$ | $d$ | $\epsilon = 10^{-8}$ <br> $MAY$ | <br> $RMS$ |
|---|---|---|---|---|
| 4 | .24250 | .5000(-2) | .40731430(-1) | .23592696(-1) |
| 8 | .11625 | .2500(-2) | .35499215(-1) | .14240717(-1) |
| 16 | .5313(-1) | .1250(-2) | .10709703(-1) | .41285311(-2) |
| 32 | .2156(-1) | .6250(-3) | .20778785(-2) | .85051166(-3) |
| 64 | .5780(-2) | .3125(-3) | .28914213(-3) | .13981863(-3) |

TABLE 8. Maximum absolute errors in $y(x) = \exp(x) + \exp(\frac{-x}{\sqrt{\epsilon}}) - x(\exp(1) + \exp(\frac{-1}{\sqrt{\epsilon}})) - 2(1-x)$ for problem 5.

| $N$ | $h$ | $d$ | $\epsilon = \frac{1}{N}$ <br> $MAY$ | <br> $RMS$ |
|---|---|---|---|---|
| 4 | .24250 | .5000(-2) | .13634264(-1) | .11896761(-1) |
| 8 | .11625 | .2500(-2) | .57421327(-2) | .48838542(-2) |
| 16 | .5313(-1) | .1250(-2) | .22060275(-2) | .17996266(-2) |
| 32 | .2156(-1) | .6250(-3) | .10864139(-2) | .64855750(-3) |
| 64 | .5780(-2) | .3125(-3) | .74291229(-3) | .28846785(-3) |
| | | | $\epsilon = \frac{1}{N-1}$ <br> $MAY$ | <br> $RMS$ |
| 64 | .5780(-2) | .3125(-3) | .73188543(-3) | .28589202(-3) |
| | | | $\epsilon = \frac{1}{N+1}$ <br> $MAY$ | <br> $RMS$ |
| 64 | .5780(-2) | .3125(-3) | .75399876(-3) | .29125056(-3) |

We have described a method for numerically solving two-point boundary value problems and several model problems considered to demonstrate the computational efficiency of the proposed method. We know that there are causes $\epsilon$, $h$, and $d$ which affect approximate solution of the problem. The proposed method efficiently and accurately simulated these physically small features / causes in computation of consequences such as *MAY* and *RMS*. Numerical result for examples 1 which is presented in table 1, for different values of $d$ and $N$ show as $\epsilon$ and $d$ are decreases and $N$ increase, with variable step size maximum absolute errors in our method decreases i.e accuracy increase in numerical solution. The numerical results for examples 2 and 4 are good for non-uniform mesh sizes. The results for examples 3 as $N$ increases and both $d$ and $\epsilon$ are decrease, accuracy in numerical solution increases but if $\epsilon$ and $N$ are increases and $d$ decrease maximum absolute error also increase. The results for examples 5 as $N$ increases and both $d$ and $\epsilon$ are decrease, accuracy in numerical solution increases. If $N$ and $d$ are fixed and $\epsilon$ varies then accuracy varies in direct proportion to $\epsilon$. Over all method (11) is convergent and accuracy of the numerical solution of the problem using method (11) depends on choice of causes $d$, $N$ and $\epsilon$.

## 6. **Conclusion**

In this article, an order variable mesh size method to find the numerical solution of two point boundary value problems has been derived. Our method based on exponential approximation, if the source function is f(x) then the system of equations from (11) is linear otherwise we will obtain nonlinear system of equations, which is always difficult to be solved. In general, the finite difference method or any other numerical method can, in principle, be applied but it is obvious that special method required for some special problem where the solution is not regular and varies rapidly. The decision to use a certain difference method in solving these problems depend on computational efficiency of the method and this is entirely a numerical issue. The method presented in this article is efficient in solving such problem without any difficulty and produces good numerical approximate solutions. Thus we can conclude from the numerical results of the model problems that the proposed method is computationally efficient and accurate. The idea presented in this article leads to the possibility to develop methods of higher order and more accurate finite-difference approximations to solve boundary value problems in ODEs and PDEs. Works in this direction is in progress.

## REFERENCES

[1] Shashkov, M., Conservative Finite-Difference Methods on General Grids. CRC Press, Florida, FL, USA(1996).

[2] Ascher, U., Mattheij, R. M. M. AND Russell, R. D., Numerical Solution of Boundary Value Problems for Ordinary Differential Equations. Classics In Applied Mathematics Vol. 13, SIAM, Philadelphia, PA, USA (1995).

[3] Atkinson K., E., Han, W., and Stewart, D. E., Numerical Solution of Ordinary Differential Equations. John Wiley & Sons Inc., New Jersey, NJ, USA (2009).

[4] Keller, H. B., Numerical Methods for Two Point Boundary Value Problems. Dover, New York, NY, (1992).

[5] Stoer, J. and Bulirsch, R., Introduction to Numerical Analysis (2/e). Springer-Verlag, Berlin Heidelberg (1991).

[6] Baxley, J. V., Nonlinear Two Point Boundary Value Problems in Ordinary and Partial Differential Equations (Everrit, W.N. and Sleeman, B.D. Eds.), Springer-Verlag, New York (1981), 46–54.

[7] Roos, H.G., Stynes, M. and Tobiska, L., Numerical Methods for Singularly Perturbed Differential Equations. Springer, New York (1996).

[8] Hairer, E., Norsett, S. P. and Wanner, G., Solving Ordinary Differential Equations I Nonstiff Problems (Second Revised Edition). Springer-Verlag New York, Inc. New York, NY, USA (1993).

[9] Jain, M.K., Numerical Solution of Difierential Equations, Wiley Eastern Limited, New Delhi, (1984).

[10] Van Niekerk, F.D., A nonlinear one step methods for initial value problems. Comp. Math. Appl. 13(1987), 367–371.

[11] Pandey, P. K., Nonlinear Explicit Method for first Order Initial Value Problems. Acta Technica Jaurinensis, Vol. 6, No. 2 (2013), 118–125.

[12] Pandey, P. K., A nonstandard variable step length method for the solution of first order initial value problems in ODEs. Neural, Parallel, and Scientific Computations 22 (2014), 239–248.

[13] Pandey, P. K., Solving Two Point Boundary Value Problems for Ordinary Differential Equations Using Exponential Finite Difference Method. Bol. Soc. Paran. Mat. (2014), **-**.

[14] Hemker, P.W. and Miller, J.J.H., Numerical Analysis of Singular Perturbation Problems. Academic Press, New York, (1979).

[15] Varga, R. S., Matrix Iterative Analysis, Second Revised and Expanded Edition. Springer-Verlag, Heidelberg, (2000).

[16] Golub, G. H. and Van Loan, C.F., Matrix Computations. John Hopkins University Press, Baltimore, USA (1989).

[17] Henrici, P., Discrete Variable Methods in Ordinary Differential Equations. John Wiley and Sons, New York, NY, USA(1962).

[18] Volkov, Y. S. and Miroshnichenko, V. L., Norm estimates for the inverses of matrices of monotone type and totally positive matrics. Siberian Mathematical Journal Vol. 50, No. 6,(2009), 982–987.

[19] Ahlberg, J. H. and Nilson, E. N., Convergence properties of the spline fit. J. SIAM, 11, No. 1, (1963), 95–104.

[20] Scott, M. R. and Watts, H. A., Computational solution of linear two point boundary value problems via orthonormalization. SIAM J. Numer. Anal. 14 (1977), 40–70.

[21] O'Riordan, E. and Stynes, M., A uniformely accurate finite element method for a singularly perturbed one-dimensional reaction-diffusion problem. Math. Comput.,47 (1986),555-570.

[22] Vukoslavčević, V. and Surla, K., Finite element method for solving self adjoint singularly perturbed boundary value problems. Math. Montisnigri,8 (1996), 89-96.

[23] Schatz, A. H. and Wahlbin, L. B., On the finite element method for singularly perturbed reaction diffusion problems in two and one dimension. Math. Comput., 40 (1983), 47–89.

[24] Kadalbajoo, M. K. and Panwar, D. K., Variable mesh finite difference method for self-adjoint singularly perturbed two-point boundary value problems. Journal of Computational Mathematics, Vol.28, No. 5 (2010), 711–724.