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APPROXIMATION BY INTERPOLATING NEURAL NETWORK OPERATORS

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ABSTRACT. Here we introduce some general interpolating neural network operators in the univariate and multivariate cases. Initially we establish the interpolation property of the operators on functions. Then we derive the approximation properties of these operators on functions. We prove first the ordinary real quantitative pointwise and uniform convergences of these operators to the unit. Smoothness of functions is taken into consideration and speed of convergence improves dramatically. As extensions we consider also the fractional, fuzzy, fuzzy-fractional, fuzzy-random, complex and iterated cases. Furthermore we give Voronovskaya type asymptotic-expansions at all studied settings for the errors of related approximations.

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1. INTRODUCTION

This article is mainly inspired by the great article of D. Costarelli [27], where he establishes interpolation and approximation properties of very specific neural network operators.

We present here the general related theory of similar general neural network operators. We expand to all possible directions.

The featured interpolation and approximation properties of our approximations is something very rare.

We mention next in very brief the initial D. Costarelli [27] theory.

We consider $C([a,b])$ the space of all continuous functions $f:[a,b]\to\mathbb{R}$, $a,b\in\mathbb{R}$, $a<b$. Let now $\sigma_R:\mathbb{R}\to[0,1]$ the ramp function defined by

$$\sigma_R(x):=\begin{cases}0, & x\leq-\frac{1}{2},\\ 1, & x\geq\frac{1}{2},\\ x+\frac{1}{2}, & -\frac{1}{2}<x<\frac{1}{2}.\end{cases}\tag{1}$$

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The ramp function is a sigmoidal function $\sigma : \mathbb{R} \to \mathbb{R}$ which is measurable with $\lim_{x\to-\infty} \sigma(x) = 0$ and $\lim_{x\to+\infty} \sigma(x) = 1$. The last features arise in the theory of neural networks, where sigmoidal functions play the role of activation functions in the networks, see [38].

In [27], the author introduces

$$\Phi_R(x) := \sigma_R\left(x+\frac{1}{2}\right) - \sigma_R\left(x-\frac{1}{2}\right), \quad x \in \mathbb{R}. \tag{2}$$

The function $\Phi_R(x)$ has the properties: it is even, non-decreasing for $x < 0$ and non-increasing for $x \geq 0$, $\operatorname{sup}p(\Phi_R) \subseteq [-1, 1]$. Notice that $\Phi_R(\pm 1) = 0$.

Thus for $f : [a, b] \to \mathbb{R}$ a bounded and measurable function D. Costarelli [27], defines the neural network interpolation operator

$$F_n(f,x) := \frac{\sum_{k=0}^{n} f(x_k)\, \Phi_R\left(\frac{n(x-x_k)}{b-a}\right)}{\sum_{k=0}^{n} \Phi_R\left(\frac{n(x-x_k)}{b-a}\right)}, \quad x \in [a,b], \tag{3}$$

where the x_k's are the uniform spaced nodes defined by $x_k := a + kh$, $k = 0, 1, \ldots, n$, with $h := \frac{b-a}{n}$.

For a bounded measurable function f he proves

$$\|F_n(f)\|_\infty \leq \|f\|_\infty < +\infty, \tag{4}$$

where $\|f\|_\infty := \sup_{x\in[a,b]} |f(x)|$.

He also proves

Theorem 1.1 ([27]). *Let $f : [a, b] \to \mathbb{R}$ a bounded measurable function and $n \in \mathbb{N}$. Then*

$$F_n(f, x_i) = f(x_i), \quad i = 0, 1, \ldots, n. \tag{5}$$

Theorem 1.2 ([27]). *Let $f \in C([a, b])$. Then*

$$\|F_n(f) - f\|_\infty \leq 4\omega_1\left(f, \frac{b-a}{n}\right), \quad \forall\, n \in \mathbb{N}. \tag{6}$$

Above he uses

$$\omega_1(f,\delta) := \sup_{\substack{x,y:\\ |x-y|\leq\delta}} |f(x) - f(y)|, \quad 0 < \delta \leq b-a, \tag{7}$$

and if $\delta > b - a$, $\omega_1(f, \delta) := \omega_1(f, b - a)$, the first modulus of continuity.

D. Costarelli [27] gives also another specific example of interpolation neural network operators with the same properties as the F_n operators.

Denote by

$$M_s(x) := \frac{1}{(s-1)!}\sum_{i=0}^{s}(-1)^i \binom{s}{i}\left(\frac{s}{2}+x-i\right)_+^{s-1}, \quad x \in \mathbb{R}, \tag{8}$$

the B-spline of order $s \in \mathbb{N}$ [25], where $(x)_+ = \max\{x,0\}$, and $\sup p(M_S) \subseteq \left[-\frac{s}{2}, \frac{s}{2}\right]$.

He defines [27] the sigmoidal functions

$$\sigma_{M_s}(x) := \int_{-\infty}^{x} M_s(t)\,dt, \quad x \in \mathbb{R}, \tag{9}$$

and the non-negative density functions:

$$\Phi_s(x) := \sigma_{M_s}\left(x+\frac{1}{2}\right) - \sigma_{M_s}\left(x-\frac{1}{2}\right), \quad x \in \mathbb{R}, \ \forall\, s \in \mathbb{N}. \tag{10}$$

The functions Φ_s have the properties: even, non-decreasing for $x < 0$ and non-increasing for $x \geq 0$, $\sup p(\Phi_s) \subseteq [-K_s, K_s] := \left[-\frac{(s+1)}{2}, \frac{(s+1)}{2}\right]$ and $\Phi_s\left(\frac{K_s}{2}\right) > 0$. Notice that $\Phi_s(\pm K_s) = 0$.

He [27] defines similarly the neural network operators

$$F_n^s(f,x) := \frac{\sum_{k=0}^{n} f(x_k)\,\Phi_s\left(K_s\frac{n(x-x_k)}{b-a}\right)}{\sum_{k=0}^{n}\Phi_s\left(K_s\frac{n(x-x_k)}{b-a}\right)}, \quad \forall\, x \in [a,b], \tag{11}$$

where $x_k := a + kh$, $k = 0,1,\ldots,n$, and $h := \frac{b-a}{n}$.

Theorem 1.3 ([27]). *Let $f : [a,b] \to \mathbb{R}$ a bounded and measurable function, $n \in \mathbb{N}$. Then*

$$F_n^s(f,x_k) = f(x_k), \quad k = 0,1,\ldots,n, \ s \in \mathbb{N}, \tag{12}$$

the interpolation property.

In addition, for $f \in C([a,b])$ we have

$$\|F_n^s(f) - f\|_\infty \leq \frac{2}{\Phi_s\left(\frac{K_s}{2}\right)}\omega_1\left(f, \frac{b-a}{n}\right), \quad \forall\, n, s \in \mathbb{N}. \tag{13}$$

Above the samples $f(x_k)$ can be viewed as the elements of the training set that can be used to train the normalized neural networks F_n, F_n^s. According to [27], the interpolation results show that the representation errors made by F_n, F_n^s on the elements of the training set are zero.

Furthermore the uniform approximation results, show the closeness property of neural network operators to well estimate elements outside the training set.

So our general theory presented in this article is the natural and complete outgrowth of [27] in very general diverse settings.

Other books and articles that inspired our work are: [12], [16], [17], [18], [19], [20], [21], [22], [23], [26], [36], [37].

The author was the first in 1997 to establish quantitative neural network approximations, see [1], [2], [3], [5], etc.

2. MAIN RESULTS

2.1. Neural Networks: Univariate theory of Interpolation and Approximation.

We need

Definition 2.1. Let $B : \mathbb{R} \rightarrow \mathbb{R}_+$, be a bell-shaped function of compact support $[-T, T]$, $T > 0$. We assume it is even, non-decreasing for $x < 0$ and non-increasing for $x \geq 0$. Suppose also that $B(0) =: B^* > 0$ is the global maximum of B. The function B may have jump discontinuities and it is measurable. Assume further that $B(\pm T) = 0$.

Examples for B can be the hat function

$$\beta(x) := \begin{cases} 1+x, & -1 \leq x \leq 0, \\ 1-x, & 0 < x \leq 1, \\ 0, & elsewhere, \end{cases}$$

the function Φ_R, see (2), and the function Φ_s, see (10). Etc.

Definition 2.2. Let $f : [a, b] \rightarrow \mathbb{R}$, $a, b \in \mathbb{R}$, $a < b$, a bounded and measurable function, $n \in \mathbb{N}$, $h := \frac{b-a}{n}$, $x_k := a + kh$, $k = 0, 1, \ldots, n$, $x \in [a, b]$.

We define the interpolation neural network operator

$$H_n(f, x) := \frac{\sum_{k=0}^{n} f(x_k) B\left(\frac{Tn(x-x_k)}{b-a}\right)}{\sum_{k=0}^{n} B\left(\frac{Tn(x-x_k)}{b-a}\right)}. \tag{14}$$

We make

Remark 2.3 (on $H_n(f, x)$)**.** We observe that

$$|H_n(f, x)| \leq \frac{\sum_{k=0}^{n} |f(x_k)| B\left(\frac{Tn(x-x_k)}{b-a}\right)}{\sum_{k=0}^{n} B\left(\frac{Tn(x-x_k)}{b-a}\right)} \leq \|f\|_\infty < +\infty. \tag{15}$$

That is

$$\|H_n(f)\|_\infty \leq \|f\|_\infty. \tag{16}$$

We make

Remark 2.4. Let $x \in [a, b]$, then $x_k \leq x \leq x_{k+1}$, for some $k \in \{0, 1, \ldots, n-1\}$, and $|x - x_k| \leq h$, $|x - x_{k+1}| \leq h$.

Notice that $B\left(\frac{Tn(x-x_k)}{b-a}\right) \neq 0$

$$\Leftrightarrow -T < \frac{Tn\,(x-x_k)}{b-a} < T$$

$$\Leftrightarrow -1 < \frac{n\,(x-x_k)}{b-a} < 1 \tag{17}$$

$$\Leftrightarrow -h < x - x_k < h$$

$$\Leftrightarrow |x-x_k| < h.$$

So when $x \in (x_k, x_{k+1})$, for some $k \in \{0,1,\dots,n-1\}$, we get both

$$B\left(\frac{Tn\,(x-x_k)}{b-a}\right),\ B\left(\frac{Tn\,(x-x_{k+1})}{b-a}\right) \neq 0.$$

When $x = x_k$, then

$$B\left(\frac{Tn\,(x_k-x_k)}{b-a}\right) = B\,(0) = B^* > 0,$$

and

$$B\left(\frac{Tn\,(x_k-x_{k+1})}{b-a}\right) = B\,(-T) = 0.$$

When $x = x_{k+1}$, then

$$B\left(\frac{Tn\,(x_{k+1}-x_k)}{b-a}\right) = B\,(T) = 0,$$

and

$$B\left(\frac{Tn\,(x_{k+1}-x_{k+1})}{b-a}\right) = B\,(0) = B^* > 0.$$

Clearly for any $x \in [x_k, x_{k+1}]$ we get that

$$B\left(\frac{Tn\,(x-x_i)}{b-a}\right) = 0, \ \text{ for all } i \neq k, k+1. \tag{18}$$

We make

Remark 2.5. For $x \in [a,b]$ we notice that

$$V\,(x) := \sum_{k=0}^{n} B\left(\frac{Tn\,(x-x_k)}{b-a}\right) = \sum_{k=0}^{n} B\left(\frac{Tn\,|x-x_k|}{b-a}\right) \geq B\left(\frac{Tn\,|x-x_i|}{b-a}\right), \tag{19}$$

where $i \in \{0,1,\dots,n\}$ is such that $|x-x_i| \leq \frac{h}{2}$. Thus

$$\frac{Tn\,|x-x_i|}{b-a} \leq \frac{Tnh}{2\,(b-a)} = \frac{T}{2}. \tag{20}$$

Therefore

$$B\left(\frac{Tn\,|x-x_i|}{b-a}\right) \geq B\left(\frac{T}{2}\right), \tag{21}$$

where $B\left(\frac{T}{2}\right) > 0$.

Thus $V\,(x) \geq B\left(\frac{T}{2}\right)$.

Consequently it holds

$$\frac{1}{V(x)}=\frac{1}{\sum_{k=0}^{n}B\left(\frac{Tn(x-x_k)}{b-a}\right)}\leq\frac{1}{B\left(\frac{T}{2}\right)}. \tag{22}$$

We state the interpolation result

Theorem 2.6. *Let $f:[a,b]\to\mathbb{R}$ be a bounded and measurable function. Then*

$$H_n(f,x_i)=f(x_i),\quad i=0,1,\ldots,n, \tag{23}$$

where $x_i:=a+ih$, $h:=\frac{b-a}{n}$, $n\in\mathbb{N}$.

Proof. Let $i\in\{0,1,\ldots,n\}$ be fixed. When $k=i$, we have that

$$B\left(\frac{Tn(x_i-x_k)}{b-a}\right)=B(0)=B^*>0. \tag{24}$$

But when $k\neq i$ we have

$$\frac{Tn|x_i-x_k|}{b-a}\geq\frac{Tnh}{b-a}=T, \tag{25}$$

hence

$$0\leq B\left(\frac{Tn(x_i-x_k)}{b-a}\right)=B\left(\frac{Tn|x_i-x_k|}{b-a}\right)\leq B(T)=0. \tag{26}$$

So we conclude that

$$B\left(\frac{Tn(x_i-x_k)}{b-a}\right)=\left\{\begin{array}{l}B^*,\ \ i=k,\\ 0,\ \ i\neq k\end{array}\right\}, \tag{27}$$

for any $i,k=0,1,\ldots,n$.

By (27) we derive that

$$H_n(f,x_i)=\frac{f(x_i)B\left(\frac{Tn(x_i-x_i)}{b-a}\right)}{B\left(\frac{Tn(x_i-x_i)}{b-a}\right)}=\frac{f(x_i)B^*}{B^*}=f(x_i),\quad i=0,1,\ldots,n, \tag{28}$$

proving the claim. □

We state our first approximation result at Jackson speed of convergence $\frac{1}{n}$.

Theorem 2.7. *Let $f\in C([a,b])$. Then*

$$\|H_n(f)-f\|_\infty\leq\frac{2B^*}{B\left(\frac{T}{2}\right)}\omega_1\left(f,\frac{b-a}{n}\right),\quad\forall\ n\in\mathbb{N}. \tag{29}$$

Proof. Let $x\in[a,b]$, we can write

$$H_n(f,x)-f(x)=\frac{\sum_{k=0}^{n}f(x_k)B\left(\frac{Tn(x-x_k)}{b-a}\right)}{\sum_{k=0}^{n}B\left(\frac{Tn(x-x_k)}{b-a}\right)}-f(x)$$

$$= \frac{\sum_{k=0}^{n} f(x_k) B\left(\frac{Tn(x-x_k)}{b-a}\right) - f(x)\left(\sum_{k=0}^{n} B\left(\frac{Tn(x-x_k)}{b-a}\right)\right)}{V(x)}$$

$$= \frac{\sum_{k=0}^{n} (f(x_k) - f(x)) B\left(\frac{Tn(x-x_k)}{b-a}\right)}{V(x)}. \tag{30}$$

Therefore it holds

$$|H_n(f,x) - f(x)| \leq \frac{\sum_{k=0}^{n} |f(x_k) - f(x)| B\left(\frac{Tn(x-x_k)}{b-a}\right)}{V(x)}$$

$$\overset{(22)}{\leq} \frac{1}{B\left(\frac{T}{2}\right)} \left\{ \sum_{k=0}^{n} |f(x_k) - f(x)| B\left(\frac{Tn(x-x_k)}{b-a}\right) \right\} =: (*). \tag{31}$$

Let now $i \in \{0, 1, \ldots, n-1\}$ such that $x_i \leq x \leq x_{i+1}$. Hence

$$(*) = \frac{1}{B\left(\frac{T}{2}\right)} \left\{ \sum_{\substack{k=0 \\ k \neq i, i+1}}^{n} |f(x_k) - f(x)| B\left(\frac{Tn(x-x_k)}{b-a}\right) \right.$$
$$\left. + |f(x_i) - f(x)| B\left(\frac{Tn(x-x_i)}{b-a}\right) + |f(x_{i+1}) - f(x)| B\left(\frac{Tn(x-x_{i+1})}{b-a}\right) \right\}$$

(32)

$$\leq \frac{1}{B\left(\frac{T}{2}\right)} \left\{0 + \omega_1(f,h) B^* + \omega_1(f,h) B^*\right\} = \frac{2B^*}{B\left(\frac{T}{2}\right)} \omega_1(f,h).$$

We derive for $f \in C([a,b])$ that it holds

$$|H_n(f,x) - f(x)| \leq \frac{2B^*}{B\left(\frac{T}{2}\right)} \omega_1\left(f, \frac{b-a}{n}\right), \quad \forall\, x \in [a,b]. \tag{33}$$

The theorem now is proved. ☐

Taking into account the smoothness of f, we present the following high order approximation result.

Theorem 2.8. *Let $f \in C^N([a,b])$, $N \in \mathbb{N}$, $x \in [a,b]$. Then*

i)

$$|H_n(f,x) - f(x)|$$
$$\leq \frac{2B^*}{B\left(\frac{T}{2}\right)} \left[\sum_{j=1}^{N} \frac{\left|f^{(j)}(x)\right|}{j!} \frac{(b-a)^j}{n^j} + \omega_1\left(f^{(N)}, \frac{b-a}{n}\right) \frac{(b-a)^N}{n^N N!} \right], \tag{34}$$

ii)

$$\|H_n(f) - f\|_\infty$$

$$\leq \frac{2B^*}{B\left(\frac{T}{2}\right)}\left[\sum_{j=1}^{N}\frac{\left\|f^{(j)}\right\|_\infty}{j!}\frac{(b-a)^j}{n^j}+\omega_1\left(f^{(N)},\frac{b-a}{n}\right)\frac{(b-a)^N}{n^N N!}\right], \tag{35}$$

iii) Assume more that $f^{(j)}(x)=0$, $j=1,\dots,N$, where $x\in[a,b]$ is fixed, we get

$$|H_n(f,x)-f(x)|\leq \frac{2B^*}{B\left(\frac{T}{2}\right)}\omega_1\left(f^{(N)},\frac{b-a}{n}\right)\frac{(b-a)^N}{n^N N!}, \tag{36}$$

a high speed $\frac{1}{n^{N+1}}$ pointwise convergence, and

iv)

$$\left|H_n(f,x)-f(x)-\sum_{j=1}^{N}\frac{f^{(j)}(x)}{j!}H_n\left((\cdot-x)^j,x\right)\right|$$
$$\leq \frac{2B^*}{B\left(\frac{T}{2}\right)}\omega_1\left(f^{(N)},\frac{b-a}{n}\right)\frac{(b-a)^N}{n^N N!}. \tag{37}$$

Proof. Let $f\in C^N([a,b])$, $N\in\mathbb{N}$. Then

$$f(x_k)=\sum_{j=0}^{N}\frac{f^{(j)}(x)}{j!}(x_k-x)^j+\int_x^{x_k}\left(f^{(N)}(t)-f^{(N)}(x)\right)\frac{(x_k-t)^{N-1}}{(N-1)!}dt. \tag{38}$$

Hence it holds

$$\frac{f(x_k)B\left(\frac{Tn(x-x_k)}{b-a}\right)}{V(x)}=\sum_{j=0}^{N}\frac{f^{(j)}(x)}{j!}(x_k-x)^j\frac{B\left(\frac{Tn(x-x_k)}{b-a}\right)}{V(x)}$$
$$+\frac{B\left(\frac{Tn(x-x_k)}{b-a}\right)}{V(x)}\int_x^{x_k}\left(f^{(N)}(t)-f^{(N)}(x)\right)\frac{(x_k-t)^{N-1}}{(N-1)!}dt. \tag{39}$$

Thus we can write

$$H_n(f,x)-f(x)=\frac{\sum_{k=0}^{n}f(x_k)B\left(\frac{Tn(x-x_k)}{b-a}\right)}{V(x)}-f(x)$$
$$=\sum_{j=1}^{N}\frac{f^{(j)}(x)}{j!}\frac{\sum_{k=0}^{n}(x_k-x)^jB\left(\frac{Tn(x-x_k)}{b-a}\right)}{V(x)}$$
$$+\frac{\sum_{k=0}^{n}B\left(\frac{Tn(x-x_k)}{b-a}\right)}{V(x)}\int_x^{x_k}\left(f^{(N)}(t)-f^{(N)}(x)\right)\frac{(x_k-t)^{N-1}}{(N-1)!}dt. \tag{40}$$

Call

$$R_n(x):=\frac{\sum_{k=0}^{n}B\left(\frac{Tn(x-x_k)}{b-a}\right)}{V(x)}\int_x^{x_k}\left(f^{(N)}(t)-f^{(N)}(x)\right)\frac{(x_k-t)^{N-1}}{(N-1)!}dt. \tag{41}$$

Also call

$$\gamma\left(x,x_{k}\right):=\left|\int_{x}^{x_{k}}\left(f^{(N)}\left(t\right)-f^{(N)}\left(x\right)\right)\frac{\left(x_{k}-t\right)^{N-1}}{(N-1)!}dt\right|. \tag{42}$$

We distinguish the cases:

(i) Let $x\leq x_{k}$, then

$$\gamma\left(x,x_{k}\right)\leq\int_{x}^{x_{k}}\left|f^{(N)}\left(t\right)-f^{(N)}\left(x\right)\right|\frac{\left(x_{k}-t\right)^{N-1}}{(N-1)!}dt$$

$$\leq\omega_{1}\left(f^{(N)},x_{k}-x\right)\frac{\left(x_{k}-x\right)^{N}}{N!}. \tag{43}$$

(ii) Let $x\geq x_{k}$, then

$$\gamma\left(x,x_{k}\right)=\left|\int_{x_{k}}^{x}\left(f^{(N)}\left(t\right)-f^{(N)}\left(x\right)\right)\frac{\left(t-x_{k}\right)^{N-1}}{(N-1)!}dt\right|$$

$$\leq\int_{x_{k}}^{x}\left|f^{(N)}\left(t\right)-f^{(N)}\left(x\right)\right|\frac{\left(t-x_{k}\right)^{N-1}}{(N-1)!}dt$$

$$\leq\omega_{1}\left(f^{(N)},x-x_{k}\right)\frac{\left(x-x_{k}\right)^{N}}{N!}. \tag{44}$$

We have found that

$$\gamma\left(x,x_{k}\right)\leq\omega_{1}\left(f^{(N)},\left|x-x_{k}\right|\right)\frac{\left|x-x_{k}\right|^{N}}{N!}. \tag{45}$$

Therefore it holds

$$\left|R_{n}\left(x\right)\right|\leq\frac{\sum_{k=0}^{n}B\left(\frac{Tn\left(x-x_{k}\right)}{b-a}\right)}{V\left(x\right)}\omega_{1}\left(f^{(N)},\left|x-x_{k}\right|\right)\frac{\left|x-x_{k}\right|^{N}}{N!}=:\left(*\right). \tag{46}$$

Given that $x_{k}\leq x\leq x_{k+1}$, for some $k\in\{0,1,\ldots,n-1\}$, we get

$$\left(*\right)=\frac{B\left(\frac{Tn\left(x-x_{k}\right)}{b-a}\right)\omega_{1}\left(f^{(N)},\left|x-x_{k}\right|\right)\frac{\left|x-x_{k}\right|^{N}}{N!}}{V\left(x\right)}$$

$$+\frac{B\left(\frac{Tn\left(x-x_{k+1}\right)}{b-a}\right)\omega_{1}\left(f^{(N)},\left|x-x_{k+1}\right|\right)\frac{\left|x-x_{k+1}\right|^{N}}{N!}}{V\left(x\right)}$$

$$\leq\frac{2B^{*}\omega_{1}\left(f^{(N)},\frac{b-a}{n}\right)\frac{(b-a)^{N}}{n^{N}N!}}{B\left(\frac{T}{2}\right)}. \tag{47}$$

We have proved that

$$\left|R_{n}\left(x\right)\right|\leq\frac{2B^{*}}{B\left(\frac{T}{2}\right)}\omega_{1}\left(f^{(N)},\frac{b-a}{n}\right)\frac{\left(b-a\right)^{N}}{n^{N}N!}. \tag{48}$$

Next we observe

$$\frac{\left|\sum_{k=0}^{n}\left(x_{k}-x\right)^{j}B\left(\frac{Tn\left(x-x_{k}\right)}{b-a}\right)\right|}{V\left(x\right)}\leq\frac{\sum_{k=0}^{n}\left|x_{k}-x\right|^{j}B\left(\frac{Tn\left(x-x_{k}\right)}{b-a}\right)}{V\left(x\right)}$$

$$\leq \frac{1}{B\left(\frac{T}{2}\right)}\left\{\left|x_k-x\right|^j B\left(\frac{Tn\left(x-x_k\right)}{b-a}\right)+\left|x_{k+1}-x\right|^j B\left(\frac{Tn\left(x-x_{k+1}\right)}{b-a}\right)\right\}$$

$$\leq \frac{2B^* \frac{(b-a)^j}{n^j}}{B\left(\frac{T}{2}\right)}. \tag{49}$$

Therefore we derive

$$\left|\sum_{j=1}^{N} \frac{f^{(j)}(x)}{j!} \frac{\sum_{k=0}^{n}\left(x_k-x\right)^j B\left(\frac{Tn\left(x-x_k\right)}{b-a}\right)}{V(x)}\right| \leq \frac{2B^*}{B\left(\frac{T}{2}\right)}\left(\sum_{j=1}^{N} \frac{\left|f^{(j)}(x)\right|}{j!} \frac{(b-a)^j}{n^j}\right). \tag{50}$$

Using (48) and (50) we derive (34)-(36).

Noticing that

$$\frac{\sum_{k=0}^{n}\left(x_k-x\right)^j B\left(\frac{Tn\left(x-x_k\right)}{b-a}\right)}{V(x)}=H_n\left((\cdot-x)^j, x\right), \tag{51}$$

we derive (37).

The theorem is proved. ∎

We present a related Voronovskaya type asymptotic expansion for the error of approximation.

Theorem 2.9. *Let $f \in C^N([a,b])$, $N \in \mathbb{N}$. Then*

$$H_n(f,x)-f(x)-\sum_{j=1}^{N-1} \frac{f^{(j)}(x)}{j!} H_n\left((\cdot-x)^j, x\right)=o\left(\frac{1}{n^{N-\varepsilon}}\right), \tag{52}$$

where $0<\varepsilon \leq N$, $n \in \mathbb{N}$.

If $N=1$, the sum above disappears.

Asymptotic expansion (52) implies

$$n^{N-\varepsilon}\left[H_n(f,x)-f(x)-\sum_{j=1}^{N-1} \frac{f^{(j)}(x)}{j!} H_n\left((\cdot-x)^j, x\right)\right] \rightarrow 0, \text{ as } n \rightarrow \infty, \tag{53}$$

$0<\varepsilon \leq N$.

When $N=1$, or $f^{(j)}(x)=0$, $j=1,\ldots,N-1$, then

$$n^{N-\varepsilon}\left[H_n(f,x)-f(x)\right] \rightarrow 0, \text{ as } n \rightarrow \infty,\ 0<\varepsilon \leq N. \tag{54}$$

Proof. Let $x \in [a,b]$, then

$$f\left(x_k\right)=\sum_{j=0}^{N-1} \frac{f^{(j)}(x)}{j!}\left(x_k-x\right)^j+\int_x^{x_k} f^{(N)}(t) \frac{\left(x_k-t\right)^{N-1}}{(N-1)!} dt. \tag{55}$$

Let here $i \in \{0,1,\ldots,n-1\}$ such that $x_i \leq x \leq x_{i+1}$.

Hence we have

$$\frac{f(x_k)\, B\left(\frac{Tn(x-x_k)}{b-a}\right)}{V(x)} = \sum_{j=0}^{N-1} \frac{f^{(j)}(x)}{j!} (x_k - x)^j \frac{B\left(\frac{Tn(x-x_k)}{b-a}\right)}{V(x)}$$

$$+ \frac{B\left(\frac{Tn(x-x_k)}{b-a}\right)}{V(x)} \int_x^{x_k} f^{(N)}(t) \frac{(x_k - t)^{N-1}}{(N-1)!} dt. \tag{56}$$

Thus it holds

$$H_n(f,x) - f(x) = \frac{\sum_{k=0}^{n} f(x_k)\, B\left(\frac{Tn(x-x_k)}{b-a}\right)}{V(x)} - f(x)$$

$$= \sum_{j=1}^{N-1} \frac{f^{(j)}(x)}{j!} \frac{\sum_{k=0}^{n} (x_k - x)^j B\left(\frac{Tn(x-x_k)}{b-a}\right)}{V(x)}$$

$$+ \frac{\sum_{k=0}^{n} B\left(\frac{Tn(x-x_k)}{b-a}\right)}{V(x)} \int_x^{x_k} f^{(N)}(t) \frac{(x_k - t)^{N-1}}{(N-1)!} dt. \tag{57}$$

Call

$$R(x) := \frac{\sum_{k=0}^{n} B\left(\frac{Tn(x-x_k)}{b-a}\right)}{V(x)} \int_x^{x_k} f^{(N)}(t) \frac{(x_k - t)^{N-1}}{(N-1)!} dt. \tag{58}$$

So that

$$H_n(f,x) - f(x) - \sum_{j=1}^{N-1} \frac{f^{(j)}(x)}{j!} H_n\left((\cdot - x)^j, x\right) = R(x). \tag{59}$$

Hence it holds

$$|R(x)| \leq \frac{\sum_{k=0}^{n} B\left(\frac{Tn(x-x_k)}{b-a}\right)}{B\left(\frac{T}{2}\right)} \left| \int_x^{x_k} f^{(N)}(t) \frac{(x_k - t)^{N-1}}{(N-1)!} dt \right| \leq (*). \tag{60}$$

But we find:

i) Let $x_k \geq x$. Then

$$\left| \int_x^{x_k} f^{(N)}(t) \frac{(x_k - t)^{N-1}}{(N-1)!} dt \right| \leq \int_x^{x_k} \left| f^{(N)}(t) \right| \frac{(x_k - t)^{N-1}}{(N-1)!} dt \leq \left\| f^{(N)} \right\|_\infty \frac{(x_k - x)^N}{N!}. \tag{61}$$

ii) Let $x_k \leq x$. Then

$$\left| \int_x^{x_k} f^{(N)}(t) \frac{(x_k - t)^{N-1}}{(N-1)!} dt \right| = \left| \int_{x_k}^{x} f^{(N)}(t) \frac{(t - x_k)^{N-1}}{(N-1)!} dt \right|$$

$$\leq \int_{x_k}^{x} \left| f^{(N)}(t) \right| \frac{(t - x_k)^{N-1}}{(N-1)!} dt$$

$$\leq \left\| f^{(N)} \right\|_{\infty} \frac{(x - x_k)^N}{N!}. \tag{62}$$

So in either case we have proved

$$\left| \int_x^{x_k} f^{(N)}(t) \frac{(x_k - t)^{N-1}}{(N-1)!} dt \right| \leq \left\| f^{(N)} \right\|_{\infty} \frac{|x - x_k|^N}{N!}. \tag{63}$$

Therefore we find

$$(*) \leq \frac{\sum_{k=0}^{n} B\left(\frac{Tn(x-x_k)}{b-a}\right)}{B\left(\frac{T}{2}\right)} \left\| f^{(N)} \right\|_{\infty} \frac{|x - x_k|^N}{N!}$$

$$\leq \frac{2B^*}{B\left(\frac{T}{2}\right)} \left\| f^{(N)} \right\|_{\infty} \frac{(b-a)^N}{N! n^N}. \tag{64}$$

We have proved that

$$|R(x)| \leq \frac{\psi}{n^N}, \tag{65}$$

where

$$\psi := \frac{2B^*}{B\left(\frac{T}{2}\right)} \frac{\left\| f^{(N)} \right\|_{\infty} (b-a)^N}{N!}. \tag{66}$$

Hence we derive

$$|R(x)| = O\left(\frac{1}{n^N}\right), \tag{67}$$

and

$$|R(x)| = o(1). \tag{68}$$

Letting $0 < \varepsilon \leq N$, we obtain

$$\frac{|R(x)|}{\left(\frac{1}{n^{N-\varepsilon}}\right)} \leq \frac{\psi}{n^{\varepsilon}} \to 0, \tag{69}$$

as $n \to \infty$. So that

$$|R(x)| = o\left(\frac{1}{n^{N-\varepsilon}}\right), \quad n \in \mathbb{N}, \tag{70}$$

proving the claim. □

We need

Definition 2.10. Let $\nu > 0$, $m = \lceil \nu \rceil$ ($\lceil \cdot \rceil$ is the ceiling of the number), $f \in AC^m([a,b])$ (space of functions f with $f^{(m-1)} \in AC([a,b])$, absolutely continuous functions). We call left Caputo fractional derivative (see [28, pp. 49–52], [31], [39]) the function

$$D_{*a}^{\nu} f(x) := \frac{1}{\Gamma(m-\nu)} \int_a^x (x-t)^{m-\nu-1} f^{(m)}(t)\, dt, \tag{71}$$

$\forall\, x \in [a,b]$, where Γ is the gamma function $\Gamma(\nu) := \int_0^{\infty} e^{-t} t^{\nu-1} dt$, $\nu > 0$.

We set $D_{*a}^{0}f(x) = f(x)$, $\forall\, x \in [a,b]$.

Lemma 2.11 ([8])**.** *Let $\nu > 0$, $\nu \notin \mathbb{N}$, $m = \lceil \nu \rceil$, $f \in C^{m-1}([a,b])$ and $f^{(m)} \in L_\infty([a,b])$. Then $D_{*a}^{\nu}f(a) = 0$.*

Definition 2.12 (see also [9], [30], [31])**.** Let $f \in AC^m([a,b])$, $m = \lceil \nu \rceil$, $\nu > 0$. The right Caputo fractional derivative of order $\nu > 0$ is given by

$$D_{b-}^{\nu}f(x) := \frac{(-1)^m}{\Gamma(m-\nu)} \int_x^b (z-x)^{m-\nu-1} f^{(m)}(z)\, dz, \tag{72}$$

$\forall\, x \in [a,b]$. We set $D_{b-}^{0}f(x) = f(x)$.

Lemma 2.13 ([8])**.** *Let $f \in C^{m-1}([a,b])$, $f^{(m)} \in L_\infty([a,b])$, $m = \lceil \nu \rceil$, $\nu > 0$. Then $D_{b-}^{\nu}f(b) = 0$.*

Convention 2.14 ([8])**.** We assume that

$$\begin{array}{l} D_{*x_0}^{\nu}f(x) = 0, \ \text{ for } x < x_0, \\ \text{and} \\ D_{x_0-}^{\nu}f(x) = 0, \ \text{ for } x > x_0, \end{array} \tag{73}$$

for all $x, x_0 \in [a,b]$.

We present the related fractional approximation result

Theorem 2.15. *Let $\beta > 0$, $N = \lceil \beta \rceil$, $\beta \notin \mathbb{N}$, $f \in AC^N([a,b])$, $f^{(N)} \in L_\infty([a,b])$. Then*

i)

$$\begin{aligned} |H_n(f,x) - f(x)| \leq \frac{B^*}{B\left(\frac{T}{2}\right)} \Bigg[2 \sum_{j=1}^{N-1} \frac{\left|f^{(j)}(x)\right|}{j!} \frac{(b-a)^j}{n^j} \\ + \frac{(b-a)^\beta}{\Gamma(\beta+1)\, n^\beta} \left[\omega_1\left(D_{x-}^{\beta}f, \frac{b-a}{n}\right) + \omega_1\left(D_{*x}^{\beta}f, \frac{b-a}{n}\right) \right] \Bigg], \end{aligned} \tag{74}$$

and

ii)

$$\begin{aligned} \|H_n(f) - f\|_\infty \leq \frac{B^*}{B\left(\frac{T}{2}\right)} \Bigg[2 \sum_{j=1}^{N-1} \frac{\left\|f^{(j)}\right\|_\infty}{j!} \frac{(b-a)^j}{n^j} \\ + \frac{(b-a)^\beta}{\Gamma(\beta+1)\, n^\beta} \left[\sup_{x\in[a,b]} \omega_1\left(D_{x-}^{\beta}f, \frac{b-a}{n}\right) + \sup_{x\in[a,b]} \omega_1\left(D_{*x}^{\beta}f, \frac{b-a}{n}\right) \right] \Bigg] < \infty. \end{aligned} \tag{75}$$

Proof. Let fixed $x \in [a,b]$ with $x_i \leq x \leq x_{i+1}$, for some $i \in \{0, 1, \ldots, n-1\}$.

We have that

$$D_{x-}^{\beta}f(x) = D_{*x}^{\beta} = 0. \tag{76}$$

By Convention 2.14, $D_{*x}^{\beta}f(z)=0$, for $z<x$; $D_{x-}^{\beta}f(z)=0$, for $z>x$, all $x,z\in[a,b]$.

From [28, p. 54], we get by the left Caputo fractional Taylor formula that

$$f(x_k)=\sum_{j=0}^{N-1}\frac{f^{(j)}(x)}{j!}(x_k-x)^j+\frac{1}{\Gamma(\beta)}\int_x^{x_k}(x_k-J)^{\beta-1}\left(D_{*x}^{\beta}f(J)-D_{*x}^{\beta}f(x)\right)dJ, \tag{77}$$

for all $x\le x_k\le b$.

Also from [9], using the right Caputo fractional Taylor formula we get

$$f(x_k)=\sum_{j=0}^{N-1}\frac{f^{(j)}(x)}{j!}(x_k-x)^j+\frac{1}{\Gamma(\beta)}\int_{x_k}^{x}(J-x_k)^{\beta-1}\left(D_{x-}^{\beta}f(J)-D_{x-}^{\beta}f(x)\right)dJ, \tag{78}$$

for all $a\le x_k\le x$.

Hence it holds

$$\frac{f(x_k)B\left(\frac{Tn(x-x_k)}{b-a}\right)}{V(x)}=\sum_{j=0}^{N-1}\frac{f^{(j)}(x)}{j!}(x_k-x)^j\frac{B\left(\frac{Tn(x-x_k)}{b-a}\right)}{V(x)}+\frac{B\left(\frac{Tn(x-x_k)}{b-a}\right)}{V(x)}\frac{1}{\Gamma(\beta)}\int_x^{x_k}(x_k-J)^{\beta-1}\left(D_{*x}^{\beta}f(J)-D_{*x}^{\beta}f(x)\right)dJ, \tag{79}$$

all $x\le x_k\le b$.

Also we have

$$\frac{f(x_k)B\left(\frac{Tn(x-x_k)}{b-a}\right)}{V(x)}=\sum_{j=0}^{N-1}\frac{f^{(j)}(x)}{j!}(x_k-x)^j\frac{B\left(\frac{Tn(x-x_k)}{b-a}\right)}{V(x)}+\frac{B\left(\frac{Tn(x-x_k)}{b-a}\right)}{V(x)}\frac{1}{\Gamma(\beta)}\int_{x_k}^{x}(J-x_k)^{\beta-1}\left(D_{x-}^{\beta}f(J)-D_{x-}^{\beta}f(x)\right)dJ, \tag{80}$$

all $a\le x_k\le x$.

Hence we derive

$$\frac{\sum_{k=i+1}^{n}f(x_k)B\left(\frac{Tn(x-x_k)}{b-a}\right)}{V(x)}=\sum_{j=0}^{N-1}\frac{f^{(j)}(x)}{j!}\frac{\sum_{k=i+1}^{n}(x_k-x)^jB\left(\frac{Tn(x-x_k)}{b-a}\right)}{V(x)}+R_1, \tag{81}$$

where

$$R_1:=\frac{\sum_{k=i+1}^{n}B\left(\frac{Tn(x-x_k)}{b-a}\right)}{V(x)}\frac{1}{\Gamma(\beta)}\int_x^{x_k}(x_k-J)^{\beta-1}\left(D_{*x}^{\beta}f(J)-D_{*x}^{\beta}f(x)\right)dJ, \tag{82}$$

all $x\le x_k\le b$.

Also it holds

$$\frac{\sum_{k=0}^{i} f(x_k) B\left(\frac{Tn(x-x_k)}{b-a}\right)}{V(x)} = \sum_{j=0}^{N-1} \frac{f^{(j)}(x)}{j!} \frac{\sum_{k=0}^{i} (x_k - x)^j B\left(\frac{Tn(x-x_k)}{b-a}\right)}{V(x)} + R_2, \tag{83}$$

where

$$R_2 := \frac{\sum_{k=0}^{i} B\left(\frac{Tn(x-x_k)}{b-a}\right)}{V(x)} \frac{1}{\Gamma(\beta)} \int_{x_k}^{x} (J - x_k)^{\beta-1} \left(D_{x-}^{\beta} f(J) - D_{x-}^{\beta} f(x)\right) dJ, \tag{84}$$

all $a \leq x_k \leq x$.

Consequently, by adding (81) and (83), we obtain

$$H_n(f, x) - f(x) = \sum_{j=1}^{N-1} \frac{f^{(j)}(x)}{j!} \left(\frac{\sum_{k=0}^{n} (x_k - x)^j B\left(\frac{Tn(x-x_k)}{b-a}\right)}{V(x)} \right) + R_1 + R_2. \tag{85}$$

Hence we find

$$|H_n(f, x) - f(x)| \leq \sum_{j=1}^{N-1} \frac{\left|f^{(j)}(x)\right|}{j!} \left(\frac{\sum_{k=0}^{n} |x_k - x|^j B\left(\frac{Tn(x-x_k)}{b-a}\right)}{V(x)} \right) + |R_1| + |R_2|$$

$$\leq \sum_{j=1}^{N-1} \frac{\left|f^{(j)}(x)\right|}{j!} \frac{2(b-a)^j B^*}{n^j B\left(\frac{T}{2}\right)} + |R_1| + |R_2|. \tag{86}$$

Next we estimate $|R_1|$, $|R_2|$.

We have that

$$|R_1| \leq \frac{\sum_{k=i+1}^{n} B\left(\frac{Tn(x-x_k)}{b-a}\right)}{B\left(\frac{T}{2}\right)} \frac{1}{\Gamma(\beta)} \int_{x}^{x_k} (x_k - J)^{\beta-1} \left|D_{*x}^{\beta} f(J) - D_{*x}^{\beta} f(x)\right| dJ \tag{87}$$

$$\leq \frac{\sum_{k=i+1}^{n} B\left(\frac{Tn(x-x_k)}{b-a}\right)}{B\left(\frac{T}{2}\right)} \frac{1}{\Gamma(\beta)} \omega_1\left(D_{*x}^{\beta} f, (x_k - x)\right) \left(\int_{x}^{x_k} (x_k - J)^{\beta-1} dJ \right)$$

$$= \frac{\sum_{k=i+1}^{n} B\left(\frac{Tn(x-x_k)}{b-a}\right)}{B\left(\frac{T}{2}\right)} \frac{1}{\Gamma(\beta)} \omega_1\left(D_{*x}^{\beta} f, x_k - x\right) \frac{(x_k - x)^{\beta}}{\beta}$$

$$= \frac{\sum_{k=i+1}^{n} B\left(\frac{Tn(x-x_k)}{b-a}\right)}{B\left(\frac{T}{2}\right) \Gamma(\beta+1)} (x_k - x)^{\beta} \omega_1\left(D_{*x}^{\beta} f, x_k - x\right) \tag{88}$$

$$\leq \frac{B\left(\frac{Tn(x-x_{i+1})}{b-a}\right)}{B\left(\frac{T}{2}\right) \Gamma(\beta+1)} \frac{(b-a)^{\beta}}{n^{\beta}} \omega_1\left(D_{*x}^{\beta} f, \frac{b-a}{n}\right). \tag{89}$$

We have proved that

$$|R_1| \leq \frac{B^*}{B\left(\frac{T}{2}\right)\Gamma(\beta+1)} \frac{(b-a)^{\beta}}{n^{\beta}} \omega_1\left(D_{*x}^{\beta} f, \frac{b-a}{n}\right). \tag{90}$$

Furthermore we observe that

$$|R_2| \leq \frac{\sum_{k=0}^{i} B\left(\frac{Tn(x-x_k)}{b-a}\right)}{B\left(\frac{T}{2}\right)} \frac{1}{\Gamma(\beta)} \left(\int_{x_k}^{x} (J-x_k)^{\beta-1} \left|D_{x-}^{\beta} f(J) - D_{x-}^{\beta} f(x)\right| dJ\right) \tag{91}$$

$$\leq \frac{\sum_{k=0}^{i} B\left(\frac{Tn(x-x_k)}{b-a}\right)}{B\left(\frac{T}{2}\right)} \frac{1}{\Gamma(\beta)} \omega_1\left(D_{x-}^{\beta} f, x-x_k\right) \frac{(x-x_k)^{\beta}}{\beta}$$

$$= \frac{B\left(\frac{Tn(x-x_i)}{b-a}\right)}{B\left(\frac{T}{2}\right)} \frac{1}{\Gamma(\beta+1)} (x-x_i)^{\beta} \omega_1\left(D_{x-}^{\beta} f, x-x_i\right) \tag{92}$$

$$\leq \frac{B^*}{B\left(\frac{T}{2}\right)} \frac{1}{\Gamma(\beta+1)} \frac{(b-a)^{\beta}}{n^{\beta}} \omega_1\left(D_{x-}^{\beta} f, \frac{b-a}{n}\right).$$

That is we have proved

$$|R_2| \leq \frac{B^*}{B\left(\frac{T}{2}\right)\Gamma(\beta+1)} \frac{(b-a)^{\beta}}{n^{\beta}} \omega_1\left(D_{x-}^{\beta} f, \frac{b-a}{n}\right). \tag{93}$$

Thus

$$|R_1| + |R_2| \leq \frac{B^*(b-a)^{\beta}}{B\left(\frac{T}{2}\right)\Gamma(\beta+1) n^{\beta}} \left[\omega_1\left(D_{x-}^{\beta} f, \frac{b-a}{n}\right) + \omega_1\left(D_{*x}^{\beta} f, \frac{b-a}{n}\right)\right]. \tag{94}$$

So by using (86) and (94) we obtain (74), which implies (75).

Next we justify that the right hand side of (75) is finite.

We have

$$\left(D_{*x}^{\beta} f\right)(t) = \frac{1}{\Gamma(N-\beta)} \int_{x}^{t} (t-z)^{N-\beta-1} f^{(N)}(z)\, dz, \quad x \leq t \leq b. \tag{95}$$

Hence

$$\left|D_{*x}^{\beta} f(t)\right| \leq \frac{\left\|f^{(N)}\right\|_{\infty}}{\Gamma(N-\beta+1)} (b-a)^{N-\beta}, \quad x \leq t \leq b. \tag{96}$$

Thus

$$\left\|D_{*x}^{\beta} f\right\|_{\infty} \leq \frac{\left\|f^{(N)}\right\|_{\infty}}{\Gamma(N-\beta+1)} (b-a)^{N-\beta}. \tag{97}$$

Similarly

$$D_{x-}^{\beta} f(t) = \frac{(-1)^N}{\Gamma(N-\beta)} \int_{t}^{x} (z-t)^{N-\beta-1} f^{(N)}(z)\, dz, \quad \text{all } a \leq t \leq x. \tag{98}$$

Hence

$$\left|D_{x-}^{\beta} f(t)\right| \leq \frac{\left\|f^{(N)}\right\|_{\infty}}{\Gamma(N-\beta+1)} (b-a)^{N-\beta}, \quad a \leq t \leq x. \tag{99}$$

Thus

$$\left\| D_{x-}^{\beta} f \right\|_{\infty} \leq \frac{\left\| f^{(N)} \right\|_{\infty}}{\Gamma (N-\beta+1)} (b-a)^{N-\beta} . \tag{100}$$

Consequently (for $\delta > 0$)

$$\begin{aligned} \omega_1 \left(D_{x-}^{\beta} f, \delta \right) &= \sup_{\substack{z_1, z_2 \\ |z_1 - z_2| \leq \delta}} \left| D_{x-}^{\beta} f (z_1) - D_{x-}^{\beta} f (z_2) \right| \\ &\leq \sup_{\substack{z_1, z_2 \\ |z_1 - z_2| \leq \delta}} \left\{ \left| D_{x-}^{\beta} f (z_1) \right| + \left| D_{x-}^{\beta} f (z_2) \right| \right\} \leq 2 \left\| D_{x-}^{\beta} f \right\|_{\infty} \\ &\leq \frac{2 \left\| f^{(N)} \right\|_{\infty}}{\Gamma (N-\beta+1)} (b-a)^{N-\beta} < +\infty. \end{aligned} \tag{101}$$

Hence it holds

$$\omega_1 \left(D_{x-}^{\beta} f, \delta \right) \leq \frac{2 \left\| f^{(N)} \right\|_{\infty}}{\Gamma (N-\beta+1)} (b-a)^{N-\beta} < +\infty. \tag{102}$$

Therefore

$$\sup_{x \in [a,b]} \omega_1 \left(D_{x-}^{\beta} f, \delta \right) \leq \frac{2 \left\| f^{(N)} \right\|_{\infty}}{\Gamma (N-\beta+1)} (b-a)^{N-\beta} < +\infty, \tag{103}$$

and, similarly, we get

$$\sup_{x \in [a,b]} \omega_1 \left(D_{*x}^{\beta} f, \delta \right) \leq \frac{2 \left\| f^{(N)} \right\|_{\infty}}{\Gamma (N-\beta+1)} (b-a)^{N-\beta} < +\infty. \tag{104}$$

The proof of the theorem now is complete. □

Corollary 2.16 (to Theorem 2.15)**.** All as in Theorem 2.15. Additionally assume that $f^{(j)}(x) = 0$, $j = 1, \ldots, N-1$. Then

$$\begin{aligned} |H_n (f,x) - f(x)| \leq{}& \frac{B^*}{B\left(\frac{T}{2}\right)} \frac{(b-a)^{\beta}}{\Gamma(\beta+1) n^{\beta}} \\ &\times \left[\omega_1 \left(D_{x-}^{\beta} f, \frac{b-a}{n} \right) + \omega_1 \left(D_{*x}^{\beta} f, \frac{b-a}{n} \right) \right]. \end{aligned} \tag{105}$$

In the last we have the high speed of pointwise convergence at $\frac{1}{n^{\beta+1}}$.

A fractional Voronovskaya type asymptotic expansion follows.

Theorem 2.17. *Let $\beta > 0$, $N = \lceil \beta \rceil$, $\beta \notin \mathbb{N}$, $f \in AC^N([a,b])$, $f^{(N)} \in L_{\infty}([a,b])$. Then*

$$H_n (f,x) - f(x) - \sum_{j=1}^{N-1} \frac{f^{(j)}(x)}{j!} H_n \left((\cdot - x)^j, x \right) = o \left(\frac{1}{n^{\beta - \varepsilon}} \right), \tag{106}$$

where $0 < \varepsilon \leq \beta$, $n \in \mathbb{N}$.

If $N = 1$, the sum above disappears.

Asymptotic expansion (106) *implies*

$$n^{\beta-\varepsilon}\left[H_n(f,x)-f(x)-\sum_{j=1}^{N-1}\frac{f^{(j)}(x)}{j!}H_n\left((\cdot-x)^j,x\right)\right]\to 0, \tag{107}$$

as $n\to\infty$, $0<\varepsilon\leq\beta$.

When $N=1$, *or* $f^{(j)}(x)=0$, $j=1,\ldots,N-1$, *then*

$$n^{\beta-\varepsilon}\left[H_n(f,x)-f(x)\right]\to 0, \tag{108}$$

as $n\to\infty$, $0<\varepsilon\leq\beta$.

Of great interest is the case $\beta=\frac{1}{2}$.

Proof. From [28, p. 54], we get by the left Caputo fractional Taylor formula that

$$f(x_k)=\sum_{j=0}^{N-1}\frac{f^{(j)}(x)}{j!}(x_k-x)^j+\frac{1}{\Gamma(\beta)}\int_x^{x_k}(x_k-J)^{\beta-1}D_{*x}^{\beta}f(J)\,dJ, \tag{109}$$

for all $x\leq x_k\leq b$.

Also from [9], using the right Caputo fractional Taylor formula we get

$$f(x_k)=\sum_{j=0}^{N-1}\frac{f^{(j)}(x)}{j!}(x_k-x)^j+\frac{1}{\Gamma(\beta)}\int_{x_k}^{x}(J-x_k)^{\beta-1}D_{x-}^{\beta}f(J)\,dJ, \tag{110}$$

for all $a\leq x_k\leq x$.

Hence

$$\frac{f(x_k)B\left(\frac{Tn(x-x_k)}{b-a}\right)}{V(x)}=\sum_{j=0}^{N-1}\frac{f^{(j)}(x)}{j!}(x_k-x)^j\frac{B\left(\frac{Tn(x-x_k)}{b-a}\right)}{V(x)}+ \tag{111}$$

$$\frac{B\left(\frac{Tn(x-x_k)}{b-a}\right)}{V(x)}\frac{1}{\Gamma(\beta)}\int_x^{x_k}(x_k-J)^{\beta-1}D_{*x}^{\beta}f(J)\,dJ,$$

all $x\leq x_k\leq b$.

Also we have

$$\frac{f(x_k)B\left(\frac{Tn(x-x_k)}{b-a}\right)}{V(x)}=\sum_{j=0}^{N-1}\frac{f^{(j)}(x)}{j!}(x_k-x)^j\frac{B\left(\frac{Tn(x-x_k)}{b-a}\right)}{V(x)}$$

$$+\frac{B\left(\frac{Tn(x-x_k)}{b-a}\right)}{V(x)}\frac{1}{\Gamma(\beta)}\int_{x_k}^{x}(J-x_k)^{\beta-1}D_{x-}^{\beta}f(J)\,dJ, \tag{112}$$

all $a\leq x_k\leq x$.

Hence $x\in[a,b]$ is fixed such that $x_i\leq x\leq x_{i+1}$, for some $i\in\{0,1,\ldots,n-1\}$.

Hence it holds

$$\frac{\sum_{k=i+1}^{n}f(x_k)B\left(\frac{Tn(x-x_k)}{b-a}\right)}{V(x)}=\sum_{j=0}^{N-1}\frac{f^{(j)}(x)}{j!}\frac{\sum_{k=i+1}^{n}(x_k-x)^jB\left(\frac{Tn(x-x_k)}{b-a}\right)}{V(x)}+R_1, \tag{113}$$

where

$$R_1 := \frac{\sum_{k=i+1}^{n} B\left(\frac{Tn(x-x_k)}{b-a}\right)}{V(x)} \frac{1}{\Gamma(\beta)} \int_x^{x_k} (x_k - J)^{\beta-1} D_{*x}^{\beta} f(J)\, dJ, \tag{114}$$

all $x \leq x_k \leq b$.

Also it holds

$$\frac{\sum_{k=0}^{i} f(x_k) B\left(\frac{Tn(x-x_k)}{b-a}\right)}{V(x)} = \sum_{j=0}^{N-1} \frac{f^{(j)}(x)}{j!} \frac{\sum_{k=0}^{i} (x_k - x)^j B\left(\frac{Tn(x-x_k)}{b-a}\right)}{V(x)} + R_2, \tag{115}$$

where

$$R_2 := \frac{\sum_{k=0}^{i} B\left(\frac{Tn(x-x_k)}{b-a}\right)}{V(x)} \frac{1}{\Gamma(\beta)} \int_{x_k}^{x} (J - x_k)^{\beta-1} D_{x-}^{\beta} f(J)\, dJ, \tag{116}$$

all $a \leq x_k \leq x$.

Hence we get

$$H_n(f,x) - f(x) - \sum_{j=1}^{N-1} \frac{f^{(j)}(x)}{j!} \left(\frac{\sum_{k=0}^{n} (x_k - x)^j B\left(\frac{Tn(x-x_k)}{b-a}\right)}{V(x)} \right) = R_1 + R_2. \tag{117}$$

Notice also that for any $x \in [a,b]$, by (97) and (100), we have

$$\left\{ \left\| D_{*x}^{\beta} f \right\|_{\infty}, \left\| D_{x-}^{\beta} f \right\|_{\infty} \right\} \leq \frac{\left\| f^{(N)} \right\|_{\infty}}{\Gamma(N-\beta+1)} (b-a)^{N-\beta} =: M, \tag{118}$$

with $M > 0$.

That is we find

$$H_n(f,x) - f(x) - \sum_{j=1}^{N-1} \frac{f^{(j)}(x)}{j!} H_n\left((\cdot - x)^j, x \right) = R_1 + R_2. \tag{119}$$

Notice that

$$\begin{aligned} |R_1| &\leq M \frac{\sum_{k=i+1}^{n} B\left(\frac{Tn(x-x_k)}{b-a}\right)}{V(x)} \frac{1}{\Gamma(\beta+1)} (x_k - x)^{\beta} \\ &\leq \frac{MB^*}{B\left(\frac{T}{2}\right)\Gamma(\beta+1)} \frac{(b-a)^{\beta}}{n^{\beta}}, \end{aligned} \tag{120}$$

that is

$$|R_1| \leq \frac{MB^*(b-a)^{\beta}}{B\left(\frac{T}{2}\right)\Gamma(\beta+1)\, n^{\beta}}. \tag{121}$$

Similarly we have

$$|R_2| \leq M \frac{\sum_{k=0}^{i} B\left(\frac{Tn(x-x_k)}{b-a}\right)}{B\left(\frac{T}{2}\right)} \frac{1}{\Gamma(\beta+1)} (x-x_k)^{\beta}$$

$$\leq \frac{MB^*}{B\left(\frac{T}{2}\right)\Gamma(\beta+1)} \frac{(b-a)^{\beta}}{n^{\beta}}. \tag{122}$$

Hence

$$|R_2| \leq \frac{MB^*(b-a)^{\beta}}{B\left(\frac{T}{2}\right)\Gamma(\beta+1)\, n^{\beta}}. \tag{123}$$

Therefore it holds

$$|R_1+R_2| \leq |R_1| + |R_2| \leq \frac{\Phi}{n^{\beta}}, \tag{124}$$

where

$$\Phi := \frac{2MB^*(b-a)^{\beta}}{B\left(\frac{T}{2}\right)\Gamma(\beta+1)}. \tag{125}$$

Thus

$$|R_1+R_2| = O\left(\frac{1}{n^{\beta}}\right), \tag{126}$$

and

$$|R_1+R_2| = o(1).$$

Letting $0 < \varepsilon \leq \beta$, we derive

$$\frac{|R_1+R_2|}{\left(\frac{1}{n^{\beta-\varepsilon}}\right)} \leq \frac{\Phi}{n^{\varepsilon}} \to 0, \tag{127}$$

as $n \to \infty$. So that

$$|R_1+R_2| = o\left(\frac{1}{n^{\beta-\varepsilon}}\right), \quad n \in \mathbb{N}, \tag{128}$$

proving the claim. □

2.2. **Neural Networks: Multivariate theory of Interpolation and Approximation.** We need

Definition 2.18. Consider the d-dimensional bell-shaped function $E : \mathbb{R}^d \to \mathbb{R}_+$ ($d \in \mathbb{N}$) with the property for all $i = 1, \ldots, d$, $\mathbb{R} \ni t \longrightarrow E(x_1, \ldots, t, \ldots, x_d)$ is a bell-shaped function, as in Definition 2.1, where $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ is arbitrary.

More precisely here E is of compact support $K := \prod_{i=1}^{d} [-T_i, T_i]$, $T_i > 0$ and it may have jump discontinuities there, also it holds

$$E(x_1, \ldots, \pm T_i, \ldots, x_d) = 0, \tag{129}$$

for any $i = 1, \ldots, d$, all $(x_1, \ldots, x_d) \in \mathbb{R}^d$.

Furthermore assume that $E(0,\ldots,0) =: E^* > 0$ is the global maximum of E, also E is assumed to be measurable. That is $E(x_1,\ldots,t,\ldots,x_d)$ in t is even, non-decreasing for $t<0$ and non-increasing for $t\geq 0$.

Clearly it holds

$$E(\pm x_1,\ldots,\pm x_d) = E(|x_1|,\ldots,|x_d|). \tag{130}$$

Also it is $E(x_1,\ldots,0,\ldots,x_d) =: E^*(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_d) > 0$, for all $i = 1,\ldots,d$, for any $(x_1,\ldots,x_d) \in \prod_{i=1}^{d}(-T_i,T_i)$.

Examples: $\prod_{i=1}^{d}\beta(x_i)$, $\prod_{i=1}^{d}\Phi_R(x_i)$, $\prod_{i=1}^{d}\Phi_s(x_i)$, etc.

Definition 2.19. Let $f : \prod_{i=1}^{d}[a_i,b_i] \to \mathbb{R}$ be a bounded and measurable function, $a_i < b_i$, $n \in \mathbb{N}$, $h_i := \frac{b_i-a_i}{n}$, $x_{k_i i} := a_i + k_i h_i$, $k_i = 0,1,\ldots,n$, $i = 1,\ldots,d$, $x = (x_1,\ldots,x_d) \in \prod_{i=1}^{d}[a_i,b_i]$.

Next we define the multivariate interpolation neural network operator:

$$M_n(f,x) := M_n(f,x_1,\ldots,x_d)$$

$$:= \frac{\sum_{k_1=0}^{n}\cdots\sum_{k_d=0}^{n} f(x_{k_1 1},\ldots,x_{k_d d})\, E\left(\frac{T_1 n\left(x_1-x_{k_1 1}\right)}{b_1-a_1},\ldots,\frac{T_d n\left(x_d-x_{k_d d}\right)}{b_d-a_d}\right)}{\sum_{k_1=0}^{n}\cdots\sum_{k_d=0}^{n} E\left(\frac{T_1 n\left(x_1-x_{k_1 1}\right)}{b_1-a_1},\ldots,\frac{T_d n\left(x_d-x_{k_d d}\right)}{b_d-a_d}\right)}. \tag{131}$$

Remark 2.20. Trivially we get that

$$|M_n(f,x)| \leq \|f\|_\infty < +\infty, \tag{132}$$

and

$$\|M_n(f)\|_\infty \leq \|f\|_\infty < +\infty. \tag{133}$$

Remark 2.21. Let now $x_{k_i i} < x_i < x_{(k_i+1)i}$, for all $i = 1,\ldots,d$, for some $(k_1,\ldots,k_d) \in \{0,1,\ldots,n-1\}^d$. Thus $|x_i - x_{k_i i}| < h_i$ and $\left|x_i - x_{(k_i+1)i}\right| < h_i$, for all $i = 1,\ldots,d$, for some $(k_1,\ldots,k_d) \in \{0,1,\ldots,n-1\}^d$.

Remark 2.22. Notice next that be given $(x_1,\ldots,x_d) \in \mathbb{R}^d$ and

$$E\left(\frac{T_1 n(x_1 - x_{k_1 1})}{b_1-a_1},\ldots,\frac{T_d n(x_d - x_{k_d d})}{b_d-a_d}\right) > 0, \tag{134}$$

for some $(k_1,\ldots,k_d) \in \{0,1,\ldots,n\}^d$, $\Leftrightarrow$ simultaneously it holds

$$-T_i < \frac{T_i n(x_i - x_{k_i i})}{b_i - a_i} < T_i,$$

for all $i=1,\ldots,d$, for some $(k_1,\ldots,k_d)\in\{0,1,\ldots,n\}^d$, $\Leftrightarrow$

$$-1<\frac{n\left(x_i-x_{k_i i}\right)}{b_i-a_i}<1, \tag{135}$$

for all $i=1,\ldots,d$, for some $(k_1,\ldots,k_d)\in\{0,1,\ldots,n\}^d$, $\Leftrightarrow$

$$-h_i<x_i-x_{k_i i}<h_i,$$

for all $i=1,\ldots,d$, for some $(k_1,\ldots,k_d)\in\{0,1,\ldots,n\}^d$, $\Leftrightarrow$

$$\left|x_i-x_{k_i i}\right|<h_i, \tag{136}$$

for all $i=1,\ldots,d$, for some $(k_1,\ldots,k_d)\in\{0,1,\ldots,n\}^d$.

Thus, when $x\in\prod_{i=1}^{d}\left[x_{k_i i},x_{(k_i+1)i}\right)$, for some $(k_1,\ldots,k_d)\in\{0,1,\ldots,n-1\}^d$, we get that

$$E\left(\frac{T_1 n\left(x_1-x_{k_1 1}\right)}{b_1-a_1},\ldots,\frac{T_d n\left(x_d-x_{k_d d}\right)}{b_d-a_d}\right)>0. \tag{137}$$

Remark 2.23. Notice that $\left(x\in\prod_{i=1}^{d}[a_i,b_i]\right)$

$$\begin{aligned}
W &:= \sum_{k_1=0}^{n}\cdots\sum_{k_d=0}^{n}E\left(\frac{T_1 n\left(x_1-x_{k_1 1}\right)}{b_1-a_1},\ldots,\frac{T_d n\left(x_d-x_{k_d d}\right)}{b_d-a_d}\right)\\
&= \sum_{k_1=0}^{n}\cdots\sum_{k_d=0}^{n}E\left(\frac{T_1 n\left|x_1-x_{k_1 1}\right|}{b_1-a_1},\ldots,\frac{T_d n\left|x_d-x_{k_d d}\right|}{b_d-a_d}\right)\\
&\geq E\left(\frac{T_1 n\left|x_1-x_{k_1 1}\right|}{b_1-a_1},\ldots,\frac{T_d n\left|x_d-x_{k_d d}\right|}{b_d-a_d}\right),
\end{aligned} \tag{138}$$

the last inequality is chosen for suitable x_i and $x_{k_i i}$, for all $i=1,\ldots,d$, and for some $(k_1,\ldots,k_d)\in\{0,1,\ldots,n\}^d$, such that $\left|x_i-x_{k_i i}\right|\leq\frac{h_i}{2}$.

Thus

$$\frac{T_i n\left|x_i-x_{k_i i}\right|}{b_i-a_i}\leq\frac{T_i n h_i}{2\left(b_i-a_i\right)}=\frac{T_i}{2}, \tag{139}$$

all $i=1,\ldots,d$.

Therefore it holds

$$\begin{aligned}
&E\left(\frac{T_1 n\left|x_1-x_{k_1 1}\right|}{b_1-a_1},\frac{T_2 n\left|x_2-x_{k_2 2}\right|}{b_2-a_2},\ldots,\frac{T_d n\left|x_d-x_{k_d d}\right|}{b_d-a_d}\right)\\
&\geq E\left(\frac{T_1}{2},\frac{T_2 n\left|x_2-x_{k_2 2}\right|}{b_2-a_2},\ldots,\frac{T_d n\left|x_d-x_{k_d d}\right|}{b_d-a_d}\right)\\
&\geq E\left(\frac{T_1}{2},\frac{T_2}{2},\ldots,\frac{T_d n\left|x_d-x_{k_d d}\right|}{b_d-a_d}\right)\geq\cdots\geq E\left(\frac{T_1}{2},\frac{T_2}{2},\ldots,\frac{T_d}{2}\right)>0.
\end{aligned} \tag{140}$$

Hence we have

$$\frac{1}{W}=\frac{1}{\sum_{k_1=0}^{n}\cdots\sum_{k_d=0}^{n}E\left(\frac{T_1n\left(x_1-x_{k_1 1}\right)}{b_1-a_1},\ldots,\frac{T_dn\left(x_d-x_{k_d d}\right)}{b_d-a_d}\right)}\leq\frac{1}{E\left(\frac{T_1}{2},\ldots,\frac{T_d}{2}\right)}. \tag{141}$$

Remark 2.24. Let all $x_i=x_{k_i i}$, $i=1,\ldots,d$, for some $(k_1,\ldots,k_d)\in\{0,1,\ldots,n\}^d$. Then

$$E\left(\frac{T_1n\left(x_1-x_{k_1 1}\right)}{b_1-a_1},\ldots,\frac{T_dn\left(x_d-x_{k_d d}\right)}{b_d-a_d}\right)=E\left(0,\ldots,0\right)=E^*>0. \tag{142}$$

Let next $\left|x_{k_i i}-x_{(k_i+j_i)i}\right|\geq h_i$, for some $i\in\{1,\ldots,d\}$, where $j_i\geq 1$ integer, and $k_i,k_i+j_i\in\{0,1,\ldots,n\}$.

Then

$$\frac{T_in\left|x_{k_i i}-x_{(k_i+j_i)i}\right|}{b_i-a_i}\geq\frac{T_inh_i}{b_i-a_i}=T_i,\quad\text{for some } i=1,\ldots,d. \tag{143}$$

Hence

$$0\leq E\left(\frac{T_1n\left|x_1-x_{k_1 1}\right|}{b_1-a_1},\ldots,\frac{T_in\left|x_{(k_i+j_i)i}-x_{k_i i}\right|}{b_i-a_i},\ldots,\frac{T_dn\left|x_d-x_{k_d d}\right|}{b_d-a_d}\right)$$

$$\leq E\left(\frac{T_1n\left|x_1-x_{k_1 1}\right|}{b_1-a_1},\ldots,T_i,\ldots,\frac{T_dn\left|x_d-x_{k_d d}\right|}{b_d-a_d}\right)=0. \tag{144}$$

Therefore it holds

$$E\left(\frac{T_1n\left(x_1-x_{k_1 1}\right)}{b_1-a_1},\ldots,\frac{T_in\left(x_{(k_i+j_i)i}-x_{k_i i}\right)}{b_i-a_i},\ldots,\frac{T_dn\left(x_d-x_{k_d d}\right)}{b_d-a_d}\right)=0, \tag{145}$$

for any arbitrary $(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_d)\in\mathbb{R}^{d-1}$.

Let now $x_i=x_{k_i i}$, for all $i=1,\ldots,d$, for some $(k_1,\ldots,k_d)\in\{0,1,\ldots,n\}^d$.

Then

$$M_n\left(f,x_{k_1 1},\ldots,x_{k_d d}\right)=\frac{f\left(x_{k_1 1},\ldots,x_{k_d d}\right)E^*}{E^*}=f\left(x_{k_1 1},\ldots,x_{k_d d}\right), \tag{146}$$

proving the interpolation property of operators M_n.

Theorem 2.25. *Operators M_n possess the interpolation property over $x_{k_i i}$, $i=1,\ldots,d$, $k_i=0,1,\ldots,n$.*

Definition 2.26. Let $f\in C\left(\prod_{i=1}^{d}[a_i,b_i]\right)$. We call

$$\omega_1\left(f,h\right):=\sup_{\substack{\text{all } x,y\in\prod_{i=1}^{d}[a_i,b_i]:\\ \|x-y\|_\infty\leq h,}}\left|f\left(x\right)-f\left(y\right)\right| \tag{147}$$

$h>0$, the first multivariate modulus of continuity of f, above $\|\cdot\|_\infty$ is the max-norm.

Approximation result follows

Theorem 2.27. *For* $f \in C\left(\prod_{i=1}^{d}[a_i,b_i]\right)$ *we have*

$$\|M_n(f)-f\|_\infty \leq \frac{2^d E^*}{E\left(\frac{T_1}{2},\dots,\frac{T_d}{2}\right)}\omega_1\left(f,\frac{\|b-a\|_\infty}{n}\right) =: \varphi_1(n), \tag{148}$$

where $\|b-a\|_\infty := \max_{i=1,\dots,d}\{b_i-a_i\}$.

Proof. Let $x \in \prod_{i=1}^{d}[a_i,b_i]$, we can write

$M_n(f,x)-f(x)$

(149)
$$= \frac{\sum_{k_1=0}^{n}\cdots\sum_{k_d=0}^{n} f(x_{k_1 1},\dots,x_{k_d d})\, E\left(\frac{T_1 n\left(x_1-x_{k_1 1}\right)}{b_1-a_1},\dots,\frac{T_d n\left(x_d-x_{k_d d}\right)}{b_d-a_d}\right)}{W} - \frac{f(x)\,W}{W}$$

(150)
$$= \frac{\sum_{k_1=0}^{n}\cdots\sum_{k_d=0}^{n} \left(f(x_{k_1 1},\dots,x_{k_d d})-f(x_1,\dots,x_d)\right) E\left(\frac{T_1 n\left(x_1-x_{k_1 1}\right)}{b_1-a_1},\dots,\frac{T_d n\left(x_d-x_{k_d d}\right)}{b_d-a_d}\right)}{W}.$$

Therefore

$$|M_n(f,x)-f(x)| \leq \frac{1}{E\left(\frac{T_1}{2},\dots,\frac{T_d}{2}\right)}$$

$$\times\left\{\sum_{k_1=0}^{n}\cdots\sum_{k_d=0}^{n}|f(x_{k_1 1},\dots,x_{k_d d})-f(x_1,\dots,x_d)|\right. \tag{151}$$

$$\left.\times\ E\left(\frac{T_1 n(x_1-x_{k_1 1})}{b_1-a_1},\dots,\frac{T_d n(x_d-x_{k_d d})}{b_d-a_d}\right)\right\} \leq \frac{1}{E\left(\frac{T_1}{2},\dots,\frac{T_d}{2}\right)}\left\{0+\right.$$

$$\sum_{\left\{\substack{\text{all } (k_1,\dots,k_d)\in\{0,1,\dots,n\}^d \\ \left|x_i-x_{k_i i}\right|<h_i,\ i=1,\dots,d}\right\}} |f(x_{k_1 1},\dots,x_{k_d d})-f(x_1,\dots,x_d)| \tag{152}$$

$$\left.\times\ E\left(\frac{T_1 n(x_1-x_{k_1 1})}{b_1-a_1},\dots,\frac{T_d n(x_d-x_{k_d d})}{b_d-a_d}\right)\right\} \leq$$

(indeed x belongs to a specific box $\prod_{i=1}^{d}\left[x_{k_i i},x_{(k_i+1)i}\right]$)

$$\frac{2^d E^*}{E\left(\frac{T_1}{2},\dots,\frac{T_d}{2}\right)}\omega_1\left(f,\frac{\|b-a\|_\infty}{n}\right), \tag{153}$$

proving the claim. ∎

Next we denote by $f_{\widetilde{\alpha}} := \frac{\partial^{\widetilde{\alpha}} f}{\partial x^{\widetilde{\alpha}}}$, where $\widetilde{\alpha} := (\alpha_1,\dots,\alpha_d)$, $\alpha_i \in \mathbb{Z}^+$, $i=1,\dots,d$, such that $|\widetilde{\alpha}| := \sum_{i=1}^{d}\alpha_i = j$, $j=1,\dots,N$.

High speed approximation using smoothness follows.

Theorem 2.28. *Let* $f \in C^N\left(\prod_{i=1}^{d}[a_i,b_i]\right)$, $N \in \mathbb{N}$, *and* $x \in \prod_{i=1}^{d}[a_i,b_i]$. *Then*

i)

$$\left|M_n(f,x) - f(x) - \sum_{j=1}^{N}\left(\sum_{|\widetilde{\alpha}|=j}\left(\frac{f_{\widetilde{\alpha}}(x)}{\prod_{i=1}^{d}\alpha_i!}\right)M_n\left(\prod_{i=1}^{d}(\cdot - x_i)^{\alpha_i}, x\right)\right)\right| \leq \frac{2^d E^*}{E\left(\frac{T_1}{2},\dots,\frac{T_d}{2}\right)}\frac{\|b-a\|_\infty^N d^N}{N!n^N}\max_{\widetilde{\alpha}:|\widetilde{\alpha}|=N}\omega_1\left(f_{\widetilde{\alpha}}, \frac{\|b-a\|_\infty}{n}\right), \tag{154}$$

ii) assume more that $f_{\widetilde{\alpha}}(x) = 0$, *for all* $\widetilde{\alpha} : |\widetilde{\alpha}| = 1,\dots,N$; *where* $x \in \prod_{i=1}^{d}[a_i,b_i]$ *is fixed, we obtain*

$$|M_n(f,x) - f(x)| \leq \frac{2^d E^*}{E\left(\frac{T_1}{2},\dots,\frac{T_d}{2}\right)}\frac{\|b-a\|_\infty^N d^N}{N!n^N}\max_{\widetilde{\alpha}:|\widetilde{\alpha}|=N}\omega_1\left(f_{\widetilde{\alpha}}, \frac{\|b-a\|_\infty}{n}\right), \tag{155}$$

with high speed of pointwise convergence at $\frac{1}{n^{N+1}}$,

iii)

$$|M_n(f,x) - f(x)| \leq \frac{2^d E^*}{E\left(\frac{T_1}{2},\dots,\frac{T_d}{2}\right)} \times \left[\sum_{j=1}^{N}\left(\frac{\|b-a\|_\infty^j}{n^j}\right)\left(\sum_{|\widetilde{\alpha}|=j}\left(\frac{|f_{\widetilde{\alpha}}(x)|}{\prod_{i=1}^{d}\alpha_i!}\right)\right) + \frac{\|b-a\|_\infty^N d^N}{N!n^N}\max_{\widetilde{\alpha}:|\widetilde{\alpha}|=N}\omega_1\left(f_{\widetilde{\alpha}}, \frac{\|b-a\|_\infty}{n}\right)\right], \tag{156}$$

iv)

$$\|M_n(f) - f\|_\infty \leq \frac{2^d E^*}{E\left(\frac{T_1}{2},\dots,\frac{T_d}{2}\right)} \times \left[\sum_{j=1}^{N}\left(\frac{\|b-a\|_\infty^j}{n^j}\right)\left(\sum_{|\widetilde{\alpha}|=j}\left(\frac{\|f_{\widetilde{\alpha}}\|_\infty}{\prod_{i=1}^{d}\alpha_i!}\right)\right) \frac{\|b-a\|_\infty^N d^N}{N!n^N}\max_{\widetilde{\alpha}:|\widetilde{\alpha}|=N}\omega_1\left(f_{\widetilde{\alpha}}, \frac{\|b-a\|_\infty}{n}\right)\right] =: \varphi_2(n). \tag{157}$$

Proof. Here $f \in C^N\left(\prod_{i=1}^{d}[a_i,b_i]\right)$, $N \in \mathbb{N}$. We call $x_k = (x_{k_1 1}, \ldots, x_{k_d d})$. Set

$$g_{x_k}(t) := f(x + t(x_k - x)), \quad 0 \le t \le 1, \tag{158}$$

$x \in \prod_{i=1}^{d}[a_i,b_i]$, $x = (x_1, \ldots, x_d)$. Then

$$g_{x_k}^{(j)}(t) = \left[\left(\sum_{i=1}^{d}(x_{k_i i} - x_i)\frac{\partial}{\partial x_i}\right)^j f\right](x_1 + t(x_{k_1 1} - x_1), \ldots, x_d + t(x_{k_d d} - x_d)), \tag{159}$$

$$g_{x_k}^{(j)}(0) = \left[\left(\sum_{i=1}^{d}(x_{k_i i} - x_i)\frac{\partial}{\partial x_i}\right)^j f\right](x), \tag{160}$$

and

$$g_{x_k}(0) = f(x).$$

By Taylor's formula, we get

$$f(x_{k_1 1}, \ldots, x_{k_d d}) = g_{x_k}(1) = \sum_{j=0}^{N}\frac{g_{x_k}^{(j)}(0)}{j!} + R_N(x_k, 0), \tag{161}$$

where

$$R_N(x_k, 0) := \int_0^1\left(\int_0^{t_1}\cdots\left(\int_0^{t_{N-1}}\left(g_{x_k}^{(N)}(t_N) - g_{x_k}^{(N)}(0)\right)dt_N\right)\cdots\right)dt_1. \tag{162}$$

Thus,

$$\frac{f(x_{k_1 1}, \ldots, x_{k_d d})\, E\left(\frac{T_1 n\left(x_1 - x_{k_1 1}\right)}{b_1 - a_1}, \ldots, \frac{T_d n\left(x_d - x_{k_d d}\right)}{b_d - a_d}\right)}{W} \tag{163}$$

$$= \sum_{j=0}^{N}\frac{g_{x_k}^{(j)}(0)}{j!}\frac{E\left(\frac{T_1 n\left(x_1 - x_{k_1 1}\right)}{b_1 - a_1}, \ldots, \frac{T_d n\left(x_d - x_{k_d d}\right)}{b_d - a_d}\right)}{W}$$

$$+ \frac{E\left(\frac{T_1 n\left(x_1 - x_{k_1 1}\right)}{b_1 - a_1}, \ldots, \frac{T_d n\left(x_d - x_{k_d d}\right)}{b_d - a_d}\right)}{W} R_N(x_k, 0). \tag{164}$$

Therefore

$$M_n(f,x) - f(x) \tag{165}$$

$$= \sum_{j=1}^{N}\frac{1}{j!}\left(\frac{\sum_{k_1=0}^{n}\cdots\sum_{k_d=0}^{n} g_{x_k}^{(j)}(0)\, E\left(\frac{T_1 n\left(x_1 - x_{k_1 1}\right)}{b_1 - a_1}, \ldots, \frac{T_d n\left(x_d - x_{k_d d}\right)}{b_d - a_d}\right)}{W}\right) + R^*,$$

where

$$R^{*}:=\frac{\sum_{k_1=0}^{n}\cdots\sum_{k_d=0}^{n}E\left(\frac{T_1n\left(x_1-x_{k_11}\right)}{b_1-a_1},\ldots,\frac{T_dn\left(x_d-x_{k_dd}\right)}{b_d-a_d}\right)}{W}R_N\left(x_k,0\right). \tag{166}$$

Consequently, we obtain

$$\begin{aligned}&\left|M_n\left(f,x\right)-f\left(x\right)\right|\\&\leq\sum_{j=1}^{N}\frac{1}{j!}\frac{\sum_{k_1=0}^{n}\cdots\sum_{k_d=0}^{n}\left|g_{x_k}^{(j)}\left(0\right)\right|E\left(\frac{T_1n\left(x_1-x_{k_11}\right)}{b_1-a_1},\ldots,\frac{T_dn\left(x_d-x_{k_dd}\right)}{b_d-a_d}\right)}{E\left(\frac{T_1}{2},\ldots,\frac{T_d}{2}\right)}+\left|R^{*}\right|\end{aligned}$$

$$\leq\sum_{j=1}^{N}\frac{1}{j!}\frac{2^d\left(\frac{\|b-a\|_\infty^j}{n^j}\right)\left(\left(\sum_{i=1}^{d}\left|\frac{\partial}{\partial x_i}\right|\right)^{j}f\left(x\right)\right)E^{*}}{E\left(\frac{T_1}{2},\ldots,\frac{T_d}{2}\right)}+\left|R^{*}\right| \tag{167}$$

$$=\frac{2^dE^{*}}{E\left(\frac{T_1}{2},\ldots,\frac{T_d}{2}\right)}\left[\sum_{j=1}^{N}\frac{1}{j!}\left(\left(\sum_{i=1}^{d}\left|\frac{\partial}{\partial x_i}\right|\right)^{j}f\left(x\right)\right)\left(\frac{\|b-a\|_\infty^j}{n^j}\right)\right]+\left|R^{*}\right|. \tag{168}$$

Next, we estimate $|R^{*}|$.

For that, we observe

$$\left|R^{*}\right|\leq\frac{\sum_{k_1=0}^{n}\cdots\sum_{k_d=0}^{n}E\left(\frac{T_1n\left(x_1-x_{k_11}\right)}{b_1-a_1},\ldots,\frac{T_dn\left(x_d-x_{k_dd}\right)}{b_d-a_d}\right)}{E\left(\frac{T_1}{2},\ldots,\frac{T_d}{2}\right)} \tag{169}$$

$$\times\left(\int_0^1\left(\int_0^{t_1}\cdots\left(\int_0^{t_{N-1}}\left|g_{x_k}^{(N)}\left(t_N\right)-g_{x_k}^{(N)}\left(0\right)\right|dt_N\right)\cdots\right)dt_1\right)$$

$$=\frac{\sum_{k_1=0}^{n}\cdots\sum_{k_d=0}^{n}E\left(\frac{T_1n\left(x_1-x_{k_11}\right)}{b_1-a_1},\ldots,\frac{T_dn\left(x_d-x_{k_dd}\right)}{b_d-a_d}\right)}{E\left(\frac{T_1}{2},\ldots,\frac{T_d}{2}\right)}$$

$$\times\left(\int_0^1\left(\int_0^{t_1}\cdots\left(\int_0^{t_{N-1}}\left|\left[\left(\sum_{i=1}^{d}\left(x_{k_ii}-x_i\right)\frac{\partial}{\partial x_i}\right)^{N}f\right]\right.\right.\right.\right.$$

$$\left(x_1+t_N\left(x_{k_11}-x_1\right),\ldots,x_d+t_N\left(x_{k_dd}-x_d\right)\right)$$

$$\left.\left.\left.\left.-\left[\left(\sum_{i=1}^{d}\left(x_{k_ii}-x_i\right)\frac{\partial}{\partial x_i}\right)^{N}f\right]\left(x_1,\ldots,x_d\right)\right|dt_N\right)\cdots\right)dt_1\right) \tag{170}$$

$$\leq\frac{2^dE^{*}}{E\left(\frac{T_1}{2},\ldots,\frac{T_d}{2}\right)}\left(\int_0^1\left(\int_0^{t_1}\cdots\left(\int_0^{t_{N-1}}\left\{\left(\frac{\|b-a\|_\infty^N}{n^N}\right)d^N\right.\right.\right.\right.$$

$$\left.\left.\left.\left.\max_{\widetilde{\alpha}:|\alpha|=N}\omega_1\left(f_{\widetilde{\alpha}},\frac{\|b-a\|_\infty}{n}\right)\right\}dt_N\right)\cdots\right)dt_1\right)$$

$$=\frac{2^dE^{*}}{N!E\left(\frac{T_1}{2},\ldots,\frac{T_d}{2}\right)}\frac{\|b-a\|_\infty^N d^N}{n^N}\max_{\widetilde{\alpha}:|\alpha|=N}\omega_1\left(f_{\widetilde{\alpha}},\frac{\|b-a\|_\infty}{n}\right). \tag{171}$$

That is

$$|R^*| \leq \frac{2^d E^*}{N! E\left(\frac{T_1}{2}, \ldots, \frac{T_d}{2}\right)} \frac{\|b-a\|_\infty^N d^N}{n^N} \max_{\widetilde{\alpha}:|\alpha|=N} \omega_1\left(f_{\widetilde{\alpha}}, \frac{\|b-a\|_\infty}{n}\right). \tag{172}$$

The proof of the Theorem now is complete. □

About Multivariate Taylor formula and estimates (see [15, pp. 284–286])

Let $\prod_{i=1}^d [a_i, b_i]$; $d \geq 2$; $z := (z_1, \ldots, z_d)$, $x_0 := (x_{01}, \ldots, x_{0d}) \in \prod_{i=1}^d [a_i, b_i]$. We consider the space of functions $AC^N\left(\prod_{i=1}^d [a_i, b_i]\right)$ with $f : \prod_{i=1}^d [a_i, b_i] \to \mathbb{R}$ be such that all partial derivatives of order $(N-1)$ are coordinatewise absolutely continuous functions on $\prod_{i=1}^d [a_i, b_i]$, $N \in \mathbb{N}$. Also $f \in C^{N-1}\left(\prod_{i=1}^d [a_i, b_i]\right)$. Each N^{th} order partial derivative is denoted by $f_{\widetilde{\alpha}} := \frac{\partial^{\widetilde{\alpha}} f}{\partial x^{\widetilde{\alpha}}}$, where $\widetilde{\alpha} := (\alpha_1, \ldots, \alpha_d)$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, \ldots, d$ and $|\widetilde{\alpha}| := \sum_{i=1}^d \alpha_i = N$. Consider $g_z(t) := f(x_0 + t(z - x_0))$, $t \geq 0$. Then

$$g_z^{(j)}(t) = \left[\left(\sum_{i=1}^d (z_i - x_{0i}) \frac{\partial}{\partial x_i}\right)^j f\right](x_{01} + t(z_1 - x_{01}), \ldots, x_{0d} + t(z_N - x_{0d})), \tag{173}$$

for all $j = 0, 1, 2, \ldots, N$.

We mention the following multivariate Taylor theorem.

Theorem 2.29. *Under the above assumptions we have*

$$f(z_1, \ldots, z_d) = g_z(1) = \sum_{j=0}^{N-1} \frac{g_z^{(j)}(0)}{j!} + R_N(z, 0), \tag{174}$$

where

$$R_N(z, 0) := \int_0^1 \left(\int_0^{t_1} \cdots \left(\int_0^{t_{N-1}} g_z^{(N)}(t_N)\, dt_N\right) \cdots\right) dt_1, \tag{175}$$

or

$$R_N(z, 0) = \frac{1}{(N-1)!} \int_0^1 (1-\theta)^{N-1} g_z^{(N)}(\theta)\, d\theta. \tag{176}$$

Notice that $g_z(0) = f(x_0)$.

We make

Remark 2.30. Assume here that

$$\|f_{\widetilde{\alpha}}\|_{\infty,N}^{\max} := \max_{|\widetilde{\alpha}|=N} \|f_{\widetilde{\alpha}}\|_\infty < \infty. \tag{177}$$

Then

$$\left\|g_z^{(N)}\right\|_{\infty,[0,1]} = \left\|\left[\left(\sum_{i=1}^d (z_i - x_{0i}) \frac{\partial}{\partial x_i}\right)^N f\right](x_0 + t(z - x_0))\right\|_{\infty,[0,1]}$$

$$\leq \left(\sum_{i=1}^{d} |z_i - x_{0i}|\right)^N \|f_{\widetilde{\alpha}}\|_{\infty,N}^{\max}, \tag{178}$$

that is

$$\left\|g_z^{(N)}\right\|_{\infty,[0,1]} \leq \left(\|z - x_0\|_{l_1}\right)^N \|f_{\widetilde{\alpha}}\|_{\infty,N}^{\max} < \infty. \tag{179}$$

Hence we get by (176) that

$$|R_N(z,0)| \leq \frac{\left\|g_z^{(N)}\right\|_{\infty,[0,1]}}{N!} < \infty. \tag{180}$$

And it holds

$$|R_N(z,0)| \leq \frac{\left(\|z - x_0\|_{l_1}\right)^N}{N!} \|f_{\widetilde{\alpha}}\|_{\infty,N}^{\max}, \tag{181}$$

$\forall\ z, x_0 \in \prod_{i=1}^{d} [a_i, b_i]$.

We will use decisively (181).

Next follows a multivariate Voronovskaya type asymptotic expansion

Theorem 2.31. *Let* $f \in AC^N\left(\prod_{i=1}^{d} [a_i, b_i]\right)$, $d \in \mathbb{N} - \{1\}$, $N \in \mathbb{N}$, *with*

$$\|f_{\widetilde{\alpha}}\|_{\infty,N}^{\max} := \max_{|\widetilde{\alpha}|=N} \|f_{\widetilde{\alpha}}\|_{\infty} < \infty. \tag{182}$$

Then

$$M_n(f,x) - f(x) - \sum_{j=1}^{N-1}\left(\sum_{\substack{\widetilde{\alpha}:=(\alpha_1,\ldots,\alpha_d),\ \alpha_i \in \mathbb{Z}^+ \\ i=1,\ldots,d,\ |\widetilde{\alpha}|:=\sum_{i=1}^{d}\alpha_i = j}} \left(\frac{f_{\widetilde{\alpha}}(x)}{\prod_{i=1}^{d}\alpha_i!}\right) M_n\left(\prod_{i=1}^{d}(\cdot - x_i)^{\alpha_i}, x\right)\right)$$

$$= o\left(\frac{1}{n^{N-\varepsilon}}\right), \quad 0 < \varepsilon \leq N. \tag{183}$$

If $N = 1$, *the sum collapses.*

The last (183) *implies*

$$n^{N-\varepsilon}\left[M_n(f,x) - f(x)\right.$$

$$\left. - \sum_{j=1}^{N-1}\left(\sum_{\substack{\widetilde{\alpha}:=(\alpha_1,\ldots,\alpha_d),\ \alpha_i \in \mathbb{Z}^+ \\ i=1,\ldots,d,\ |\widetilde{\alpha}|:=\sum_{i=1}^{d}\alpha_i = j}} \left(\frac{f_{\widetilde{\alpha}}(x)}{\prod_{i=1}^{d}\alpha_i!}\right) M_n\left(\prod_{i=1}^{d}(\cdot - x_i)^{\alpha_i}, x\right)\right)\right] \to 0, \tag{184}$$

as $n \to \infty$, $0 < \varepsilon \leq N$.

When $N=1$ *or* $f_{\widetilde{\alpha}}(x)=0$, *all* $\widetilde{\alpha}:|\widetilde{\alpha}|=j=1,\dots,N-1$, *then*

$$n^{N-\varepsilon}\left[\left(M_n(f)\right)(x)-f(x)\right]\to 0, \tag{185}$$

as $n\to\infty$, $0<\varepsilon\le N$.

Proof. We call $x_k=(x_{k_1 1},\dots,x_{k_d d})$. Set

$$g_{x_k}(t):=f\left(x+t\left(x_k-x\right)\right),\quad 0\le t\le 1, \tag{186}$$

$x\in\prod_{i=1}^{d}[a_i,b_i]$. Then

$$g_{x_k}^{(j)}(t)=\left[\left(\sum_{i=1}^{d}\left(x_{k_i i}-x_i\right)\frac{\partial}{\partial x_i}\right)^j f\right]\left(x_1+t\left(x_{k_1 1}-x_1\right),\dots,x_d+t\left(x_{k_d d}-x_d\right)\right), \tag{187}$$

and

$$g_{x_k}(0)=f(x).$$

By Taylor's formula, we get

$$f(x_k)=g_{x_k}(1)=\sum_{j=0}^{N-1}\frac{g_{x_k}^{(j)}(0)}{j!}+R_N(x_k,0), \tag{188}$$

where

$$R_N(x_k,0):=\frac{1}{(N-1)!}\int_0^1(1-\theta)^{N-1}g_{x_k}^{(N)}(\theta)\,d\theta. \tag{189}$$

Here we denote by $f_{\widetilde{\alpha}}:=\frac{\partial^{\widetilde{\alpha}}f}{\partial x^{\widetilde{\alpha}}}$, $\widetilde{\alpha}:=(\alpha_1,\dots,\alpha_d)$, $\alpha_i\in\mathbb{Z}^+$, $i=1,\dots,d$, such that $|\widetilde{\alpha}|:=\sum_{i=1}^{d}\alpha_i=N$. Thus

$$\frac{f(x_k)\,E\left(\frac{T_1n\left(x_1-x_{k_1 1}\right)}{b_1-a_1},\dots,\frac{T_dn\left(x_d-x_{k_d d}\right)}{b_d-a_d}\right)}{W} \tag{190}$$

$$=\sum_{j=0}^{N-1}\frac{g_{x_k}^{(j)}(0)}{j!}\frac{E\left(\frac{T_1n\left(x_1-x_{k_1 1}\right)}{b_1-a_1},\dots,\frac{T_dn\left(x_d-x_{k_d d}\right)}{b_d-a_d}\right)}{W}$$

$$+\frac{E\left(\frac{T_1n\left(x_1-x_{k_1 1}\right)}{b_1-a_1},\dots,\frac{T_dn\left(x_d-x_{k_d d}\right)}{b_d-a_d}\right)}{W}R_N(x_k,0). \tag{191}$$

Therefore it holds

$$M_n(f,x)-f(x) \tag{192}$$

$$-\sum_{j=1}^{N-1}\frac{1}{j!}\frac{\left(\sum_{k_1=0}^{n}\cdots\sum_{k_d=0}^{n}g_{x_k}^{(j)}(0)\,E\left(\frac{T_1n\left(x_1-x_{k_1 1}\right)}{b_1-a_1},\dots,\frac{T_dn\left(x_d-x_{k_d d}\right)}{b_d-a_d}\right)\right)}{W}=R^*,$$

where

$$R^{*}:=\frac{\sum_{k_1=0}^{n}\cdots\sum_{k_d=0}^{n}E\left(\frac{T_1 n\left(x_1-x_{k_1 1}\right)}{b_1-a_1},\ldots,\frac{T_d n\left(x_d-x_{k_d d}\right)}{b_d-a_d}\right)}{W}R_N\left(x_k,0\right). \tag{193}$$

Hence

$$M_n\left(f,x\right)-f\left(x\right)-\sum_{j=1}^{N-1}\left(\sum_{\substack{\widetilde{\alpha}:=(\alpha_1,\ldots,\alpha_d),\ \alpha_i\in\mathbb{Z}^{+}\\ i=1,\ldots,d,\ |\widetilde{\alpha}|:=\sum_{i=1}^{d}\alpha_i=j}}\left(\frac{f_{\widetilde{\alpha}}\left(x\right)}{\prod_{i=1}^{d}\alpha_i!}\right)M_n\left(\prod_{i=1}^{d}\left(\cdot-x_i\right)^{\alpha_i},x\right)\right)$$

$$=R^{*}. \tag{194}$$

Notice that

$$R^{*}=\frac{\sum_{k_1=0}^{n}\cdots\sum_{k_d=0}^{n}E\left(\frac{T_1 n\left(x_1-x_{k_1 1}\right)}{b_1-a_1},\ldots,\frac{T_d n\left(x_d-x_{k_d d}\right)}{b_d-a_d}\right)}{W}$$

$$N\int_0^1\left(1-\theta\right)^{N-1}\sum_{\substack{\widetilde{\alpha}:=(\alpha_1,\ldots,\alpha_d),\ \alpha_i\in\mathbb{Z}^{+}\\ i=1,\ldots,d,\ |\widetilde{\alpha}|:=\sum_{i=1}^{d}\alpha_i=N}}\left(\frac{1}{\prod_{i=1}^{d}\alpha_i!}\right)$$

$$\left(\prod_{i=1}^{d}\left(x_{k_i i}-x_i\right)^{\alpha_i}\right)f_{\widetilde{\alpha}}\left(x+\theta\left(x_k-x\right)\right)d\theta. \tag{195}$$

Hence it holds

$$\left|R^{*}\right|\overset{(181)}{\leq}\frac{\sum_{k_1=0}^{n}\cdots\sum_{k_d=0}^{n}E\left(\frac{T_1 n\left(x_1-x_{k_1 1}\right)}{b_1-a_1},\ldots,\frac{T_d n\left(x_d-x_{k_d d}\right)}{b_d-a_d}\right)}{E\left(\frac{T_1}{2},\ldots,\frac{T_d}{2}\right)} \tag{196}$$

$$\frac{\left(\left\|x_k-x\right\|_{l_1}\right)^{N}}{N!}\left\|f_{\widetilde{\alpha}}\right\|_{\infty,N}^{\max}\leq\frac{2^{d}E^{*}}{E\left(\frac{T_1}{2},\ldots,\frac{T_d}{2}\right)}\left(d\frac{\left\|b-a\right\|_{\infty}}{n}\right)^{N}\frac{\left\|f_{\widetilde{\alpha}}\right\|_{\infty,N}^{\max}}{N!}. \tag{197}$$

That is

$$\left|R^{*}\right|\leq\frac{\delta}{n^{N}}, \tag{198}$$

where

$$\delta:=\frac{2^{d}E^{*}d^{N}\left\|b-a\right\|_{\infty}^{N}\left\|f_{\widetilde{\alpha}}\right\|_{\infty,N}^{\max}}{E\left(\frac{T_1}{2},\ldots,\frac{T_d}{2}\right)N!}<+\infty. \tag{199}$$

That is

$$\left|R^{*}\right|=O\left(\frac{1}{n^{N}}\right), \tag{200}$$

and

$$|R^*| = o(1). \tag{201}$$

And letting $0 < \varepsilon \leq N$, we derive

$$\frac{|R^*|}{\left(\frac{1}{n^{N-\varepsilon}}\right)} \leq \frac{\delta}{n^{\varepsilon}} \to 0, \tag{202}$$

as $n \to \infty$.

I.e.

$$|R^*| = o\left(\frac{1}{n^{N-\varepsilon}}\right). \tag{203}$$

The proof is completed. □

2.3. **Neural Networks Iterated Approximation and Interpolation.** We make

Remark 2.32. Here E is assumed additionally to be continuous.

Let $f \in C\left(\prod_{i=1}^{d}[a_i, b_i]\right)$. We (see (138), (140)) proved that $W > 0$. Hence $M_n(f) \in C\left(\prod_{i=1}^{d}[a_i, b_i]\right)$. Furthermore $M_n(f) - f \in C\left(\prod_{i=1}^{d}[a_i, b_i]\right)$.

We proved earlier (133) that

$$\|M_n(f)\|_\infty \leq \|f\|_\infty < +\infty. \tag{204}$$

Clearly then

$$\left\|M_n^2(f)\right\|_\infty = \|M_n(M_n(f))\|_\infty \leq \|M_n(f)\|_\infty \leq \|f\|_\infty. \tag{205}$$

Therefore

$$\left\|M_n^k(f)\right\|_\infty \leq \|f\|_\infty, \quad \forall\, k \in \mathbb{N}. \tag{206}$$

Also we see that

$$\left\|M_n^k(f)\right\|_\infty \leq \left\|M_n^{k-1}(f)\right\|_\infty \leq \cdots \leq \|M_n(f)\|_\infty \leq \|f\|_\infty. \tag{207}$$

Also it holds

$$M_n(1) = 1, \;\; M_n^k(1) = 1, \;\; \forall\, k \in \mathbb{N}. \tag{208}$$

Here M_n^k are positive linear operators.

Call $x_k = (x_{k_1 1}, \ldots, x_{k_d d})$, we proved (146), that

$$(M_n(f))(x_k) = f(x_k), \tag{209}$$

the interpolation property of M_n.

Hence we get

$$\left(M_n^2(f)\right)(x_k) = (M_n(M_n(f)))(x_k)$$

(by Theorem 2.25)

$$= (M_n(f))(x_k) = f(x_k), \tag{210}$$

In general it holds

$$\left(M_n^k(f)\right)(x_k) = f(x_k), \quad \forall\, k \in \mathbb{N}, \tag{211}$$

proving interpolation of the operators M_n^k.

Remark 2.33. Let $r \in \mathbb{N}$ and M_n as above. We observe that

$$\begin{aligned} M_n^r f - f = {} & \left(M_n^r f - M_n^{r-1} f\right) + \left(M_n^{r-1} f - M_n^{r-2} f\right) \\ & + \left(M_n^{r-2} f - M_n^{r-3} f\right) + \cdots + \left(M_n^2 f - M_n f\right) + (M_n f - f). \end{aligned} \tag{212}$$

Then

$$\begin{aligned} \|M_n^r f - f\|_\infty \leq {} & \left\|M_n^r f - M_n^{r-1} f\right\|_\infty + \left\|M_n^{r-1} f - M_n^{r-2} f\right\|_\infty \\ & + \left\|M_n^{r-2} f - M_n^{r-3} f\right\|_\infty + \cdots + \left\|M_n^2 f - M_n f\right\|_\infty + \|M_n f - f\|_\infty \\ = {} & \left\|M_n^{r-1}(M_n f - f)\right\|_\infty + \left\|M_n^{r-2}(M_n f - f)\right\|_\infty + \left\|M_n^{r-3}(M_n f - f)\right\|_\infty \\ & + \cdots + \|M_n(M_n f - f)\|_\infty + \|M_n f - f\|_\infty \leq r\|M_n f - f\|_\infty. \end{aligned} \tag{213}$$

That is

$$\|M_n^r f - f\|_\infty \leq r\|M_n f - f\|_\infty. \tag{214}$$

Conclusion 2.34. Thus, the speed of convergence to the unit operator of M_n^r is not worse than of M_n.

Remark 2.35. Let $m_1, \ldots, m_r \in \mathbb{N} : m_1 \leq m_2 \leq \cdots \leq m_r$, $r \in \mathbb{N}$.

Let M_{m_i} as above, $i = 1, \ldots, r$.

Then it holds

$$\begin{aligned} & M_{m_r}\left(M_{m_{r-1}}(\ldots M_{m_2}(M_{m_1}(f)))\right) - f \\ & = \left[M_{m_r}\left(M_{m_{r-1}}(\ldots M_{m_2}(M_{m_1}(f)))\right) - M_{m_r}\left(M_{m_{r-1}}(\ldots M_{m_2}(f))\right)\right] \\ & \quad + \left[M_{m_r}\left(M_{m_{r-1}}(\ldots M_{m_3}(M_{m_2}(f)))\right) - M_{m_r}\left(M_{m_{r-1}}(\ldots M_{m_3}(f))\right)\right] \\ & \quad + \left[M_{m_r}\left(M_{m_{r-1}}(\ldots M_{m_4}(M_{m_3}(f)))\right) - M_{m_r}\left(M_{m_{r-1}}(\ldots M_{m_4}(f))\right)\right] \\ & \quad + \cdots + \left[M_{m_r}\left(M_{m_{r-1}} f\right) - M_{m_r} f\right] + \left[M_{m_r} f - f\right] \end{aligned} \tag{215}$$

$$\begin{aligned} & = \left[M_{m_r}\left(M_{m_{r-1}}(\ldots M_{m_2})\right)(M_{m_1} f - f)\right] \\ & \quad + \left[M_{m_r}\left(M_{m_{r-1}}(\ldots M_{m_3})\right)(M_{m_2} f - f)\right] \\ & \quad + \left[M_{m_r}\left(M_{m_{r-1}}(\ldots M_{m_4})\right)(M_{m_3} f - f)\right] + \cdots \\ & \quad + \left[M_{m_r}\left(M_{m_{r-1}} f - f\right)\right] + \left[M_{m_r} f - f\right]. \end{aligned} \tag{216}$$

Therefore

$$\left\|M_{m_r}\left(M_{m_{r-1}}\left(\ldots M_{m_2}\left(M_{m_1}\left(f\right)\right)\right)\right)-f\right\|_\infty$$
$$\leq \left\|M_{m_r}\left(M_{m_{r-1}}\left(\ldots M_{m_2}\right)\right)\left(M_{m_1}f-f\right)\right\|_\infty \tag{217}$$
$$+\left\|M_{m_r}\left(M_{m_{r-1}}\left(\ldots M_{m_3}\right)\right)\left(M_{m_2}f-f\right)\right\|_\infty$$
$$+\left\|M_{m_r}\left(M_{m_{r-1}}\left(\ldots M_{m_4}\right)\right)\left(M_{m_3}f-f\right)\right\|_\infty+\cdots$$
$$+\left\|M_{m_r}\left(M_{m_{r-1}}f-f\right)\right\|_\infty+\left\|M_{m_r}f-f\right\|_\infty$$
$$\leq \left\|M_{m_1}f-f\right\|_\infty+\left\|M_{m_2}f-f\right\|_\infty+\left\|M_{m_3}f-f\right\|_\infty \tag{218}$$
$$+\cdots+\left\|M_{m_{r-1}}f-f\right\|_\infty+\left\|M_{m_r}f-f\right\|_\infty=\sum_{i=1}^{r}\left\|M_{m_i}f-f\right\|_\infty . \tag{219}$$

We have proved that

$$\left\|M_{m_r}\left(M_{m_{r-1}}\left(\ldots M_{m_2}\left(M_{m_1}\left(f\right)\right)\right)\right)-f\right\|_\infty \leq \sum_{i=1}^{r}\left\|M_{m_i}f-f\right\|_\infty . \tag{220}$$

Using (214) we derive

Theorem 2.36. *Let* $f\in C\left(\prod_{i=1}^{d}\left[a_i,b_i\right]\right)$, $r\in\mathbb{N}$. *Then*

$$\left\|M_n^r f-f\right\|_\infty \leq \frac{r2^dE^*}{E\left(\frac{T_1}{2},\ldots,\frac{T_d}{2}\right)}\omega_1\left(f,\frac{\left\|b-a\right\|_\infty}{n}\right). \tag{221}$$

Proof. Also use of (148). □

Theorem 2.37. *Let* $f\in C^N\left(\prod_{i=1}^{d}\left[a_i,b_i\right]\right)$, $N\in\mathbb{N}$, $r\in\mathbb{N}$. *Then*

$$\left\|M_n^r f-f\right\|_\infty \leq r\varphi_2\left(n\right), \tag{222}$$

where $\varphi_2\left(n\right)$ *is as in* (157).

Proof. Use also of (157). □

Next we use (220).

Theorem 2.38. *Let* $m_1,\ldots,m_r\in\mathbb{N}:m_1\leq m_2\leq\cdots\leq m_r$, $r\in\mathbb{N}$, $f\in C\left(\prod_{i=1}^{d}\left[a_i,b_i\right]\right)$. *Then*

$$\left\|M_{m_r}\left(M_{m_{r-1}}\left(\ldots M_{m_2}\left(M_{m_1}\left(f\right)\right)\right)\right)-f\right\|_\infty \leq \sum_{i=1}^{r}\varphi_1\left(m_i\right) \tag{223}$$
$$\leq \frac{r2^dE^*}{E\left(\frac{T_1}{2},\ldots,\frac{T_d}{2}\right)}\omega_1\left(f,\frac{\left\|b-a\right\|_\infty}{m_1}\right),$$

where φ_1 *as in* (148).

Proof. Use also of (148). □

Theorem 2.39. *Let* $m_1, \ldots, m_r \in \mathbb{N} : m_1 \leq m_2 \leq \cdots \leq m_r$*,* $r \in \mathbb{N}$*,* $f \in C^N\left(\prod_{i=1}^{d}[a_i, b_i]\right)$*,* $N \in \mathbb{N}$*. Then*

$$\left\| M_{m_r}\left(M_{m_{r-1}}\left(\ldots M_{m_2}\left(M_{m_1}(f)\right)\right)\right) - f \right\|_\infty \leq \sum_{i=1}^{r} \varphi_2(m_i)$$

$$\leq \frac{r 2^d E^*}{E\left(\frac{T_1}{2}, \ldots, \frac{T_d}{2}\right)} \left[\sum_{j=1}^{N} \left(\frac{\|b-a\|_\infty^j}{m_1^j} \right) \left(\sum_{|\widetilde{\alpha}|=j} \frac{\|f_{\widetilde{\alpha}}\|_\infty}{\prod_{i=1}^{d} \alpha_i!} \right) \right.$$

$$\left. + \frac{\|b-a\|_\infty^N d^N}{N! m_1^N} \max_{\widetilde{\alpha}:|\widetilde{\alpha}|=N} \omega_1\left(f_{\widetilde{\alpha}}, \frac{\|b-a\|_\infty}{m_1}\right) \right], \tag{224}$$

where φ_2 *as in* (157).

Proof. Also use of (157). ☐

2.4. **Complex Multivariate Neural Network Approximation and Interpolation.** We make

Remark 2.40. Let $f : \prod_{i=1}^{d}[a_i, b_i] \to \mathbb{C}$ with real and imaginary parts $f_1, f_2 : f = f_1 + i f_2$, $i = \sqrt{-i}$. Clearly f is continuous iff f_1 and f_2 are continuous.

Given that $f_1, f_2 \in C^N\left(\prod_{i=1}^{d}[a_i, b_i]\right)$, $N \in \mathbb{N}$, it holds

$$f_{\widetilde{\alpha}}(x) = f_{1,\widetilde{\alpha}}(x) + i f_{2,\widetilde{\alpha}}(x), \tag{225}$$

where $\widetilde{\alpha}$ indicates a partial derivative of any order and arrangement.

Let $f \in C\left(\prod_{i=1}^{d}[a_i, b_i], \mathbb{C}\right)$ the space of continuous functions $f : \prod_{i=1}^{d}[a_i, b_i] \to \mathbb{C}$. Then $f_1, f_2 \in C\left(\prod_{i=1}^{d}[a_i, b_i]\right)$, and thus both are bounded, implying that f is bounded.

We define

$$M_n^{\mathbb{C}}(f, x) := M_n(f_1, x) + i M_n(f_2, x), \quad \forall\, x \in \prod_{i=1}^{d}[a_i, b_i]. \tag{226}$$

We observe that

$$\left| M_n^{\mathbb{C}}(f, x) - f(x) \right| \leq |M_n(f_1, x) - f_1(x)| + |M_n(f_2, x) - f_2(x)|, \tag{227}$$

and

$$\left\| M_n^{\mathbb{C}}(f) - f \right\|_\infty \leq \|M_n(f_1) - f_1\|_\infty + \|M_n(f_2) - f_2\|_\infty. \tag{228}$$

If f is bounded then f_1, f_2 are also bounded.

For the interpolation property we assume that f is bounded and measurable. Thus f_1, f_2 are measurable.

We have (for any $(k_1, \ldots, k_d) \in \{0, 1, \ldots, n\}^d$)

$$M_n^{\mathbb{C}} (f, x_{k_1 1}, \ldots, x_{k_d d}) = M_n (f_1, x_{k_1 1}, \ldots, x_{k_d d}) + iM_n (f_2, x_{k_1 1}, \ldots, x_{k_d d})$$
$$= f_1 (x_{k_1 1}, \ldots, x_{k_d d}) + if_2 (x_{k_1 1}, \ldots, x_{k_d d})$$
$$= f (x_{k_1 1}, \ldots, x_{k_d d}), \tag{229}$$

proving interpolation of $M_n^{\mathbb{C}}$.

Theorem 2.41. *Let $f \in C\left(\prod_{i=1}^{d} [a_i, b_i], \mathbb{C}\right)$, such that $f = f_1 + if_2$, $n \in \mathbb{N}$. Then*

$$\left\| M_n^{\mathbb{C}} (f) - f \right\|_\infty \leq \frac{2^d E^*}{E\left(\frac{T_1}{2}, \ldots, \frac{T_d}{2}\right)} \times \left[\omega_1 \left(f_1, \frac{\|b-a\|_\infty}{n} \right) + \omega_1 \left(f_2, \frac{\|b-a\|_\infty}{n} \right) \right]. \tag{230}$$

Proof. By Theorem 2.27. □

Theorem 2.42. *Let $f : \prod_{i=1}^{d} [a_i, b_i] \to \mathbb{C}$, such that $f = f_1 + if_2$. Assume $f_1, f_2 \in C^N\left(\prod_{i=1}^{d} [a_i, b_i]\right)$, $N \in \mathbb{N}$, $n \in \mathbb{N}$. Then*

$$\begin{aligned} \left| M_n^{\mathbb{C}} (f, x) - f (x) \right| &\leq \frac{2^d E^*}{E\left(\frac{T_1}{2}, \ldots, \frac{T_d}{2}\right)} \\ &\times \left[\sum_{j=1}^{N} \frac{1}{j!} \left(\frac{\|b-a\|_\infty^j}{n^j} \right) \left[\left(\left(\sum_{i=1}^{d} \left| \frac{\partial}{\partial x_i} \right| \right)^j f_1 (x) \right) \right.\right. \\ &+ \left.\left(\left(\sum_{i=1}^{d} \left| \frac{\partial}{\partial x_i} \right| \right)^j f_2 (x) \right) \right] + \frac{\|b-a\|_\infty^N d^N}{N! n^N} \\ &\times \left.\left[\max_{\widetilde{\alpha}: |\widetilde{\alpha}| = N} \omega_1 \left(f_{1,\widetilde{\alpha}}, \frac{\|b-a\|_\infty}{n} \right) + \max_{\widetilde{\alpha}: |\widetilde{\alpha}| = N} \omega_1 \left(f_{2,\widetilde{\alpha}}, \frac{\|b-a\|_\infty}{n} \right) \right]\right] \end{aligned} \tag{231}$$

$$\begin{aligned} &= \frac{2^d E^*}{E\left(\frac{T_1}{2}, \ldots, \frac{T_d}{2}\right)} \left[\sum_{j=1}^{N} \left(\frac{\|b-a\|_\infty^j}{n^j} \right) \left(\sum_{|\widetilde{\alpha}| = j} \left(\frac{|f_{1,\widetilde{\alpha}} (x)| + |f_{2,\widetilde{\alpha}} (x)|}{\prod_{i=1}^{d} \alpha_i!} \right) \right) \right. \\ &+ \left. \frac{\|b-a\|_\infty^N d^N}{N! n^N} \left[\max_{\widetilde{\alpha}: |\widetilde{\alpha}| = N} \omega_1 \left(f_{1,\widetilde{\alpha}}, \frac{\|b-a\|_\infty}{n} \right) + \max_{\widetilde{\alpha}: |\widetilde{\alpha}| = N} \omega_1 \left(f_{2,\widetilde{\alpha}}, \frac{\|b-a\|_\infty}{n} \right) \right] \right]. \end{aligned} \tag{232}$$

Proof. By (156). □

2.5. **Fuzzy Fractional Mathematical Analysis Background.** We need the following basic background

Definition 2.43 (see [41])**.** Let $\mu : \mathbb{R} \to [0,1]$ with the following properties:

(i) is normal, i.e., $\exists\ x_0 \in \mathbb{R};\ \mu(x_0) = 1$.

(ii) $\mu(\lambda x + (1-\lambda) y) \geq \min\{\mu(x), \mu(y)\}$, $\forall\ x, y \in \mathbb{R}$, $\forall\ \lambda \in [0,1]$ (μ is called a convex fuzzy subset).

(iii) μ is upper semicontinuous on $\mathbb{R}$, i.e. $\forall\ x_0 \in \mathbb{R}$ and $\forall\ \varepsilon > 0$, $\exists$ neighborhood $V(x_0) : \mu(x) \leq \mu(x_0) + \varepsilon$, $\forall\ x \in V(x_0)$.

(iv) The set $\overline{\mathrm{supp}(\mu)}$ is compact in $\mathbb{R}$ (where $\mathrm{supp}(\mu) := \{x \in \mathbb{R} : \mu(x) > 0\}$).

We call μ a fuzzy real number. Denote the set of all μ with $\mathbb{R}_{\mathcal{F}}$.

E.g. $\chi_{\{x_0\}} \in \mathbb{R}_{\mathcal{F}}$, for any $x_0 \in \mathbb{R}$, where $\chi_{\{x_0\}}$ is the characteristic function at x_0.

For $0 < r \leq 1$ and $\mu \in \mathbb{R}_{\mathcal{F}}$ define

$$[\mu]^r := \{x \in \mathbb{R} : \mu(x) \geq r\}$$

and

$$[\mu]^0 := \overline{\{x \in \mathbb{R} : \mu(x) \geq 0\}}.$$

Then it is well known that for each $r \in [0,1]$, $[\mu]^r$ is a closed and bounded interval on $\mathbb{R}$ [33].

For $u, v \in \mathbb{R}_{\mathcal{F}}$ and $\lambda \in \mathbb{R}$, we define uniquely the sum $u \oplus v$ and the product $\lambda \odot u$ by

$$[u \oplus v]^r = [u]^r + [v]^r, \quad [\lambda \odot u]^r = \lambda [u]^r, \quad \forall\ r \in [0,1],$$

where

$[u]^r + [v]^r$ means the usual addition of two intervals (as substes of $\mathbb{R}$) and

$\lambda [u]^r$ means the usual product between a scalar and a subset of $\mathbb{R}$ (see, e.g. [41]).

Notice $1 \odot u = u$ and it holds

$$u \oplus v = v \oplus u,\ \lambda \odot u = u \odot \lambda.$$

If $0 \leq r_1 \leq r_2 \leq 1$ then

$$[u]^{r_2} \subseteq [u]^{r_1}.$$

Actually $[u]^r = \left[u_-^{(r)}, u_+^{(r)}\right]$, where $u_-^{(r)} \leq u_+^{(r)}$, $u_-^{(r)}$, $u_+^{(r)} \in \mathbb{R}$, $\forall\ r \in [0,1]$.

For $\lambda > 0$ one has $\lambda u_\pm^{(r)} = (\lambda \odot u)_\pm^{(r)}$, respectively.

Define $D : \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_{\mathcal{F}}$ by

$$D(u,v) := \sup_{r \in [0,1]} \max\left\{\left|u_-^{(r)} - v_-^{(r)}\right|, \left|u_+^{(r)} - v_+^{(r)}\right|\right\},$$

where

$$[v]^r = \left[v_-^{(r)}, v_+^{(r)}\right]; \quad u, v \in \mathbb{R}_{\mathcal{F}}.$$

We have that D is a metric on $\mathbb{R}_{\mathcal{F}}$.

Then $(\mathbb{R}_{\mathcal{F}}, D)$ is a complete metric space, see [41], [42].

Here $\overset{*}{\sum}$ stands for fuzzy summation and $\widetilde{o} := \chi_{\{0\}} \in \mathbb{R}_{\mathcal{F}}$ is the neural element with respect to $\oplus$, i.e.,

$$u \oplus \widetilde{0} = \widetilde{0} \oplus u = u, \ \ \forall \, u \in \mathbb{R}_{\mathcal{F}}.$$

Denote

$$D^{*}(f,g) = \sup_{x \in X \subseteq \mathbb{R}} D(f,g),$$

where $f, g : X \to \mathbb{R}_{\mathcal{F}}$.

We mention

Definition 2.44. Let $f : X \subseteq \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$, X interval, we define the (first) fuzzy modulus of continuity of f by

$$\omega_1^{(\mathcal{F})}(f,\delta)_X = \sup_{x,y \in X, \ |x-y| \leq \delta} D(f(x), f(y)), \ \ \delta > 0.$$

When $g : X \subseteq \mathbb{R} \to \mathbb{R}$, we define

$$\omega_1(g,\delta)_X = \sup_{x,y \in X, \ |x-y| \leq \delta} |g(x) - g(y)|.$$

We define by $C_{\mathcal{F}}^{U}(\mathbb{R})$ the space of fuzzy uniformly continuous functions from $\mathbb{R} \to \mathbb{R}_{\mathcal{F}}$, also $C_{\mathcal{F}}(\mathbb{R})$ is the space of fuzzy continuous functions on $\mathbb{R}$, and $C_b(\mathbb{R}, \mathbb{R}_{\mathcal{F}})$ is the fuzzy continuous and bounded functions.

We mention

Proposition 2.45 ([7])**.** Let $f \in C_{\mathcal{F}}^{U}(X)$. Then $\omega_1^{(\mathcal{F})}(f,\delta)_X < \infty$, for any $\delta > 0$.

By [11, p. 129], we have that $C_{\mathcal{F}}^{U}([a,b]) = C_{\mathcal{F}}([a,b])$, fuzzy continuous functions on $[a,b] \subset \mathbb{R}$.

Proposition 2.46 ([7])**.** It holds

$$\lim_{\delta \to 0} \omega_1^{(\mathcal{F})}(f,\delta)_X = \omega_1^{(\mathcal{F})}(f,0)_X = 0,$$

iff $f \in C_{\mathcal{F}}^{U}(X)$.

Proposition 2.47 ([7])**.** Here $[f]^r = \left[f_{-}^{(r)}, f_{+}^{(r)}\right]$, $r \in [0,1]$. Let $f \in C_{\mathcal{F}}(\mathbb{R})$. Then $f_{\pm}^{(r)}$ are equicontinuous with respect to $r \in [0,1]$ over $\mathbb{R}$, respectively in $\pm$.

Note 2.48. It is clear by Propositions 2.46, 2.47, that if $f \in C_{\mathcal{F}}^{U}(\mathbb{R})$, then $f_{\pm}^{(r)} \in C_U(\mathbb{R})$ (uniformly continuous on $\mathbb{R}$). Also if $f \in C_b(\mathbb{R}, \mathbb{R}_{\mathcal{F}})$ implies $f_{\pm}^{(r)} \in C_b(\mathbb{R})$ (continuous and bounded functions on $\mathbb{R}$).

Proposition 2.49. Let $f:\mathbb{R}\to\mathbb{R}_{\mathcal{F}}$. Assume that $\omega_1^{\mathcal{F}}(f,\delta)_X$, $\omega_1\left(f_-^{(r)},\delta\right)_X$, $\omega_1\left(f_+^{(r)},\delta\right)_X$ are finite for any $\delta>0$, $r\in[0,1]$, where X any interval of $\mathbb{R}$.

Then

$$\omega_1^{(\mathcal{F})}(f,\delta)_X=\sup_{r\in[0,1]}\max\left\{\omega_1\left(f_-^{(r)},\delta\right)_X,\omega_1\left(f_+^{(r)},\delta\right)_X\right\}.$$

Proof. Similar to Proposition 14.15, [11, p. 246]. □

We need

Remark 2.50 ([4])**.** Here $r\in[0,1]$, $x_i^{(r)},y_i^{(r)}\in\mathbb{R}$, $i=1,\dots,m\in\mathbb{N}$. Suppose that

$$\sup_{r\in[0,1]}\max\left(x_i^{(r)},y_i^{(r)}\right)\in\mathbb{R},\text{ for } i=1,\dots,m.$$

Then one sees easily that

$$\sup_{r\in[0,1]}\max\left(\sum_{i=1}^m x_i^{(r)},\sum_{i=1}^m y_i^{(r)}\right)\leq\sum_{i=1}^m\sup_{r\in[0,1]}\max\left(x_i^{(r)},y_i^{(r)}\right). \tag{233}$$

We need

Definition 2.51. Let $x,y\in\mathbb{R}_{\mathcal{F}}$. If there exists $z\in\mathbb{R}_{\mathcal{F}}:x=y\oplus z$, then we call z the H-difference on x and y, denoted $x-y$.

Definition 2.52 ([40])**.** Let $T:=[x_0,x_0+\beta]\subset\mathbb{R}$, with $\beta>0$. A function $f:T\to\mathbb{R}_{\mathcal{F}}$ is H-difference at $x\in T$ if there exists an $f'(x)\in\mathbb{R}_{\mathcal{F}}$ such that the limits (with respect to D)

$$\lim_{h\to0+}\frac{f(x+h)-f(x)}{h},\quad\lim_{h\to0+}\frac{f(x)-f(x-h)}{h} \tag{234}$$

exist and are equal to $f'(x)$.

We call f' the H-derivative or fuzzy derivative of f at x.

Above is assumed that the H-differences $f(x+h)-f(x)$, $f(x)-f(x-h)$ exists in $\mathbb{R}_{\mathcal{F}}$ in a neighborhood of x.

Higher order H-fuzzy derivatives are defined the obvious way, like in the real case.

We denote by $C_{\mathcal{F}}^N(\mathbb{R})$, $N\geq1$, the space of all N-times continuously H-fuzzy differentiable functions from $\mathbb{R}$ into $\mathbb{R}_{\mathcal{F}}$, similarly is defined $C_{\mathcal{F}}^N([a,b])$, $[a,b]\subset\mathbb{R}$.

We mention

Theorem 2.53 ([34])**.** *Let $f:\mathbb{R}\to\mathbb{R}_{\mathcal{F}}$ be H-fuzzy differentiable. Let $t\in\mathbb{R}$, $0\leq r\leq 1$. Clearly*

$$[f(t)]^r=\left[f(t)_-^{(r)},f(t)_+^{(r)}\right]\subseteq\mathbb{R}.$$

Then $(f(t))_{\pm}^{(r)}$ *are differentiable and*

$$[f'(t)]^r = \left[\left(f(t)_-^{(r)}\right)', \left(f(t)_+^{(r)}\right)'\right].$$

I.e.

$$(f')_{\pm}^{(r)} = \left(f_{\pm}^{(r)}\right)', \quad \forall\, r \in [0,1].$$

Remark 2.54 ([6]). Let $f \in C_{\mathcal{F}}^{N}(\mathbb{R})$, $N \geq 1$. Then by Theorem 2.53 we obtain

$$\left[f^{(i)}(t)\right]^r = \left[\left(f(t)_-^{(r)}\right)^{(i)}, \left(f(t)_+^{(r)}\right)^{(i)}\right],$$

for $i = 0, 1, 2, \ldots, N$, and in particular we have that

$$\left(f^{(i)}\right)_{\pm}^{(r)} = \left(f_{\pm}^{(r)}\right)^{(i)},$$

for any $r \in [0,1]$, all $i = 0, 1, 2, \ldots, N$.

Note 2.55 ([6]). Let $f \in C_{\mathcal{F}}^{N}(\mathbb{R})$, $N \geq 1$. Then by Theorem 2.53 we have $f_{\pm}^{(r)} \in C^N(\mathbb{R})$, for any $r \in [0,1]$.

Items 56–58 are valid also on $[a,b]$.

By [11, p. 131], if $f \in C_{\mathcal{F}}([a,b])$, then f is a fuzzy bounded function.

For the definition of general fuzzy integral we follow [35] next.

Definition 2.56. Let (Ω, Σ, μ) be a complete σ-finite measure space. We call $F : \Omega \to R_{\mathcal{F}}$ measurable iff $\forall$ closed $B \subseteq \mathbb{R}$ the function $F^{-1}(B) : \Omega \to [0,1]$ defined by

$$F^{-1}(B)(w) := \sup_{x \in B} F(w)(x), \text{ all } w \in \Omega$$

is measurable, see [35].

Theorem 2.57 ([35]). *For* $F : \Omega \to \mathbb{R}_{\mathcal{F}}$,

$$F(w) = \left\{\left(F_-^{(r)}(w), F_+^{(r)}(w)\right) \mid 0 \leq r \leq 1\right\},$$

the following are equivalent

(1) F *is measurable,*

(2) $\forall\, r \in [0,1]$, $F_-^{(r)}$, $F_+^{(r)}$ *are measurable.*

Following [35], given that for each $r \in [0,1]$, $F_-^{(r)}$, $F_+^{(r)}$ are integrable we have that the parametrized representation

$$\left\{\left(\int_A F_-^{(r)} d\mu, \int_A F_+^{(r)} d\mu\right) \mid 0 \leq r \leq 1\right\} \tag{235}$$

is a fuzzy real number for each $A \in \Sigma$.

The last fact leads to

Definition 2.58 ([35])**.** A measurable function $F : \Omega \to \mathbb{R}_{\mathcal{F}}$,

$$F(w) = \left\{ \left(F_{-}^{(r)}(w), F_{+}^{(r)}(w) \right) \mid 0 \leq r \leq 1 \right\}$$

is integrable if for each $r \in [0,1]$, $F_{\pm}^{(r)}$ are integrable, or equivalently, if $F_{\pm}^{(0)}$ are integrable.

In this case, the fuzzy integral of F over $A \in \Sigma$ is defined by

$$\int_A F d\mu := \left\{ \left(\int_A F_{-}^{(r)} d\mu, \int_A F_{+}^{(r)} d\mu \right) \mid 0 \leq r \leq 1 \right\}.$$

By [35], F is integrable iff $w \to \|F(w)\|_{\mathcal{F}}$ is real-valued integrable.

Here denote

$$\|u\|_{\mathcal{F}} := D\left(u, \widetilde{0}\right), \ \ \forall \ u \in \mathbb{R}_{\mathcal{F}}.$$

We need also

Theorem 2.59 ([35])**.** *Let $F, G : \Omega \to \mathbb{R}_{\mathcal{F}}$ be integrable. Then*

(1) Let $a, b \in \mathbb{R}$, then $aF + bG$ is integrable and for each $A \in \Sigma$,

$$\int_A (aF + bG) \, d\mu = a \int_A F d\mu + b \int_A G d\mu;$$

(2) $D(F,G)$ is a real- valued integrable function and for each $A \in \Sigma$,

$$D\left(\int_A F d\mu, \int_A G d\mu \right) \leq \int_A D(F,G) \, d\mu.$$

In particular,

$$\left\| \int_A F d\mu \right\|_{\mathcal{F}} \leq \int_A \|F\|_{\mathcal{F}} \, d\mu.$$

Above μ could be the Lebesgue measure, with all the basic properties valid here too.

Basically here we have

$$\left[\int_A F d\mu \right]^r = \left[\int_A F_{-}^{(r)} d\mu, \int_A F_{+}^{(r)} d\mu \right], \tag{236}$$

i.e.

$$\left(\int_A F d\mu \right)_{\pm}^{(r)} = \int_A F_{\pm}^{(r)} d\mu, \ \ \forall \ r \in [0,1]. \tag{237}$$

We need

Definition 2.60 ([13])**.** Let $f \in C_{\mathcal{F}}([a,b])$ (fuzzy continuous on $[a,b] \subset \mathbb{R}$), $\nu > 0$.

We define the Fuzzy Fractional left Riemann-Liouville operator as

$$J_a^{\nu} f(x) := \frac{1}{\Gamma(\nu)} \odot \int_a^x (x-t)^{\nu-1} \odot f(t) \, dt, \ \ x \in [a,b], \tag{238}$$

$$J_a^0 f := f.$$

Also, we define the Fuzzy Fractional right Riemann-Liouville operator as

$$I_{b-}^{\nu} f(x) := \frac{1}{\Gamma(\nu)} \odot \int_x^b (t-x)^{\nu-1} \odot f(t)\, dt, \quad x \in [a,b], \tag{239}$$

$$I_{b-}^{0} f := f.$$

We need

Definition 2.61 ([13])**.** We define the Fuzzy Fractional left Caputo derivative, $x \in [a,b]$.

Let $f \in C_{\mathcal{F}}^{n}([a,b])$, $n = \lceil \nu \rceil$, $\nu > 0$ ($\lceil \cdot \rceil$ denotes the ceiling). We define

$$D_{*a}^{\nu\mathcal{F}} f(x) := \frac{1}{\Gamma(n-\nu)} \odot \int_a^x (x-t)^{n-\nu-1} \odot f^{(n)}(t)\, dt \tag{240}$$

$$= \left\{ \left(\frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} \left(f^{(n)}\right)_-^{(r)}(t)\, dt, \right.\right.$$

$$\left.\left. \frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} \left(f^{(n)}\right)_+^{(r)}(t)\, dt \right) \mid 0 \le r \le 1 \right\}$$

$$= \left\{ \left(\frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} \left(f_-^{(r)}\right)^{(n)}(t)\, dt, \right.\right.$$

$$\left.\left. \frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} \left(f_+^{(r)}\right)^{(n)}(t)\, dt \right) \mid 0 \le r \le 1 \right\}. \tag{241}$$

So, we get

$$\left[D_{*a}^{\nu\mathcal{F}} f(x)\right]^r = \left[\left(\frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} \left(f_-^{(r)}\right)^{(n)}(t)\, dt, \right.\right.$$

$$\left.\left. \frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} \left(f_+^{(r)}\right)^{(n)}(t)\, dt \right) \right], \quad 0 \le r \le 1. \tag{242}$$

That is

$$\left(D_{*a}^{\nu\mathcal{F}} f(x)\right)_{\pm}^{(r)} = \frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} \left(f_{\pm}^{(r)}\right)^{(n)}(t)\, dt = \left(D_{*a}^{\nu}\left(f_{\pm}^{(r)}\right)\right)(x),$$

see [10], [28].

I.e. we get that

$$\left(D_{*a}^{\nu\mathcal{F}} f(x)\right)_{\pm}^{(r)} = \left(D_{*a}^{\nu}\left(f_{\pm}^{(r)}\right)\right)(x), \tag{243}$$

$\forall\, x \in [a,b]$, in short

$$\left(D_{*a}^{\nu\mathcal{F}} f\right)_{\pm}^{(r)} = D_{*a}^{\nu}\left(f_{\pm}^{(r)}\right), \quad \forall\, r \in [0,1]. \tag{244}$$

We need

Lemma 2.62 ([13])**.** *$D_{*a}^{\nu\mathcal{F}} f(x)$ is fuzzy continuous in $x \in [a,b]$.*

We need

Definition 2.63 ([13])**.** We define the Fuzzy Fractional right Caputo derivative, $x \in [a,b]$.

Let $f \in C_{\mathcal{F}}^{n}([a,b])$, $n = \lceil \nu \rceil$, $\nu > 0$. We define

$$(245)\quad \begin{aligned} D_{b-}^{\nu\mathcal{F}} f(x) &:= \frac{(-1)^n}{\Gamma(n-\nu)} \odot \int_x^b (t-x)^{n-\nu-1} \odot f^{(n)}(t)\, dt \\ &= \left\{ \left(\frac{(-1)^n}{\Gamma(n-\nu)} \int_x^b (t-x)^{n-\nu-1} \left(f^{(n)}\right)_-^{(r)}(t)\, dt, \right.\right. \\ &\qquad \left.\left. \frac{(-1)^n}{\Gamma(n-\nu)} \int_x^b (t-x)^{n-\nu-1} \left(f^{(n)}\right)_+^{(r)}(t)\, dt \right) \mid 0 \le r \le 1 \right\} \\ &= \left\{ \left(\frac{(-1)^n}{\Gamma(n-\nu)} \int_x^b (t-x)^{n-\nu-1} \left(f_-^{(r)}\right)^{(n)}(t)\, dt, \right.\right. \\ &\qquad \left.\left. \frac{(-1)^n}{\Gamma(n-\nu)} \int_x^b (t-x)^{n-\nu-1} \left(f_+^{(r)}\right)^{(n)}(t)\, dt \right) \mid 0 \le r \le 1 \right\}. \end{aligned}$$

We get

$$\left[D_{b-}^{\nu\mathcal{F}} f(x)\right]^r = \left[\left(\frac{(-1)^n}{\Gamma(n-\nu)} \int_x^b (t-x)^{n-\nu-1} \left(f_-^{(r)}\right)^{(n)}(t)\, dt, \right.\right. \\ \left.\left. \frac{(-1)^n}{\Gamma(n-\nu)} \int_x^b (t-x)^{n-\nu-1} \left(f_+^{(r)}\right)^{(n)}(t)\, dt \right) \right], \quad 0 \le r \le 1.$$

That is

$$\left(D_{b-}^{\nu\mathcal{F}} f(x)\right)_\pm^{(r)} = \frac{(-1)^n}{\Gamma(n-\nu)} \int_x^b (t-x)^{n-\nu-1} \left(f_\pm^{(r)}\right)^{(n)}(t)\, dt = \left(D_{b-}^{\nu}\left(f_\pm^{(r)}\right)\right)(x),$$

see [9].

I.e. we get that

$$(246)\qquad \left(D_{b-}^{\nu\mathcal{F}} f(x)\right)_\pm^{(r)} = \left(D_{b-}^{\nu}\left(f_\pm^{(r)}\right)\right)(x),$$

$\forall\, x \in [a,b]$, in short

$$(247)\qquad \left(D_{b-}^{\nu\mathcal{F}} f\right)_\pm^{(r)} = D_{b-}^{\nu}\left(f_\pm^{(r)}\right), \quad \forall\, r \in [0,1].$$

Clearly,

$$D_{b-}^{\nu}\left(f_-^{(r)}\right) \le D_{b-}^{\nu}\left(f_+^{(r)}\right), \quad \forall\, r \in [0,1].$$

We need

Lemma 2.64 ([13])**.** *$D_{b-}^{\nu\mathcal{F}} f(x)$ is fuzzy continuous in $x \in [a,b]$.*

2.6. **Fuzzy and Fuzzy-Fractional Univariate Neural Network Approximation and Interpolation.** We give

Definition 2.65. Let $f \in C_{\mathcal{F}}([a,b])$. We set

$$\left(H_n^{\mathcal{F}}(f)\right)(x) := \frac{\sum_{k=0}^{n^*} f(x_k) \odot B\left(\frac{Tn(x-x_k)}{b-a}\right)}{\sum_{k=0}^{n} B\left(\frac{Tn(x-x_k)}{b-a}\right)}, \tag{248}$$

and we call it fuzzy interpolation univariate Neural Network operator.

Comment: We observe that

$$\begin{aligned}\left[\left(H_n^{\mathcal{F}}(f)\right)(x)\right]^r &= \sum_{k=0}^{n} [f(x_k)]^r \frac{B\left(\frac{Tn(x-x_k)}{b-a}\right)}{V(x)} \\ &= \sum_{k=0}^{n}\left[f_-^{(r)}(x_k), f_+^{(r)}(x_k)\right] \frac{B\left(\frac{Tn(x-x_k)}{b-a}\right)}{V(x)} \\ &= \left[\sum_{k=0}^{n} f_-^{(r)}(x_k) \frac{B\left(\frac{Tn(x-x_k)}{b-a}\right)}{V(x)}, \sum_{k=0}^{n} f_+^{(r)}(x_k) \frac{B\left(\frac{Tn(x-x_k)}{b-a}\right)}{V(x)}\right] \\ &= \left[\left(H_n\left(f_-^{(r)}\right)\right)(x), \left(H_n\left(f_+^{(r)}\right)\right)(x)\right]. \end{aligned} \tag{249}$$

We have proved that

$$\left(H_n^{\mathcal{F}}(f)\right)_{\pm}^{(r)} = H_n\left(f_{\pm}^{(r)}\right), \tag{250}$$

$\forall\ r \in [0,1]$, respectively.

Comment: We notice also that

$$\left(\left(H_n^{\mathcal{F}}(f)\right)(x_i)\right)_{\pm}^{(r)} = \left(H_n\left(f_{\pm}^{(r)}\right)\right)(x_i) = f_{\pm}^{(r)}(x_i), \ \ i = 0,1,\dots,n, \ \forall\ r \in [0,1]. \tag{251}$$

Conclusion 2.66 (by [33], [35])**.**

$$\left(H_n^{\mathcal{F}}(f)\right)(x_i) = f(x_i), \ \ i = 0,1,\dots,n,$$

the interpolation property is true at fuzzy setting.

We make

Remark 2.67. Let $f \in C_{\mathcal{F}}([a,b])$. We notice that

$$\begin{aligned} &D\left(\left(H_n^{\mathcal{F}}(f)\right)(x), f(x)\right) \\ &= \sup_{r\in[0,1]} \max\left\{\left|\left(H_n(f)\right)_-^{(r)}(x) - f_-^{(r)}(x)\right|, \left|\left(H_n(f)\right)_+^{(r)}(x) - f_+^{(r)}(x)\right|\right\} \\ &= \sup_{r\in[0,1]} \max\left\{\left|\left(H_n\left(f_-^{(r)}\right)\right)(x) - f_-^{(r)}(x)\right|, \left|\left(H_n\left(f_+^{(r)}\right)\right)(x) - f_+^{(r)}(x)\right|\right\} \leq \end{aligned} \tag{252}$$

(hence $f_{\pm}^{(r)} \in C([a,b])$)

$$\frac{2B^*}{B\left(\frac{T}{2}\right)} \sup_{r\in[0,1]} \max\left\{\omega_1\left(f_-^{(r)}, \frac{b-a}{n}\right), \omega_1\left(f_+^{(r)}, \frac{b-a}{n}\right)\right\} =$$

(by Theorem 2.7 and Proposition 2.49)

$$\frac{2B^*}{B\left(\frac{T}{2}\right)} \omega_1^{(\mathcal{F})}\left(f, \frac{b-a}{n}\right). \tag{253}$$

We have proved that

Theorem 2.68. *Let $f \in C_{\mathcal{F}}([a,b])$, $x \in [a,b]$. Then*

1)

$$D\left(\left(H_n^{\mathcal{F}}(f)\right)(x), f(x)\right) \leq \frac{2B^*}{B\left(\frac{T}{2}\right)} \omega_1^{(\mathcal{F})}\left(f, \frac{b-a}{n}\right), \tag{254}$$

so that $\left(H_n^{\mathcal{F}}(f)\right)(x) \xrightarrow{D} f(x)$, as $n \to \infty$, pointwise, and

2)

$$D^*\left(H_n^{\mathcal{F}}(f), f\right) \leq \frac{2B^*}{B\left(\frac{T}{2}\right)} \omega_1^{(\mathcal{F})}\left(f, \frac{b-a}{n}\right), \tag{255}$$

so that $H_n^{\mathcal{F}}(f) \xrightarrow{D^*} f$, as $n \to \infty$, uniformly.

Taking into account fuzzy smoothness of f we give

Theorem 2.69. *Let $f \in C_{\mathcal{F}}^N([a,b])$, $N \in \mathbb{N}$, $x \in [a,b]$. Then*

1)

$$D\left(\left(H_n^{\mathcal{F}}(f)\right)(x), f(x)\right)$$
$$\leq \frac{2B^*}{B\left(\frac{T}{2}\right)} \left\{\sum_{j=1}^{N} \frac{(b-a)^j}{j!n^j} D\left(f^{(j)}(x), \widetilde{o}\right) + \frac{(b-a)^N}{N!n^N} \omega_1^{(\mathcal{F})}\left(f^{(N)}, \frac{b-a}{n}\right)\right\}, \tag{256}$$

2) assume more that $D\left(f^{(j)}(x), \widetilde{o}\right) = 0$, $j = 1, \ldots, N$, where $x \in [a,b]$ is fixed, we get

$$D\left(\left(H_n^{\mathcal{F}}(f)\right)(x), f(x)\right) \leq \frac{2B^*}{B\left(\frac{T}{2}\right)} \frac{(b-a)^N}{N!n^N} \omega_1^{(\mathcal{F})}\left(f^{(N)}, \frac{b-a}{n}\right), \tag{257}$$

a fuzzy pointwise convergence at high speed $\frac{1}{n^{N+1}}$,

3)

$$D^*\left(H_n^{\mathcal{F}}(f), f\right)$$
$$\leq \frac{2B^*}{B\left(\frac{T}{2}\right)} \left\{\sum_{j=1}^{N} \frac{(b-a)^j}{j!n^j} D^*\left(f^{(j)}, \widetilde{o}\right) + \frac{(b-a)^N}{N!n^N} \omega_1^{(\mathcal{F})}\left(f^{(N)}, \frac{b-a}{n}\right)\right\}. \tag{258}$$

Proof. Here clearly $f_{\pm}^{(r)} \in C^N([a,b])$, $\forall\, r \in [0,1]$. Then

$$D\left(\left(H_n^{\mathcal{F}}(f)\right)(x), f(x)\right)$$

$$= \sup_{r\in[0,1]} \max\left\{\left|\left(H_n^{\mathcal{F}}(f)\right)_-^{(r)}(x) - f_-^{(r)}(x)\right|, \left|\left(H_n^{\mathcal{F}}(f)\right)_+^{(r)}(x) - f_+^{(r)}(x)\right|\right\}$$

(259)

$$= \sup_{r\in[0,1]} \max\left\{\left|\left(H_n\left(f_-^{(r)}\right)\right)(x) - f_-^{(r)}(x)\right|, \left|\left(H_n\left(f_+^{(r)}\right)\right)(x) - f_+^{(r)}(x)\right|\right\}$$

$$\overset{\text{(by (34))}}{\leq} \frac{2B^*}{B\left(\frac{T}{2}\right)} \sup_{r\in[0,1]} \max\left\{\sum_{j=1}^{N} \frac{\left|\left(f_-^{(r)}\right)^{(j)}(x)\right|}{j!} \frac{(b-a)^j}{n^j} + \omega_1\left(\left(f_-^{(r)}\right)^{(N)}, \frac{b-a}{n}\right) \frac{(b-a)^N}{N!n^N},\right.$$

$$\left.\sum_{j=1}^{N} \frac{\left|\left(f_+^{(r)}\right)^{(j)}(x)\right|}{j!} \frac{(b-a)^j}{n^j} + \omega_1\left(\left(f_+^{(r)}\right)^{(N)}, \frac{b-a}{n}\right) \frac{(b-a)^N}{N!n^N}\right\}$$

(260)

$$= \frac{2B^*}{B\left(\frac{T}{2}\right)} \sup_{r\in[0,1]} \max\left\{\sum_{j=1}^{N} \frac{\left|\left(f_-^{(j)}\right)^{(r)}(x)\right|}{j!} \frac{(b-a)^j}{n^j} + \omega_1\left(\left(f_-^{(N)}\right)^{(r)}, \frac{b-a}{n}\right) \frac{(b-a)^N}{N!n^N},\right.$$

$$\left.\sum_{j=1}^{N} \frac{\left|\left(f_+^{(j)}\right)^{(r)}(x)\right|}{j!} \frac{(b-a)^j}{n^j} + \omega_1\left(\left(f_+^{(N)}\right)^{(r)}, \frac{b-a}{n}\right) \frac{(b-a)^N}{N!n^N}\right\}$$

$$\leq \frac{2B^*}{B\left(\frac{T}{2}\right)} \left\{\sum_{j=1}^{N} \frac{(b-a)^j}{j!n^j} \sup_{r\in[0,1]} \max\left\{\left|\left(f_-^{(j)}\right)^{(r)}(x)\right|, \left|\left(f_+^{(j)}\right)^{(r)}(x)\right|\right\}\right.$$

(261)

$$\left.+ \frac{(b-a)^N}{N!n^N} \sup_{r\in[0,1]} \max\left\{\omega_1\left(\left(f_-^{(N)}\right)^{(r)}, \frac{b-a}{n}\right), \omega_1\left(\left(f_+^{(N)}\right)^{(r)}, \frac{b-a}{n}\right)\right\}\right\}$$

$$= \frac{2B^*}{B\left(\frac{T}{2}\right)} \left\{\sum_{j=1}^{N} \frac{(b-a)^j}{j!n^j} D\left(f^{(j)}(x), \widetilde{o}\right) + \frac{(b-a)^N}{N!n^N} \omega_1^{(\mathcal{F})}\left(f^{(N)}, \frac{b-a}{n}\right)\right\},$$

proving theorem. ∎

The related fuzzy-fractional results follow.

Theorem 2.70. *Let $\beta > 0$, $N = \lceil \beta \rceil$, $\beta \notin \mathbb{N}$, $f \in C_{\mathcal{F}}^N([a,b])$, $x \in [a,b]$. Then*

$$D\left(\left(H_n^{\mathcal{F}}(f)\right)(x), f(x)\right)$$

$$\leq \frac{B^*}{B\left(\frac{T}{2}\right)} \left[2 \sum_{j=1}^{N-1} \frac{D\left(f^{(j)}(x), \widetilde{o}\right)}{j!} \frac{(b-a)^j}{n^j}\right. \tag{262}$$

$$+\frac{(b-a)^{\beta}}{\Gamma(\beta+1)n^{\beta}}\left[\omega_1^{(\mathcal{F})}\left(\left(D_{x-}^{\beta\mathcal{F}}f\right),\frac{b-a}{n}\right)+\omega_1^{(\mathcal{F})}\left(\left(D_{*x}^{\beta\mathcal{F}}f\right),\frac{b-a}{n}\right)\right]\Bigg].$$

Proof. We get that $f_{\pm}^{(r)}\in C^N([a,b])$, $\forall\, r\in[0,1]$, and $D_{x-}^{\beta\mathcal{F}}f$, $D_{*x}^{\beta\mathcal{F}}f$ are fuzzy continuous on $[a,b]$, $\forall\, x\in[a,b]$, so that $\left(D_{x-}^{\beta\mathcal{F}}f\right)_{\pm}^{(r)}$, $\left(D_{*x}^{\beta\mathcal{F}}f\right)_{\pm}^{(r)}\in C([a,b])$, $\forall\, x\in[a,b]$, $\forall\, r\in[0,1]$. By (74) we get

$$\left|H_n\left(f_{\pm}^{(r)},x\right)-f_{\pm}^{(r)}(x)\right|$$

$$\leq\frac{B^*}{B\left(\frac{T}{2}\right)}\left[2\sum_{j=1}^{N-1}\frac{\left|\left(f_{\pm}^{(r)}\right)^{(j)}(x)\right|}{j!}\frac{(b-a)^j}{n^j}\right. \tag{263}$$

$$\left.+\frac{(b-a)^{\beta}}{\Gamma(\beta+1)n^{\beta}}\left[\omega_1\left(D_{x-}^{\beta}\left(f_{\pm}^{(r)}\right),\frac{b-a}{n}\right)+\omega_1\left(D_{*x}^{\beta}\left(f_{\pm}^{(r)}\right),\frac{b-a}{n}\right)\right]\right]$$

$$=\frac{B^*}{B\left(\frac{T}{2}\right)}\left[2\sum_{j=1}^{N-1}\frac{\left|\left(f^{(j)}(x)\right)_{\pm}^{(r)}\right|}{j!}\frac{(b-a)^j}{n^j}\right.$$

$$\left.+\frac{(b-a)^{\beta}}{\Gamma(\beta+1)n^{\beta}}\left[\omega_1\left(\left(D_{x-}^{\beta\mathcal{F}}f\right)_{\pm}^{(r)},\frac{b-a}{n}\right)+\omega_1\left(\left(D_{*x}^{\beta\mathcal{F}}f\right)_{\pm}^{(r)},\frac{b-a}{n}\right)\right]\right] \tag{264}$$

$$\leq\frac{B^*}{B\left(\frac{T}{2}\right)}\left[2\sum_{j=1}^{N-1}\frac{D\left(f^{(j)}(x),\widetilde{o}\right)}{j!}\frac{(b-a)^j}{n^j}\right.$$

$$\left.+\frac{(b-a)^{\beta}}{\Gamma(\beta+1)n^{\beta}}\left[\omega_1^{(\mathcal{F})}\left(\left(D_{x-}^{\beta\mathcal{F}}f\right),\frac{b-a}{n}\right)+\omega_1^{(\mathcal{F})}\left(\left(D_{*x}^{\beta\mathcal{F}}f\right),\frac{b-a}{n}\right)\right]\right], \tag{265}$$

proving the claim. ∎

Corollary 2.71 (to Theorem 2.70)**.** Assume more that $D\left(f^{(j)}(x),\widetilde{o}\right)=0$, for $j=1,\dots,N-1$, for a fixed $x\in[a,b]$. Then

$$D\left(\left(H_n^{\mathcal{F}}(f)\right)(x),f(x)\right)\leq\frac{B^*}{B\left(\frac{T}{2}\right)}\frac{(b-a)^{\beta}}{\Gamma(\beta+1)n^{\beta}}$$

$$\times\left[\omega_1^{(\mathcal{F})}\left(\left(D_{x-}^{\beta\mathcal{F}}f\right),\frac{b-a}{n}\right)+\omega_1^{(\mathcal{F})}\left(\left(D_{*x}^{\beta\mathcal{F}}f\right),\frac{b-a}{n}\right)\right], \tag{266}$$

fuzzy pointwise convergence at high speed of $\frac{1}{n^{\beta+1}}$.

Theorem 2.72. *Let $\beta>0$, $N=\lceil\beta\rceil$, $\beta\notin\mathbb{N}$, $f\in C_{\mathcal{F}}^N([a,b])$. Then*

$$D^*\left(H_n^{\mathcal{F}}(f),f\right)$$

$$\leq\frac{B^*}{B\left(\frac{T}{2}\right)}\left[2\sum_{j=1}^{N-1}\frac{D^*\left(f^{(j)},\widetilde{o}\right)}{j!}\frac{(b-a)^j}{n^j}+\frac{(b-a)^{\beta}}{\Gamma(\beta+1)n^{\beta}}\right.$$

$$\times\left[\sup_{x\in[a,b]}\omega_1^{(\mathcal{F})}\left(\left(D_{x-}^{\beta\mathcal{F}}f\right),\frac{b-a}{n}\right)+\sup_{x\in[a,b]}\omega_1^{(\mathcal{F})}\left(\left(D_{*x}^{\beta\mathcal{F}}f\right),\frac{b-a}{n}\right)\right]\Bigg]<+\infty. \tag{267}$$

Proof. We notice the following

$$\begin{aligned}\left(D_{x-}^{\beta\mathcal{F}}f\right)_{\pm}^{(r)}(t)&=\left(D_{x-}^{\beta}\left(f_{\pm}^{(r)}\right)\right)(t)\\&=\frac{(-1)^N}{\Gamma(N-\beta)}\int_t^x(s-t)^{N-\beta-1}\left(f_{\pm}^{(r)}\right)^{(N)}(s)\,ds,\end{aligned}\tag{268}$$

all $a\le t\le x$.

Hence it holds

$$\begin{aligned}\left|\left(D_{x-}^{\beta\mathcal{F}}f\right)_{\pm}^{(r)}(t)\right|&\le\frac{1}{\Gamma(N-\beta)}\int_t^x(s-t)^{N-\beta-1}\left|\left(f_{\pm}^{(r)}\right)^{(N)}(s)\right|ds\\&\le\frac{\left\|\left(f^{(N)}\right)_{\pm}^{(r)}\right\|_{\infty}(b-a)^{N-\beta}}{\Gamma(N-\beta+1)}\le\frac{D^*\left(f^{(N)},\widetilde{o}\right)}{\Gamma(N-\beta+1)}(b-a)^{N-\beta},\end{aligned}\tag{269}$$

$a\le t\le x$.

Thus

$$\left\|\left(D_{x-}^{\beta\mathcal{F}}f\right)_{\pm}^{(r)}\right\|_{\infty}\le\frac{D^*\left(f^{(N)},\widetilde{o}\right)}{\Gamma(N-\beta+1)}(b-a)^{N-\beta}\tag{270}$$

(notice $\left(D_{x-}^{\beta\mathcal{F}}f\right)_{\pm}^{(r)}(t)=0$, for $x\le t\le b$), $\forall\ r\in[0,1]$.

So that

$$D^*\left(\left(D_{x-}^{\beta\mathcal{F}}f\right),\widetilde{o}\right)\le\frac{D^*\left(f^{(N)},\widetilde{o}\right)}{\Gamma(N-\beta+1)}(b-a)^{N-\beta}.\tag{271}$$

Similarly we have

$$\begin{aligned}\left(D_{*x}^{\beta\mathcal{F}}f\right)_{\pm}^{(r)}(t)&=\left(D_{*x}^{\beta}\left(f_{\pm}^{(r)}\right)\right)(t)\\&=\frac{1}{\Gamma(N-\beta)}\int_x^t(t-s)^{N-\beta-1}\left(f_{\pm}^{(r)}\right)^{(N)}(s)\,ds,\end{aligned}\tag{272}$$

where $x\le t\le b$.

Thus

$$\begin{aligned}\left|\left(D_{*x}^{\beta\mathcal{F}}f\right)_{\pm}^{(r)}(t)\right|&\le\frac{1}{\Gamma(N-\beta)}\int_x^t(t-s)^{N-\beta-1}\left|\left(f_{\pm}^{(r)}\right)^{(N)}(s)\right|ds\\&=\frac{1}{\Gamma(N-\beta)}\int_x^t(t-s)^{N-\beta-1}\left|\left(f^{(N)}\right)_{\pm}^{(r)}(s)\right|ds\end{aligned}\tag{273}$$

$$\begin{aligned}&\le\frac{\left\|\left(f^{(N)}\right)_{\pm}^{(r)}\right\|_{\infty}}{\Gamma(N-\beta+1)}(b-a)^{N-\beta}\\&\le\frac{D^*\left(f^{(N)},\widetilde{o}\right)}{\Gamma(N-\beta+1)}(b-a)^{N-\beta},\quad x\le t\le b.\end{aligned}\tag{274}$$

So that

$$\left|\left(D_{*x}^{\beta\mathcal{F}}f\right)_{\pm}^{(r)}(t)\right| \leq \frac{D^{*}\left(f^{(N)},\widetilde{o}\right)}{\Gamma(N-\beta+1)}(b-a)^{N-\beta}, \tag{275}$$

$x \leq t \leq b$.

(Notice $\left(D_{*x}^{\beta\mathcal{F}}f\right)_{\pm}^{(r)}(t)=0$, for $a \leq t \leq x$, $\forall\ r \in [0,1]$.)

Thus

$$\left\|\left(D_{*x}^{\beta\mathcal{F}}f\right)_{\pm}^{(r)}\right\|_{\infty} \leq \frac{D^{*}\left(f^{(N)},\widetilde{o}\right)}{\Gamma(N-\beta+1)}(b-a)^{N-\beta}, \tag{276}$$

$\forall\ r \in [0,1]$.

Therefore

$$D^{*}\left(\left(D_{*x}^{\beta\mathcal{F}}f\right),\widetilde{o}\right) \leq \frac{D^{*}\left(f^{(N)},\widetilde{o}\right)}{\Gamma(N-\beta+1)}(b-a)^{N-\beta}. \tag{277}$$

We have proved that

$$\left\{\begin{array}{c} D^{*}\left(\left(D_{x-}^{\beta\mathcal{F}}f\right),\widetilde{o}\right) \\ D^{*}\left(\left(D_{*x}^{\beta\mathcal{F}}f\right),\widetilde{o}\right) \end{array}\right. \leq \frac{D^{*}\left(f^{(N)},\widetilde{o}\right)}{\Gamma(N-\beta+1)}(b-a)^{N-\beta}. \tag{278}$$

Next we see that

(279)

$$\omega_{1}^{(\mathcal{F})}\left(\left(D_{x-}^{\beta\mathcal{F}}f\right),\frac{b-a}{n}\right) = \sup_{\substack{z_1,z_2\in[a,b] \\ :|z_1-z_2|\leq\frac{b-a}{n}}} D\left(\left(D_{x-}^{\beta\mathcal{F}}f\right)(z_1),\left(D_{x-}^{\beta\mathcal{F}}f\right)(z_2)\right)$$

$$\leq \sup_{\substack{z_1,z_2\in[a,b] \\ :|z_1-z_2|\leq\frac{b-a}{n}}} \left\{D\left(\left(D_{x-}^{\beta\mathcal{F}}f\right)(z_1),\widetilde{o}\right)+D\left(\left(D_{x-}^{\beta\mathcal{F}}f\right)(z_2),\widetilde{o}\right)\right\}$$

$$\leq 2D^{*}\left(\left(D_{x-}^{\beta\mathcal{F}}f\right),\widetilde{o}\right) \leq \frac{2D^{*}\left(f^{(N)},\widetilde{o}\right)}{\Gamma(N-\beta+1)}(b-a)^{N-\beta} =: \gamma < \infty. \tag{280}$$

Therefore it holds

$$\sup_{x\in[a,b]} \omega_{1}^{(\mathcal{F})}\left(\left(D_{x-}^{\beta\mathcal{F}}f\right),\frac{b-a}{n}\right) \leq \gamma < \infty. \tag{281}$$

Totally similar we get

$$\sup_{x\in[a,b]} \omega_{1}^{(\mathcal{F})}\left(\left(D_{*x}^{\beta\mathcal{F}}f\right),\frac{b-a}{n}\right) \leq \gamma < \infty. \tag{282}$$

Using (262), (281), (282) we have established (267). □

2.7. **Multivariate Fuzzy Analysis background.** Let $f, g : \prod_{i=1}^{d} [a_i, b_i] \to \mathbb{R}_{\mathcal{F}}$. We define the distance

$$D^* (f, g) := \sup_{x \in \prod_{i=1}^{d} [a_i, b_i]} D (f (x), g (x)) . \tag{283}$$

Definition 2.73. Let $f \in C \left(\prod_{i=1}^{d} [a_i, b_i] \right)$, $d \in \mathbb{N}$, we define $(h > 0)$

$$\omega_1 (f, h) := \sup_{\text{all } x_i, x_i' \in [a_i, b_i], \left| x_i - x_i' \right| \leq h, \text{ for } i=1,\ldots,d} \left| f (x_1, \ldots, x_d) - f (x_1', \ldots, x_d') \right| . \tag{284}$$

For convenience call $Q := \prod_{i=1}^{d} [a_i, b_i]$.

Definition 2.74. Let $f : Q \to \mathbb{R}_{\mathcal{F}}$, we define the fuzzy modulus of continuity of f by

$$\omega_1^{(\mathcal{F})} (f, \delta) = \sup_{x, y \in Q, |x_i - y_i| \leq \delta, \text{ for } i=1,\ldots,d} D (f (x), f (y)) , \quad \delta > 0, \tag{285}$$

where $x = (x_1, \ldots, x_d)$, $y = (y_1, \ldots, y_d)$.

For $f : Q \to \mathbb{R}_{\mathcal{F}}$, we use

$$[f]^r = \left[f_-^{(r)}, f_+^{(r)} \right] , \tag{286}$$

where $f_{\pm}^{(r)} : Q \to \mathbb{R}$, $\forall\, r \in [0, 1]$.

We need

Proposition 2.75. Let $f : Q \to \mathbb{R}_{\mathcal{F}}$. Assume that $\omega_1^{\mathcal{F}} (f, \delta)$, $\omega_1 \left(f_-^{(r)}, \delta \right)$, $\omega_1 \left(f_+^{(r)}, \delta \right)$ are finite for any $\delta > 0$, $r \in [0, 1]$.

Then

$$\omega_1^{(\mathcal{F})} (f, \delta) = \sup_{r \in [0,1]} \max \left\{ \omega_1 \left(f_-^{(r)}, \delta \right), \omega_1 \left(f_+^{(r)}, \delta \right) \right\} . \tag{287}$$

Proof. By [11, p. 128]. □

We define $C_{\mathcal{F}} (Q)$ the space of fuzzy continuous functions on Q.

We mention

Proposition 2.76. Let $f \in C_{\mathcal{F}} (Q)$. Then $\omega_1^{(\mathcal{F})} (f, \delta) < \infty$, for any $\delta > 0$.

Proof. By [11, p. 129]. □

Proposition 2.77. It holds

$$\lim_{\delta \to 0} \omega_1^{(\mathcal{F})} (f, \delta) = \omega_1^{(\mathcal{F})} (f, 0) = 0, \tag{288}$$

iff $f \in C_{\mathcal{F}} (Q)$.

Proof. By [11, p. 129]. □

Proposition 2.78. Let $f \in C_{\mathcal{F}}(Q)$. Then $f_{\pm}^{(r)}$ are equicontinuous with respect to $r \in [0,1]$ over Q, respectively in $\pm$. Also f is a fuzzy bounded function.

Proof. By [11, pp. 131, 132]. □

We call $C_{\mathcal{F}}^{N}(Q)$, $N \in \mathbb{N}$, the space of all N-times fuzzy continuously differentiable functions from Q into $\mathbb{R}_{\mathcal{F}}$.

Let $f \in C_{\mathcal{F}}^{N}(Q)$, denote $f_{\widetilde{\alpha}} := \frac{\partial^{\widetilde{\alpha}} f}{\partial x^{\widetilde{\alpha}}}$, where $\widetilde{\alpha} := (\alpha_1, \ldots, \alpha_d)$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, \ldots, d$ and

$$0 < |\widetilde{\alpha}| := \sum_{i=1}^{d} \alpha_i \leq N, \quad N > 1.$$

Then by Theorem 2.53 we get that

$$\left(f_{\pm}^{(r)}\right)_{\widetilde{\alpha}} = (f_{\widetilde{\alpha}})_{\pm}^{(r)}, \ \ \forall\, r \in [0,1], \tag{289}$$

and any $\widetilde{\alpha} : |\widetilde{\alpha}| \leq N$. Here $f_{\pm}^{(r)} \in C^{N}(Q)$.

Notation 2.79. We denote

$$\left(\sum_{i=1}^{2} D\left(\frac{\partial}{\partial x_i}, \widetilde{0}\right)\right)^{2} f(x)$$
$$:= D\left(\frac{\partial^2 f(x_1,x_2)}{\partial x_1^2}, \widetilde{0}\right) + D\left(\frac{\partial^2 f(x_1,x_2)}{\partial x_2^2}, \widetilde{0}\right) + 2D\left(\frac{\partial^2 f(x_1,x_2)}{\partial x_1 \partial x_2}, \widetilde{0}\right). \tag{290}$$

In general we denote $(j = 1, \ldots, N)$

$$\left(\sum_{i=1}^{d} D\left(\frac{\partial}{\partial x_i}, \widetilde{0}\right)\right)^{j} f(x)$$
$$:= \sum_{(j_1,\ldots,j_d) \in \mathbb{Z}_+^d : \sum_{i=1}^{d} j_i = j} \frac{j!}{j_1! j_2! \cdots j_d!} D\left(\frac{\partial^j f(x_1,\ldots,x_d)}{\partial x_1^{j_1} \partial x_2^{j_2} \cdots \partial x_d^{j_d}}, \widetilde{0}\right). \tag{291}$$

Let

$$f_{\widetilde{\alpha}}(x) = \widetilde{o}, \text{ for all } \widetilde{\alpha} : |\widetilde{\alpha}| = 1, \ldots, N,$$

for $x \in Q$ fixed.

The last implies $D(f_{\widetilde{\alpha}}(x), \widetilde{o}) = 0$, and by (291) we obtain

$$\left[\left(\sum_{i=1}^{d} D\left(\frac{\partial}{\partial x_i}, \widetilde{o}\right)\right)^{j} f(x)\right] = 0, \tag{292}$$

for $j = 1, \ldots, N$.

2.8. **Multivariate Fuzzy Neural Network Approximation and Interpolation.** Let $f \in C_{\mathcal{F}}\left(\prod_{i=1}^{d}[a_i,b_i]\right)$, $x \in \prod_{i=1}^{d}[a_i,b_i]$, we define

$$M_n^{\mathcal{F}}(f,x) := M_n^{\mathcal{F}}(f,x_1,\ldots,x_d)$$

$$:= \frac{\sum_{k_1=0}^{n*}\cdots\sum_{k_d=0}^{n*} f(x_{k_1 1},\ldots,x_{k_d d}) \odot E\left(\frac{T_1 n\left(x_1-x_{k_1 1}\right)}{b_1-a_1},\ldots,\frac{T_d n\left(x_d-x_{k_d d}\right)}{b_d-a_d}\right)}{\sum_{k_1=0}^{n}\cdots\sum_{k_d=0}^{n} E\left(\frac{T_1 n\left(x_1-x_{k_1 1}\right)}{b_1-a_1},\ldots,\frac{T_d n\left(x_d-x_{k_d d}\right)}{b_d-a_d}\right)}, \tag{293}$$

the multivariate fuzzy neural network interpolation operator, $\forall\, n \in \mathbb{N}$.

Remark 2.80. We observe that

$$\left[M_n^{\mathcal{F}}(f,x)\right]^r = \frac{\sum_{k_1=0}^{n}\cdots\sum_{k_d=0}^{n}\left[f(x_{k_1 1},\ldots,x_{k_d d})\right]^r E\left(\frac{T_1 n\left(x_1-x_{k_1 1}\right)}{b_1-a_1},\ldots,\frac{T_d n\left(x_d-x_{k_d d}\right)}{b_d-a_d}\right)}{W} \tag{294}$$

$$= \sum_{k_1=0}^{n}\cdots\sum_{k_d=0}^{n}\left[f_-^{(r)}(x_{k_1 1},\ldots,x_{k_d d}), f_+^{(r)}(x_{k_1 1},\ldots,x_{k_d d})\right] \times \frac{E\left(\frac{T_1 n\left(x_1-x_{k_1 1}\right)}{b_1-a_1},\ldots,\frac{T_d n\left(x_d-x_{k_d d}\right)}{b_d-a_d}\right)}{W}$$

$$= \left[\sum_{k_1=0}^{n}\cdots\sum_{k_d=0}^{n} f_-^{(r)}(x_{k_1 1},\ldots,x_{k_d d})\frac{E(\gg)}{W}, \sum_{k_1=0}^{n}\cdots\sum_{k_d=0}^{n} f_+^{(r)}(x_{k_1 1},\ldots,x_{k_d d})\frac{E(\gg)}{W}\right] \tag{295}$$

$$= \left[\left(M_n\left(f_-^{(r)}\right)\right)(x), \left(M_n\left(f_+^{(r)}\right)\right)(x)\right].$$

Hence it holds

$$\left(M_n^{\mathcal{F}}(f)\right)_{\pm}^{(r)} = M_n\left(f_{\pm}^{(r)}\right), \tag{296}$$

$\forall\, r \in [0,1]$, respectively.

Remark 2.81. Let $(k_1,\ldots,k_d) \in \{0,1,\ldots,n\}^d$. Then

$$\left(M_n^{\mathcal{F}}(f,x_{k_1 1},\ldots,x_{k_d d})\right)_{\pm}^{(r)} = M_n\left(f_{\pm}^{(r)}\right)(x_{k_1 1},\ldots,x_{k_d d}) = f_{\pm}^{(r)}(x_{k_1 1},\ldots,x_{k_d d}), \quad \forall\, r \in [0,1], \tag{297}$$

proving

$$M_n^{\mathcal{F}}(f,x_{k_1 1},\ldots,x_{k_d d}) = f(x_{k_1 1},\ldots,x_{k_d d}), \tag{298}$$

the interpolation property.

Remark 2.82. Let $f \in C_{\mathcal{F}}\left(\prod_{i=1}^{d}[a_i,b_i]\right)$. Then

$$D\left(\left(M_n^{\mathcal{F}}(f)\right)(x), f(x)\right)$$
$$= \sup_{r\in[0,1]} \max\left\{\left|\left(M_n^{\mathcal{F}}(f)\right)_-^{(r)}(x) - f_-^{(r)}(x)\right|, \left|\left(M_n^{\mathcal{F}}(f)\right)_+^{(r)}(x) - f_+^{(r)}(x)\right|\right\}$$

(299)

$$= \sup_{r\in[0,1]} \max\left\{\left|\left(M_n^{\mathcal{F}}\left(f_-^{(r)}\right)\right)(x) - f_-^{(r)}(x)\right|, \left|\left(M_n^{\mathcal{F}}\left(f_+^{(r)}\right)\right)(x) - f_+^{(r)}(x)\right|\right\} \overset{(148)}{\leq}$$

(we have $f_{\pm}^{r} \in C\left(\prod_{i=1}^{d}[a_i,b_i]\right)$)

$$\sup_{r\in[0,1]} \max\left\{\frac{2^d E^*}{E\left(\frac{T_1}{2},\ldots,\frac{T_d}{2}\right)}\omega_1\left(f_-^{(r)}, \frac{\|b-a\|_\infty}{n}\right),\right.$$

$$\left.\frac{2^d E^*}{E\left(\frac{T_1}{2},\ldots,\frac{T_d}{2}\right)}\omega_1\left(f_+^{(r)}, \frac{\|b-a\|_\infty}{n}\right)\right\} \tag{300}$$

$$= \frac{2^d E^*}{E\left(\frac{T_1}{2},\ldots,\frac{T_d}{2}\right)} \sup_{r\in[0,1]} \max\left\{\omega_1\left(f_-^{(r)}, \frac{\|b-a\|_\infty}{n}\right), \omega_1\left(f_+^{(r)}, \frac{\|b-a\|_\infty}{n}\right)\right\} \tag{301}$$

$$\overset{(287)}{=} \frac{2^d E^*}{E\left(\frac{T_1}{2},\ldots,\frac{T_d}{2}\right)}\omega_1^{(\mathcal{F})}\left(f, \frac{\|b-a\|_\infty}{n}\right).$$

We have proved

Theorem 2.83. *Let* $f \in C_{\mathcal{F}}\left(\prod_{i=1}^{d}[a_i,b_i]\right)$. *Then*

$$D\left(\left(M_n^{\mathcal{F}}(f)\right)(x), f(x)\right) \leq \frac{2^d E^*}{E\left(\frac{T_1}{2},\ldots,\frac{T_d}{2}\right)}\omega_1^{(\mathcal{F})}\left(f, \frac{\|b-a\|_\infty}{n}\right) =: \lambda, \tag{302}$$

and

$$D^*\left(M_n^{\mathcal{F}}(f), f\right) \leq \lambda. \tag{303}$$

We make

Remark 2.84. Let $f \in C_{\mathcal{F}}^{N}\left(\prod_{i=1}^{d}[a_i,b_i]\right)$, $N \in \mathbb{N}$, $x \in \prod_{i=1}^{d}[a_i,b_i]$ (so that $f_{\pm}^{(r)} \in C^N\left(\prod_{i=1}^{d}[a_i,b_i]\right)$).

We get

$$\left|M_n\left(f_{\pm}^{(r)}, x\right) - f_{\pm}^{(r)}(x)\right|$$

$$\overset{(156)}{\leq} \frac{2^d E^*}{E\left(\frac{T_1}{2},\ldots,\frac{T_d}{2}\right)}\left[\sum_{j=1}^{N}\frac{1}{j!}\left(\frac{\|b-a\|_\infty^j}{n^j}\right)\left(\left(\sum_{i=1}^{d}\left|\frac{\partial}{\partial x_i}\right|\right)^j f_{\pm}^{(r)}(x)\right)\right. \tag{304}$$

$$+\frac{\|b-a\|_{\infty}^{N} d^{N}}{N! n^{N}} \max_{\widetilde{\alpha}:|\widetilde{\alpha}|=N} \omega_1\left(\left(f_{\pm}^{(r)}\right)_{\widetilde{\alpha}}, \frac{\|b-a\|_{\infty}}{n}\right)\Bigg]$$

$$= \frac{2^d E^*}{E\left(\frac{T_1}{2},\ldots,\frac{T_d}{2}\right)}\left[\sum_{j=1}^{N}\frac{1}{j!}\left(\frac{\|b-a\|_{\infty}^{j}}{n^j}\right)\left(\left(\sum_{i=1}^{d}\left|\frac{\partial}{\partial x_i}\right|\right)^j f(x)\right)_{\pm}^{(r)}\right. \tag{305}$$

$$\left.+\frac{\|b-a\|_{\infty}^{N} d^{N}}{N! n^{N}} \max_{\widetilde{\alpha}:|\widetilde{\alpha}|=N} \omega_1\left(\left(f_{\widetilde{\alpha}}\right)_{\pm}^{(r)}, \frac{\|b-a\|_{\infty}}{n}\right)\right]$$

$$\leq \frac{2^d E^*}{E\left(\frac{T_1}{2},\ldots,\frac{T_d}{2}\right)}\left[\sum_{j=1}^{N}\frac{1}{j!}\left(\frac{\|b-a\|_{\infty}^{j}}{n^j}\right)\left[\left(\sum_{i=1}^{d} D\left(\frac{\partial}{\partial x_i},\widetilde{o}\right)\right)^j f(x)\right]\right. \tag{306}$$

$$\left.+\frac{\|b-a\|_{\infty}^{N} d^{N}}{N! n^{N}} \max_{\widetilde{\alpha}:|\widetilde{\alpha}|=N} \omega_1^{(\mathcal{F})}\left(f_{\widetilde{\alpha}}, \frac{\|b-a\|_{\infty}}{n}\right)\right].$$

We have proved

Theorem 2.85. *Let $f \in C_{\mathcal{F}}^{N}\left(\prod_{i=1}^{d}[a_i,b_i]\right)$, $N \in \mathbb{N}$, $x \in \prod_{i=1}^{d}[a_i,b_i]$. Then*

$$D\left(M_n^{\mathcal{F}}(f)(x), f(x)\right)$$

$$\leq \frac{2^d E^*}{E\left(\frac{T_1}{2},\ldots,\frac{T_d}{2}\right)}\left[\sum_{j=1}^{N}\frac{1}{j!}\left(\frac{\|b-a\|_{\infty}^{j}}{n^j}\right)\left[\left(\sum_{i=1}^{d} D\left(\frac{\partial}{\partial x_i},\widetilde{o}\right)\right)^j f(x)\right]\right. \tag{307}$$

$$\left.+\frac{\|b-a\|_{\infty}^{N} d^{N}}{N! n^{N}} \max_{\widetilde{\alpha}:|\widetilde{\alpha}|=N} \omega_1^{(\mathcal{F})}\left(f_{\widetilde{\alpha}}, \frac{\|b-a\|_{\infty}}{n}\right)\right].$$

Corollary 2.86. (to Theorem 2.85) Additionally assume that $f_{\widetilde{\alpha}}(x) = \widetilde{o}$, for all $\widetilde{\alpha} : |\widetilde{\alpha}| = 1,\ldots,N$, where $x \in \prod_{i=1}^{d}[a_i,b_i]$ is fixed.

[Then $D\left(f_{\widetilde{\alpha}}(x), \widetilde{o}\right) = 0$, and $\left[\left(\sum_{i=1}^{d} D\left(\frac{\partial}{\partial x_i},\widetilde{o}\right)\right)^j f(x)\right] = 0$, $j = 1,\ldots,N$].

Hence

$$D\left(M_n^{\mathcal{F}}(f)(x), f(x)\right)$$

$$\leq \frac{2^d E^*}{E\left(\frac{T_1}{2},\ldots,\frac{T_d}{2}\right)}\frac{\|b-a\|_{\infty}^{N} d^{N}}{N! n^{N}} \max_{\widetilde{\alpha}:|\widetilde{\alpha}|=N} \omega_1^{(\mathcal{F})}\left(f_{\widetilde{\alpha}}, \frac{\|b-a\|_{\infty}}{n}\right). \tag{308}$$

Corollary 2.87 (to Theorem 2.85)**.** We get

$$D^*\left(M_n^{\mathcal{F}}(f), f\right)$$

$$\leq \frac{2^d E^*}{E\left(\frac{T_1}{2},\ldots,\frac{T_d}{2}\right)}\left[\sum_{j=1}^{N}\frac{1}{j!}\left(\frac{\|b-a\|_{\infty}^{j}}{n^j}\right)\left\|\left(\sum_{i=1}^{d} D\left(\frac{\partial}{\partial x_i},\widetilde{o}\right)\right)^j f(x)\right\|_{\infty}\right. \tag{309}$$

$$+\frac{\|b-a\|_{\infty}^{N} d^{N}}{N! n^{N}} \max_{\widetilde{\alpha}:|\widetilde{\alpha}|=N} \omega_1^{(\mathcal{F})}\left(f_{\widetilde{\alpha}}, \frac{\|b-a\|_{\infty}}{n}\right)\right].$$

Corollary 2.88 (to Theorem 2.85)**.** Case of $N=1$. We derive

$$D\left(\left(M_n^{\mathcal{F}}(f)\right)(x), f(x)\right) \leq \frac{2^d E^* \|b-a\|_{\infty}}{nE\left(\frac{T_1}{2}, \ldots, \frac{T_d}{2}\right)}$$

$$\times\left[\sum_{i=1}^{d} D\left(\frac{\partial f}{\partial x_i}, \widetilde{o}\right) + d \max_{i\in\{1,\ldots,d\}} \omega_1^{(\mathcal{F})}\left(\frac{\partial f}{\partial x_i}, \frac{\|b-a\|_{\infty}}{n}\right)\right]. \tag{310}$$

2.9. **Fuzzy-Random Analysis background.** Define

$$D: \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \rightarrow \mathbb{R}_{+} \cup \{0\}$$

by

$$D(u,v) := \sup_{r\in[0,1]} \max\left\{\left|u_-^{(r)} - v_-^{(r)}\right|, \left|u_+^{(r)} - v_+^{(r)}\right|\right\}, \tag{311}$$

where $[v]^r = \left[v_-^{(r)}, v_+^{(r)}\right]$; $u, v \in \mathbb{R}_{\mathcal{F}}$. We have that D is a metric on $\mathbb{R}_{\mathcal{F}}$. Then $(\mathbb{R}_{\mathcal{F}}, D)$ is a complete metric space, see [40], with the properties

$$\begin{aligned} D(u\oplus w, v\oplus w) &= D(u,v), \quad \forall\, u,v,w \in \mathbb{R}_{\mathcal{F}}, \\ D(k\odot u, k\odot v) &= |k|\, D(u,v), \quad \forall\, u,v \in \mathbb{R}_{\mathcal{F}},\ \forall\, k\in\mathbb{R}, \\ D(u\oplus v, w\oplus e) &\leq D(u,w) + D(v,e), \quad \forall\, u,v,w,e \in \mathbb{R}_{\mathcal{F}}. \end{aligned}$$

Let $U^* := \prod_{i=1}^{d} [a_i, b_i]$, $d \in \mathbb{N}$, $f, g : U^* \rightarrow \mathbb{R}_{\mathcal{F}}$ be fuzzy real number valued functions. The distance between f, g is defined by

$$D^*(f,g) := \sup_{x\in U^*} D(f(x), g(x)).$$

On $\mathbb{R}_{\mathcal{F}}$ we define a partial order by "$\leq$": $u, v \in \mathbb{R}_{\mathcal{F}}$, $u \leq v$ iff $u_-^{(r)} \leq v_-^{(r)}$ and $u_+^{(r)} \leq v_+^{(r)}$, $\forall\, r \in [0,1]$.

We need

Lemma 2.89 ([24])**.** *For any $a, b \in \mathbb{R} : a\cdot b \geq 0$ and any $u \in \mathbb{R}_{\mathcal{F}}$ we have*

$$D(a\odot u, b\odot u) \leq |a-b| \cdot D(u, \widetilde{o}), \tag{312}$$

where $\widetilde{o} \in \mathbb{R}_{\mathcal{F}}$ is defined by $\widetilde{o} := \chi_{\{0\}}$.

Lemma 2.90 ([24])**.** (i) *If we denote $\widetilde{o} := \chi_{\{0\}}$, then $\widetilde{o} \in \mathbb{R}_{\mathcal{F}}$ is the neutral element with respect to $\oplus$, i.e., $u \oplus \widetilde{o} = \widetilde{o} \oplus u = u$, $\forall\, u \in \mathbb{R}_{\mathcal{F}}$.*

(ii) *With respect to $\widetilde{o}$, none of $u \in \mathbb{R}_{\mathcal{F}}$, $u \neq \widetilde{o}$ has opposite in $\mathbb{R}_{\mathcal{F}}$.*

(iii) *Let $a, b \in \mathbb{R} : a\cdot b \geq 0$, and any $u \in \mathbb{R}_{\mathcal{F}}$, we have $(a+b)\odot u = a\odot u \oplus b\odot u$. For general $a, b \in \mathbb{R}$, the above property is false.*

(iv) *For any $\lambda \in \mathbb{R}$ and any $u, v \in \mathbb{R}_{\mathcal{F}}$, we have $\lambda\odot(u\oplus v) = \lambda\odot u \oplus \lambda\odot v$.*

(v) *For any* $\lambda, \mu \in \mathbb{R}$ *and* $u \in \mathbb{R}_{\mathcal{F}}$, *we have* $\lambda \odot (\mu \odot u) = (\lambda \cdot \mu) \odot u$.

(vi) *If we denote* $\|u\|_{\mathcal{F}} := D(u, \widetilde{o})$, $\forall\, u \in \mathbb{R}_{\mathcal{F}}$, *then* $\|\cdot\|_{\mathcal{F}}$ *has the properties of a usual norm on* $\mathbb{R}_{\mathcal{F}}$, *i.e.*,

$$\|u\|_{\mathcal{F}} = 0 \text{ iff } u = \widetilde{o}, \quad \|\lambda \odot u\|_{\mathcal{F}} = |\lambda| \cdot \|u\|_{\mathcal{F}},$$

$$\|u \oplus v\|_{\mathcal{F}} \leq \|u\|_{\mathcal{F}} + \|v\|_{\mathcal{F}}, \quad \|u\|_{\mathcal{F}} - \|v\|_{\mathcal{F}} \leq D(u, v). \tag{313}$$

Notice that $(\mathbb{R}_{\mathcal{F}}, \oplus, \odot)$ is not a linear space over $\mathbb{R}$; and consequently $(\mathbb{R}_{\mathcal{F}}, \|\cdot\|_{\mathcal{F}})$ is not a normed space.

As in Remark 4.4 [24] one can show easily that a sequence of operators of the form

$$L_n(f)(x) := \sum_{k=0}^{n*} f(x_{k_n}) \odot w_{n,k}(x), \quad n \in \mathbb{N}, \tag{314}$$

($\overset{*}{\sum}$ denotes the fuzzy summation) where $f : U^* \to \mathbb{R}_{\mathcal{F}}$, $x_{k_n} \in U^*$, $w_{n,k}(x)$ real valued weights, are linear over U^*, i.e.,

$$L_n(\lambda \odot f \oplus \mu \odot g)(x) = \lambda \odot L_n(f)(x) \oplus \mu \odot L_n(g)(x), \tag{315}$$

$\forall \lambda, \mu \in \mathbb{R}$, any $x \in U^*$; $f, g : U^* \to \mathbb{R}_{\mathcal{F}}$. (Proof based on Lemma 2.90 (iv).)

We further need

Definition 2.91 (see also [32, Definition 13.16, p. 654])**.** Let $(X, \mathcal{B}, P)$ be a probability space. A fuzzy-random variable is a $\mathcal{B}$-measurable mapping $g : X \to \mathbb{R}_{\mathcal{F}}$ (i.e., for any open set $Z \subseteq \mathbb{R}_{\mathcal{F}}$, in the topology of $\mathbb{R}_{\mathcal{F}}$ generated by the metric D, we have

$$g^{-1}(Z) = \{s \in X; g(s) \in Z\} \in \mathcal{B}). \tag{316}$$

The set of all fuzzy-random variables is denoted by $\mathcal{L}_{\mathcal{F}}(X, \mathcal{B}, P)$. Let $g_n, g \in \mathcal{L}_{\mathcal{F}}(X, \mathcal{B}, P)$, $n \in \mathbb{N}$ and $0 < q < +\infty$. We say $g_n(s) \underset{n \to +\infty}{\overset{\text{"}q\text{-mean"}}{\longrightarrow}} g(s)$ if

$$\lim_{n \to +\infty} \int_X D(g_n(s), g(s))^q P(ds) = 0. \tag{317}$$

Remark 2.92 (see [32, p. 654])**.** If $f, g \in \mathcal{L}_{\mathcal{F}}(X, \mathcal{B}, P)$, let us denote $F : X \to \mathbb{R}_+ \cup \{0\}$ by $F(s) = D(f(s), g(s))$, $s \in X$. Here, F is $\mathcal{B}$-measurable, because $F = G \circ H$, where $G(u, v) = D(u, v)$ is continuous on $\mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}}$, and $H : X \to \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}}$, $H(s) = (f(s), g(s))$, $s \in X$, is $\mathcal{B}$-measurable. This shows that the above convergence in q-mean makes sense.

Definition 2.93 (see [32, p. 654, Definition 13.17])**.** Let $(T, \mathcal{T})$ be a topological space. A mapping $f : T \to \mathcal{L}_{\mathcal{F}}(X, \mathcal{B}, P)$ will be called fuzzy-random function (or fuzzy-stochastic process) on T. We denote $f(t)(s) = f(t, s)$, $t \in T$, $s \in X$.

Remark 2.94 (see [32, p. 655]). Any usual fuzzy real function $f : T \to \mathbb{R}_{\mathcal{F}}$ can be identified with the degenerate fuzzy-random function $f(t,s) = f(t)$, $\forall\, t \in T$, $s \in X$.

Remark 2.95 (see [32, p. 655]). Fuzzy-random functions that coincide with probability one for each $t \in T$ will be considered equivalent.

Remark 2.96 (see [32, p. 655]). Let $f, g : T \to \mathcal{L}_{\mathcal{F}}(X, \mathcal{B}, P)$. Then $f \oplus g$ and $k \odot f$ are defined pointwise, i.e.,

$$(f \oplus g)(t,s) = f(t,s) \oplus g(t,s),$$
$$(k \odot f)(t,s) = k \odot f(t,s), \quad t \in T, s \in X.$$

Definition 2.97 (see also [32, Definition 13.18, pp. 655-656]). For a fuzzy-random function $f : U^* \to \mathcal{L}_{\mathcal{F}}(X, \mathcal{B}, P)$, $d \in \mathbb{N}$, we define the (first) fuzzy-random modulus of continuity

$$\Omega_1^{(\mathcal{F})}(f,\delta)_{L^q} = \sup\left\{ \left(\int_X D^q\left(f(x,s), f(y,s)\right) P(ds) \right)^{\frac{1}{q}} : x, y \in U^*,\ \ \|x-y\|_{l_1} \le \delta \right\}, \tag{318}$$

$0 < \delta$, $1 \le q < \infty$.

Definition 2.98 (as in [22]). Here $1 \le q < +\infty$. Let $f : U^* \to \mathcal{L}_{\mathcal{F}}(X, \mathcal{B}, P)$, $d \in \mathbb{N}$, be a fuzzy random function. We call f a (q-mean) uniformly continuous fuzzy random function over U^*, iff $\forall\, \varepsilon > 0\ \exists\, \delta > 0$:whenever $\|x-y\|_{l_1} \le \delta$, $x, y \in U^*$, implies that

$$\int_X \left(D\left(f(x,s), f(y,s)\right)\right)^q P(ds) \le \varepsilon. \tag{319}$$

We denote it as $f \in C_{FR}^{U_q}(U^*)$.

Proposition 2.99 (as in [22]). Let $f \in C_{FR}^{U_q}(U^*)$. Then $\Omega_1^{(\mathcal{F})}(f,\delta)_{L^q} < \infty$, any $\delta > 0$.

Proposition 2.100 (as in [22]). Let $f, g : U^* \to \mathcal{L}_{\mathcal{F}}(X, \mathcal{B}, P)$, $d \in \mathbb{N}$, be fuzzy random functions. It holds

(i) $\Omega_1^{(\mathcal{F})}(f,\delta)_{L^q}$ is nonnegative and nondecreasing in $\delta > 0$.

(ii) $\lim_{\delta \downarrow 0} \Omega_1^{(\mathcal{F})}(f,\delta)_{L^q} = \Omega_1^{(\mathcal{F})}(f,0)_{L^q} = 0$, iff $f \in C_{FR}^{U_q}(U^*)$.

(iii) $\Omega_1^{(\mathcal{F})}(f,\delta_1+\delta_2)_{L^q} \le \Omega_1^{(\mathcal{F})}(f,\delta_1)_{L^q} + \Omega_1^{(\mathcal{F})}(f,\delta_2)_{L^q}$, $\delta_1, \delta_2 > 0$.

(iv) $\Omega_1^{(\mathcal{F})}(f,n\delta)_{L^q} \le n\Omega_1^{(\mathcal{F})}(f,\delta)_{L^q}$, $\delta > 0$, $n \in \mathbb{N}$.

(v) $\Omega_1^{(\mathcal{F})}(f,\lambda\delta)_{L^q} \le \lceil \lambda \rceil \Omega_1^{(\mathcal{F})}(f,\delta)_{L^q} \le (\lambda+1)\Omega_1^{(\mathcal{F})}(f,\delta)_{L^q}$, $\lambda > 0$, $\delta > 0$, where $\lceil \cdot \rceil$ is the ceilling of the number.

(vi) $\Omega_1^{(\mathcal{F})}(f \oplus g,\delta)_{L^q} \le \Omega_1^{(\mathcal{F})}(f,\delta)_{L^q} + \Omega_1^{(\mathcal{F})}(g,\delta)_{L^q}$, $\delta > 0$. Here $f \oplus g$ is a fuzzy random function.

(vii) $\Omega_1^{(\mathcal{F})}(f,\cdot)_{L^q}$ is continuous on $\mathbb{R}_+$, for $f \in C_{FR}^{U_q}(U^*)$.

According to [29, p. 94] we have the following

Definition 2.101. Let $(Y, \mathcal{T})$ be a topological space, with its σ-algebra of Borel sets $\mathcal{B} := \mathcal{B}(Y, \mathcal{T})$ generated by $\mathcal{T}$. If $(X, \mathcal{S})$ is a measurable space, a function $f : X \to Y$ is called measurable iff $f^{-1}(B) \in \mathcal{S}$ for all $B \in \mathcal{B}$.

By Theorem 4.1.6 of [29, p. 89] f as above is measurable iff

$$f^{-1}(C) \in \mathcal{S} \text{ for all } C \in \mathcal{T}.$$

We would need

Theorem 2.102 (see [29, p. 95])**.** *Let $(X, \mathcal{S})$ be a measurable space and (Y, d) be a metric space. Let f_n be measurable functions from X into Y such that for all $x \in X$, $f_n(x) \to f(x)$ in Y. Then f is measurable. I.e., $\lim_{n\to\infty} f_n = f$ is measurable.*

We need also

Proposition 2.103. Let f, g be fuzzy random variables from $\mathcal{S}$ into $\mathbb{R}_{\mathcal{F}}$. Then

(i) Let $c \in \mathbb{R}$, then $c \odot f$ is a fuzzy random variable.

(ii) $f \oplus g$ is a fuzzy random variable.

2.10. **Multivariate Fuzzy Random Neural Network Approximation and Interpolation.** We need

Definition 2.104. Let here $(X, \mathcal{B}, P)$ be a probability space, $s \in X$, $n \in \mathbb{N}$, $f \in C^{U_q}_{\mathcal{F}R}\left(\prod_{i=1}^{d}[a_i, b_i]\right)$, $1 \le q < \infty$, and $x \in \prod_{i=1}^{d}[a_i, b_i]$.

We define

$$M_n^{\mathcal{F}R}(f, x, s) := M_n^{\mathcal{F}R}(f, x_1, \dots, x_d, s)$$

$$:= \frac{\sum_{k_1=0}^{n*} \cdots \sum_{k_d=0}^{n*} f(x_{k_1 1}, \dots, x_{k_d d}, s) \odot E\left(\frac{T_1 n\left(x_1 - x_{k_1 1}\right)}{b_1 - a_1}, \dots, \frac{T_d n\left(x_d - x_{k_d d}\right)}{b_d - a_d}\right)}{\sum_{k_1=0}^{n} \cdots \sum_{k_d=0}^{n} E\left(\frac{T_1 n\left(x_1 - x_{k_1 1}\right)}{b_1 - a_1}, \dots, \frac{T_d n\left(x_d - x_{k_d d}\right)}{b_d - a_d}\right)}. \tag{320}$$

We make

Remark 2.105. Clearly here it holds

$$\begin{aligned} M_n^{\mathcal{F}R}(f, x_{k_1 1}, \dots, x_{k_d d}, s) &= \frac{f(x_{k_1 1}, \dots, x_{k_d d}, s) \odot E^*}{E^*} \\ &= f(x_{k_1 1}, \dots, x_{k_d d}, s) \odot 1 \\ &= f(x_{k_1 1}, \dots, x_{k_d d}, s), \end{aligned} \tag{321}$$

proving the interpolation property of operators $M_n^{\mathcal{F}R}$.

We make

Remark 2.106. Let $f \in C_{\mathcal{F}R}^{U_q}\left(\prod_{i=1}^{d}[a_i,b_i]\right)$, $1 \leq q < \infty$, $x \in \prod_{i=1}^{d}[a_i,b_i]$, $n \in \mathbb{N}$. We observe that

$$D\left(M_n^{\mathcal{F}R}(f,x,s), f(x,s)\right)$$

(322)

$$= D\left(\sum_{k_1=0}^{n*}\cdots\sum_{k_d=0}^{n*} f\left(x_{k_1 1},\ldots,x_{k_d d},s\right) \odot \frac{E\left(\frac{T_1 n\left(x_1-x_{k_1 1}\right)}{b_1-a_1},\ldots,\frac{T_d n\left(x_d-x_{k_d d}\right)}{b_d-a_d}\right)}{W},\right.$$
$$\left. f(x,s) \odot \frac{W}{W}\right)$$

(323)

$$= D\left(\sum_{k_1=0}^{n*}\cdots\sum_{k_d=0}^{n*} f\left(x_{k_1 1},\ldots,x_{k_d d},s\right) \odot \frac{E(>>)}{W}, \sum_{k_1=0}^{n*}\cdots\sum_{k_d=0}^{n*} f(x,s) \odot \frac{E(>>)}{W}\right)$$
$$\leq \frac{\sum_{k_1=0}^{n}\cdots\sum_{k_d=0}^{n} E(>>)}{W} D\left(f\left(x_{k_1 1},\ldots,x_{k_d d},s\right), f(x,s)\right).$$

So it holds

$$D\left(M_n^{\mathcal{F}R}(f,x,s), f(x,s)\right)$$

(324)

$$\leq \frac{\sum_{k_1=0}^{n}\cdots\sum_{k_d=0}^{n} E\left(\frac{T_1 n\left(x_1-x_{k_1 1}\right)}{b_1-a_1},\ldots,\frac{T_d n\left(x_d-x_{k_d d}\right)}{b_d-a_d}\right)}{W} D\left(f\left(x_{k_1 1},\ldots,x_{k_d d},s\right), f(x,s)\right).$$

Therefore we derive

$$\left(\int_X D^q\left(\left(M_n^{\mathcal{F}R}(f,x,s), f(x,s)\right)\right) P(ds)\right)^{\frac{1}{q}}$$
$$\leq \frac{\sum_{k_1=0}^{n}\cdots\sum_{k_d=0}^{n} E\left(\frac{T_1 n\left(x_1-x_{k_1 1}\right)}{b_1-a_1},\ldots,\frac{T_d n\left(x_d-x_{k_d d}\right)}{b_d-a_d}\right)}{W}$$
$$\times\left(\int_X D^q\left(f\left(x_{k_1 1},\ldots,x_{k_d d},s\right), f(x,s)\right) P(ds)\right)^{\frac{1}{q}} \tag{325}$$
$$\leq \frac{2^d E^*}{E\left(\frac{T_1}{2},\ldots,\frac{T_d}{2}\right)} \Omega_1^{(\mathcal{F})}\left(f, \frac{\sum_{i=1}^{d}(b_i-a_i)}{n}\right)_{L^q}.$$

We have proved the following approximation result.

Theorem 2.107. *Let* $(X, \mathcal{B}, P)$ *probability space,* $f \in C_{\mathcal{F}R}^{U_q}\left(\prod_{i=1}^{d}[a_i, b_i]\right)$, $1 \leq q < \infty$. *Then*

$$\left\| \left(\int_X D^q \left(\left(M_n^{\mathcal{F}R}(f, x, s), f(x, s) \right) \right) P(ds) \right)^{\frac{1}{q}} \right\|_{\infty, x} \tag{326}$$

$$\leq \frac{2^d E^*}{E\left(\frac{T_1}{2}, \ldots, \frac{T_d}{2}\right)} \Omega_1^{(\mathcal{F})} \left(f, \frac{\sum_{i=1}^{d}(b_i - a_i)}{n} \right)_{L^q},$$

where $x \in \prod_{i=1}^{d}[a_i, b_i]$, $\forall\ n \in \mathbb{N}$.

REFERENCES

[1] G. A. Anastassiou, Rate of convergence of some neural network operators to the unit-univariate case, *Journal of Mathematical Analysis and Application*, Vol. 212 (1997), 237–262.

[2] G. A. Anastassiou, Rate of Convergence of some Multivariate Neural Network Operators to the Unit, *Computers and Mathematics*, 40 (2000), 1–19.

[3] G. A. Anastassiou, *Quantitative Approximation*, Chapmann and Hall/CRC, Boca Raton, New York, 2001.

[4] G. A. Anastassiou,Fuzzy Approximation by Fuzzy Convolution type Operators, *Computers and Mathematics*, 48(2004), 1369–1386.

[5] G. A. Anastassiou, Higher order Fuzzy Approximation by Fuzzy Wavelet type and Neural Network Operators, *Computers and Mathematics*, 48 (2004), 1387–1401.

[6] G. A. Anastassiou, Higher order Fuzzy Korovkin Theory via inequalities, *Communications in Applied Analysis*, 10(2006), No. 2, 359–392.

[7] G. A. Anastassiou, Fuzzy Korovkin Theorems and Inequalities, *Journal of Fuzzy Mathematics*, 15(2007), No. 1, 169–205.

[8] G. A. Anastassiou, Fractional Korovkin theory, *Chaos, Solitons & Fractals*, Vol. 42, No. 4 (2009), 2080–2094.

[9] G. A. Anastassiou, On Right Fractional Calculus, *Chaos, solitons and fractals*, 42 (2009), 365–376.

[10] G. A. Anastassiou, *Fractional Differentiation Inequalities*, Springer, New York, 2009.

[11] G. A. Anastassiou, *Fuzzy Mathematics: Approximation Theory*, Springer, Heildelberg, New York, 2010.

[12] G. A. Anastassiou, *Inteligent Systems: Approximation by Artificial Neural Networks*, Springer, Heidelberg, 2011.

[13] G. A. Anastassiou, Fuzzy fractional Calculus and Ostrowski inequality, *J. Fuzzy Math.*, 19 (2011), no. 3, 577–590.

[14] G. A. Anastassiou, Multivariate hyperbolic tangent neural network approximation, *Computers and Mathematics*, 61(2011), 809–821.

[15] G. A. Anastassiou, *Advanced Inequalities*, World Scientific Publishing Corporation, Singapore, New Jersey, 2011.

[16] G. A. Anastassiou, Univariate hyperbolic tangent neural network approximation, *Mathematics and Computer Modelling*, 53(2011), 1111–1132.

[17] G. A. Anastassiou, Multivariate sigmoidal neural network approximation, *Neural Networks*, 24(2011), 378–386.

[18] G. A. Anastassiou, Higher order multivariate fuzzy approximation by multivariate fuzzy wavelet type and neural network operators, *J. of Fuzzy Mathematics*, 19(2011), no. 3, 601–618.

[19] G. A. Anastassiou, Univariate sigmoidal neural network approximation, *J. of Computational Analysis and Applications*, Vol. 14(4), (2012), 659–690.

[20] G. A. Anastassiou, High degree multivariate fuzzy approximation by quasi-interpolation neural network operators, *Discontinuity, Nonlinearity and Complexity*, 2 (2), 2013, 125–146.

[21] G. A. Anastassiou, Approximation by Neural Network Iterates, in *Advances in Applied Mathematics and Approximation Theory: Contributions from AMAT 2012*, pp. 1–20, Editors: G. Anastassiou and O. Duman, Springer NY, 2013.

[22] G. A. Anastassiou, Multivariate Fuzzy-Random normalized Neural Network Approximation Operators, *Annals of Fuzzy Mathematics and Informatics*, Vol. 6, No. 1, 2013, 191–212.

[23] G. A. Anastassiou, Multivariate fuzzy-random quasi-interpolation neural network approximation operators, *J. Fuzzy Math.*, 22 (2014), no. 1, 167–184.

[24] G. A. Anastassiou, S. Gal, On a fuzzy trigonometric approximation theorem of Weierstrass-type, *the Journal of Fuzzy Mathematics*, 9, No. 3 (2001), 701–708.

[25] P. L. Butzer, R. J. Nessel, *Fourier Analysis and Approximation*, *Pure and Applied Mathematics* 40, Academic Press, New York-London, 1971.

[26] F. L. Cao, Y. Q. Zhang, Interpolation and approximation by neural networks in metric spaces. (Chinese), *Acta Math. Sinica (Chin. Ser.)*, 51 (2008), no. 1, 91–98.

[27] D. Costarelli, Interpolation by neural network operators activated by ramp functions, *J. Math. Anal. Appl.*, 419 (2014), no. 1, 574–582.

[28] K. Diethelm, *The Analysis of Fractional Differential Equations*, Lecture Notes in Mathematics 2004, Springer-Verlag, Berlin, Heidelberg, 2010.

[29] R. M. Dudley, *Real Analysis and Probability*, Wadsworth & Brooks / Cole Mathematics Series, Pacific Grove, California, 1989.

[30] A. M. A. El-Sayed and M. Gaber, On the finite Caputo and finite Riesz derivatives, *Electronic Journal of Theoretical Physics*, Vol. 3, No. 12 (2006), 81–95.

[31] G. S. Frederico and D. F. M. Torres, Fractional Optimal Control in the sense of Caputo and the fractional Noether's theorem, *International Mathematical Forum*, Vol. 3, No. 10 (2008), 479–493.

[32] S. Gal, *Approximation Theory in Fuzzy Setting*, Chapter 13 in *Handbook of Analytic-Computational Methods in Applied Mathematics*, 617-666, edited by G. Anastassiou, Chapman & Hall/CRC, Boca Raton, New York, 2000.

[33] R. Goetschel Jr., W. Voxman, Elementary fuzzy calculus, *Fuzzy Sets and Systems*, 18(1986), 31–43.

[34] O. Kaleva, Fuzzy differential equations, *Fuzzy Sets and Systems*, 24(1987), 301–317.

[35] Y. K. Kim, B. M. Ghil, Integrals of fuzzy-number-valued functions, *Fuzzy Sets and Systems*, 86(1997), 213–222.

[36] B. Lenze, Local behaviour of neural network operators-approximation and interpolation, *Analysis*, 13 (1993), no. 4, 377–387.

[37] B. Lenze, One-sided approximation and interpolation operators generating hyperbolic sigma-pi neural networks. Multivariate approximation and splines (Mannheim, 1996), 99-112, *Internat. Ser. Numer. Math.*, 125, Birkhäuser, Basel, 1997.

[38] A. Pinkus, Approximation theory of the MLP model in neural networks, *Acta Numer.*, 8 (1999), 143–195.

[39] S. G. Samko, A. A. Kilbas and O. I. Marichev, *Fractional Integrals and Derivatives, Theory and Applications*, (Gordon and Breach, Amsterdam, 1993) [English translation from the Russian, Integrals and Derivatives of Fractional Order and Some of Their Applications (Nauka i Tekhnika, Minsk, 1987)].

[40] Wu Congxin, Gong Zengtai, On Henstock integrals of interval-valued functions and fuzzy valued functions, *Fuzzy Sets and Systems*, Vol. 115, No. 3, 2000, 377–391.

[41] C. Wu, Z. Gong, On Henstock integral of fuzzy-number-valued functions (I), *Fuzzy Sets and Systems*, 120, No. 3, (2001), 523–532.

[42] C. Wu, M. Ma, On embedding problem of fuzzy numer spaces: Part 1, *Fuzzy Sets and Systems*, 44 (1991), 33–38.