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PIECEWISE LINEAR LEAST SQUARES APPROXIMATIONS OF INVARIANT MEASURES FOR RANDOM MAPS

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ABSTRACT. Let $\{\tau_1, \tau_2, \ldots, \tau_K\}$ be a collection of nonsingular maps on $[0,1]$ into $[0,1]$ and $\{p_1, p_2, \ldots, p_K\}$ be a set of probabilities on $[0,1]$. We consider a class of random maps $T = \{\tau_1, \tau_2, \ldots, \tau_K; p_1, p_2, \ldots, p_K\}$ and we prove the existence of an absolutely continuous invariant measure μ of T with density f^*. Based on least squares approximations, a piecewise linear approximation method for f^* is developed and a proof of weak convergence of the piecewise linear method is presented. Moreover, we present numerical examples.

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1. Introduction

A random dynamical system is a measure-theoretic formulation of a dynamical system with an element of randomness. A random dynamical system of special interest is a random map consisting of a number of dynamical systems from a set into itself where the process switches from one dynamical system to another according to fixed probabilities [13, 9] or, more generally, position dependent probabilities [7, 8]. The existence and properties of invariant measures for random maps reflect their long time behavior and play an important role in understanding their chaotic nature. Random maps have application in the study of fractals [1], in modeling interference effects in quantum mechanics [3], in computing metric entropy [16], and in forecasting the financial markets [15, 14]. Invariant measures [2, 7, 11] of dynamical systems and random maps reflect their long time behavior and play an important role in understanding their chaotic nature.

The Frobenius–Perron operator is one of the main tools for establishing the existence of invariant measures for dynamical systems [17, 12]. Fixed points of the Frobenius–Perron operator [13] of random maps are the invariant densities of absolutely continuous invariant measure (acim) for the system. The Frobenius–Perron

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equation of a random map is a functional equation and it is difficult to solve this equation except for some simple cases [7, 10]. Thus it is important to approximate the F–P operator by a finite dimensional matrix operator [7, 6, 4]. In this paper we consider a class of random maps T and prove that T preserves an absolutely continuous invariant measure μ with density f^*. Based on least squares approximations, a piecewise linear approximation method for f^* is developed and a proof of weak convergence of the piecewise linear method is presented. Our piecewise linear approximation method for random maps is a generalization of the piecewise linear approximation method of single maps described in [5]. We present numerical examples.

In Section 2 we present the notation and summarize results we shall need in the sequel. In Section 3, we present the proof of existence of absolutely continuous invariant measures for a class of random maps. Based on least squares approximations, we describe a piecewise linear approximation for random maps with constant probabilities in Section 4. We present a proof of the weak convergence of our method. We present numerical examples in Section 5.

2. Random maps, Frobenius-Perron operator and invariant measure.

Let $(I = [0,1], \mathcal{B}, \lambda)$ be a measure space, λ be the usual Lebesgue measure and $\tau_k : I \to I$, $k = 1, 2, \ldots, K$ be piecewise one-to-one and differentiable, nonsingular transformations on a common partition $\mathcal{J} = \{J_1, J_2, \ldots, J_N\}$ of I. For $1 \leq p < \infty$, let $L^p(0,1)$ be the Banach space of all real functions f defined on $[0,1]$ such that the functions $|f|^p$ is Lebesgue integrable with the L^p-norm $\| f \|_p = \left(\int_0^1 |f|^p d\lambda\right)^{\frac{1}{p}}$ for $f \in L^p(0,1)$. For $L^2(0,1) \subset L^1(0,1)$ and $L^1(0,1)$ we consider the inner product $\langle f, g\rangle = \int_0^1 f(x)g(x)$. A random map T with constant probabilities is defined as

$$T = \{\tau_1, \tau_2, \ldots, \tau_K; p_1, p_2, \ldots, p_K\},$$

where $\{p_1, p_2, \ldots, p_K\}$ is a set of constant probabilities on I. For any $x \in X$, $T(x) = \tau_k(x)$ with probability p_k and, for any non-negative integer N, $T^N(x) = \tau_{k_N} \circ \tau_{k_{N-1}} \circ \ldots \circ \tau_{k_1}(x)$ with probability $\Pi_{j=1}^N p_{k_j}$. A measure μ is $T-$invariant if and only if it satisfies the following condition [13]:

$$\mu(E) = \sum_{k=1}^{K} p_k \mu(\tau_k^{-1}(E)), \tag{2.1}$$

for any $E \in \mathcal{B}$. The Frobenius-Perron operator of T is given by [13]:

$$P_T f(x) = \sum_{k=1}^{K} p_k (P_{\tau_k} f)(x), \tag{2.2}$$

where P_{τ_k} is the Frobenius-Perron operator associated with the transformation τ_k. The properties of P_T resemble the properties of the traditional Frobenius-Perron operator P_{τ_k} [2].

3. Existence of absolutely continuous invariant measures for a class of random maps

Consider a random map $T = \{\tau_1, \tau_2, \ldots, \tau_K; p_1, p_2, \ldots, p_K\}$ on $I = [0,1]$ with constant probabilities $p_1, p_2, \ldots, p_K$ and assume that the Frobenius-Perron operator operator $P_T : L^1(0,1) \to L^1(0,1)$ satisfies the following condition

Condition (A): $P_T f \in L^2(0,1)$, for all $f \in L^2(0,1)$ and there exists constants $\alpha < 1$ and β such that

$$\| P_T f \|_2 \leq \alpha \| f \|_2 + \beta \| f \|_1 . \tag{3.1}$$

Lemma 3.1 ([4]). *Let $(X, \mathcal{B}, m)$ be a measure space. Then any bounded subset B of $L^2(X)$ is weakly precompact in $L^2(X)$. If $m(X) < \infty$ then B is weakly precompact in $L^1(X)$.*

Theorem 3.2. *Let $T = \{\tau_1, \tau_2, \ldots, \tau_K; p_1, p_2, \ldots, p_K\}$ be a random map satisfying the above condition (Condition A). Then there is a fixed point $f^* \in L^2(0,1)$ of the Frobenius-Perron operator P_T.*

Proof. The proof follows from the standard techniques in [4] (see Theorem 2.1). □

4. Piecewise linear least squares approximations of invariant measures for random maps

In this section we consider least squares approximations techniques [5] of integrable functions and describe a method for the approximation of the fixed point of the Frobenius–Perron operator for random maps with constant probabilities. We closely follow [5] and generalize the piecewise linear least squares approximation scheme in [5] for a single map to a piecewise least squares scheme for random maps with constant probabilities. Let $T = \{\tau_1, \tau_2, \ldots, \tau_K; p_1(x), p_2(x), \ldots, p_K(x)\}$ be a random map satisfying conditions of Theorem 3.2. Hence by Theorem 3.2, the random map T has an absolutely continuous invariant measure μ with respect to Lebesgue measure λ. The invariant density $f^* \in L^2(0,1)$ of μ is the fixed point of the Frobenius-Perron operator P_T.

Let $\mathcal{P}^{(n)} = \{I_1, I_2, \ldots, I_n\}$ be a partition of $I = [0,1]$ where $I_i = [x_i, x_{i+1}], i = 0, 1, 2, \ldots n$ of equal length $h = \frac{1}{n}$ and $x_i = ih$. Let Δ_n be the vector space of continuous piecewise linear functions with respect to partition $\mathcal{P}^{(n)} = \{I_1, I_2, \ldots, I_n\}$. It is easy to show that Δ_n is a $n+1$ dimensional subspace of $L^2(0,1) \subset L^1(0,1)$ with the above nodes and the following canonical basis:

$$\phi_i(x) = g\left(\frac{x - x_i}{h}\right), \quad i = 0, 1, 2, \ldots, n, \tag{4.1}$$

where the function $g : \mathbb{R} \to [0,1]$ is defined by

$$g(x) = \begin{cases} 1+x, & -1 \le x \le 0, \\ 1-x, & 0 \le x \le 1, \\ 0, & x \notin [-1,1]. \end{cases} \tag{4.2}$$

The canonical basis $\{\phi_i\}_{i=0}^n$ has the following properties

- $\phi_i(x_i) = 1$ and $\phi_i(x_j) = 0$ for $j \ne i$;
- $\parallel \phi_i \parallel = \frac{1}{n}$ for $i = 1, 2, \dots n-1$, $\parallel \phi_0 \parallel = \parallel \phi_n \parallel = \frac{1}{2n}$;
- $\phi_i \ge 0$ for $i = 0, 1, 2, \dots, n$ and $\sum_{i=0}^n \phi_i(x) = 1$;
- Suppose that $f \in \Delta_n$. Then

$$f = \sum_{i=0}^{n} \alpha_i \phi_i(x) \text{ if and only if } f(x_i) = \alpha_i \text{ for } i = 0, 1, 2, \dots, n.$$

For each $n \ge 0$, let $Q_n : L^1([0,1]) \to \Delta_n$ be an operator satisfying

$$\langle Q_n f, \phi_i \rangle = \langle f, \phi_i \rangle, \quad i = 0, 1, \dots, n. \tag{4.3}$$

The coefficients of the linear combination $Q_n f = \sum_{j=0}^n c_j \phi_j$ can be uniquely determined by solving

$$\sum_{j=0}^{n} c_j \langle \phi_j, \phi_i \rangle = \langle f, \phi_i \rangle, \quad i = 0, 1, \dots, n. \tag{4.4}$$

We assume that the random map $T = \{\tau_1, \tau_2, \dots, \tau_K; p_1, p_2, \dots, p_K\}$ satisfies conditions in Theorem 3.2. Then T preserves an absolutely continuous invariant measure μ with density $f^* \in L^2(0,1)$. Note that the Frobenius-Perron operator for the random map T is given by

$$(P_T f)(x) = \sum_{k=1}^{K} p_k \left(P_{\tau_k}(f)\right)(x).$$

The density f^* is the unique solution of the Frobenius-Perron equation

$$P_T f = f, \;\; f \in L^2(0,1). \tag{4.5}$$

Let $P_n : L^1([0,1]) \to \Delta_n$ be an operator defined by

$$P_n f = Q_n \circ P_T f, \tag{4.6}$$

where Q_n is the least square operator defined in (4.3). For our piecewise linear least square approximation method we are interested in finding a finite dimensional approximation operator P_n of the Frobenius-Perron operator P_T to solve the discretized operator equation

$$P_n f = f, \;\; f \in \Delta_n \tag{4.7}$$

The equation (4.7) is equivalent to the following system of equations

$$\langle Q_n P_T f, \phi_i \rangle = \langle f, \phi_i \rangle, \quad i = 0, 1, \dots, n, \tag{4.8}$$

for $f \in \Delta_n$. Since

$$\langle Q_n P_T f, \phi_i \rangle = \langle P_T f, \phi_i \rangle, \quad i = 0, 1, \ldots, n \tag{4.9}$$

equations in (4.8) become

$$\langle P_T f, \phi_i \rangle = \langle f, \phi_i \rangle, \quad i = 0, 1, \ldots, n, \tag{4.10}$$

for $f \in \Delta_n$. Let $f(x) = \sum_{j=0}^{n} v_j \phi_j(x)$. Then (4.10) can be written as

$$\sum_{j=0}^{n} v_j \langle \phi_j, \phi_i \rangle = \sum_{j=0}^{n} v_j \langle P_T \phi_j, \phi_i \rangle = \sum_{k=1}^{K} p_k \sum_{j=0}^{n} v_j \langle P_{\tau_k} \phi_j, \phi_i \rangle, \quad i, j = 0, 1, \ldots, n, \tag{4.11}$$

For each $k = 1, 2, \ldots K$, let $A_k = (a_{k,ij})$ be the $n+1 \times n+1$ matrix defined by $a_{k,ij} = p_k \langle P_{\tau_k} \phi_j, \phi_i \rangle$, $i, j = 0, 1, \ldots, n$. Moreover, let $B = (b_{ij})$ be the $n+1 \times n+1$ matrix defined by

$$b_{ij} = \langle \phi_j, \phi_i \rangle, \quad i, j = 0, 1, \ldots, n$$

and $A = (a^*_{ij}) = \sum_{k=1}^{K} A_k$. The linear algebraic system in (4.11) can be written as

$$(A - B)v = 0, \quad v = (v_0, v_1, \ldots, v_n)' \in \mathbb{R}^{n+1}. \tag{4.12}$$

It is well known that the Koopman operator $\mathcal{K}_{\tau_k}$ [2] of τ_k is the adjoint operator of the Frobenius-Perron operator P_{τ_k}. Thus, for each $k = 1, 2, \ldots K$,

$$a_{k,ij} = p_k \langle P_{\tau_k} \phi_j, \phi_i \rangle = p_k \int_{\mathrm{supp}\ \phi_{\mathrm{j}}} e_i(\tau_k(x)) \phi_j(x) d\lambda(x), \quad i, j = 0, 1, \ldots, n.$$

Lemma 4.1. *The equation* (4.12) *has a nonzero solution* $v = (v_0, v_1, \ldots, v_n)' \in \mathbb{R}^{n+1}$.

Proof.

$$\begin{aligned}
\sum_{i=0}^{n} a^*_{ij} &= \sum_{i=0}^{n}\left(\sum_{k=1}^{K} a_{ij}\right) \\
&= \sum_{i=0}^{n}\left(\sum_{k=1}^{K} p_k\langle P_{\tau_k}\phi_j, \phi_i\rangle\right) \\
&= \sum_{k=1}^{K} p_k\langle P_{\tau_k}\phi_j, \sum_{i=0}^{n}\phi_i\rangle \\
&= \sum_{k=1}^{K} p_k\langle \phi_j, 1\rangle \\
&= \langle \phi_j, 1\rangle \sum_{k=1}^{K} p_k \\
&= \langle \phi_j, 1\rangle \\
&= \langle \phi_j, \sum_{i=0}^{n}\phi_i\rangle \\
&= \sum_{i=0}^{n}\langle \phi_j, \phi_i\rangle \\
&= \sum_{i=0}^{n} b_{ij}
\end{aligned}$$

Therefore, the vector $(1, 1, \ldots, 1)$ is a left eigenvector of $A - B$ corresponding to the eigenvalue 0. Thus, there is a right eigenvector v corresponding the same eigenvalue 0. $\square$

Lemma 4.1 ensures that for any n, there is a nonzero function $f_n \in \Delta_n$. Let v^* be the normalized vector of v such that

$$f_n = \sum_{i=0}^{n} v^*_i \phi_i(x) \tag{4.13}$$

has L^1 norm $\| f_n \|_1 = 1$.

Theorem 4.2. *Let T be a random map satisfying conditions of Theorem 3.2 and f^* be a unique fixed point of P_T. For each nonnegative integer n, let $f_n \in \Delta_n$ be a fixed point of the operator P_n and $\| f_n \|_1 = 1$. Then, $\lim_{n\to\infty} f_n = f^*$ weakly.*

Proof. For each nonnegative integer n, $Q_n : L^2(0,1) \to L^2(0,1)$ is an orthogonal projection and it is easy the check that $\| Q_n \|_2 = 1$. By the definition of P_n, it is easy to show that the operators P_n satisfies inequalities similar to (3.1) with the same constants. The Theorem follows by standard technique in [5] (see Theorem 3.1). $\square$

5. Examples

In this section, we consider a random maps with constant probabilities where the actual invariant density f^* of absolutely continuous measure of random maps are known. We present the L^1 norm $\parallel f_n - f^* \parallel$ of the difference of the actual density f^* and approximate density f_n.

Example 5.1. Consider the random map $T = \{\tau_1(x), \tau_2(x); p_1, p_2\}$ where $\tau_1, \tau_2 : [0,1] \to [0,1]$ are defined by

$$\tau_1(x) = \begin{cases} 4x, & 0 \le x < \frac{1}{4}, \\ 2 - 4x, & \frac{1}{4} \le x < \frac{1}{2}, \\ 4x - 2, & \frac{1}{2} \le x < \frac{3}{4}, \\ 4 - 4x, & \frac{3}{4} \le x \le 1. \end{cases}$$

$$\tau_2(x) = \begin{cases} 3x + \frac{1}{4}, & 0 \le x < \frac{1}{4}, \\ 3x - \frac{3}{4}, & \frac{1}{4} \le x < \frac{1}{2}, \\ -3x + \frac{9}{4}, & \frac{1}{2} \le x < \frac{3}{4}, \\ -3x + \frac{13}{4}, & \frac{3}{4} \le x \le 1. \end{cases}$$

and $p_1, p_2 : [0,1] \to [0,1]$ defined by $p_1(x) = \frac{1}{2}$ and $p_2(x) = 1 - p_1(x)$. τ_1 and τ_2 are piecewise linear and Markov transformations. It can be easily check that the random map T satisfies conditions of Theorem 3.2 and thus T has an invariant density f^*. In fact, $f^* = \left[\frac{8}{9}, \frac{7}{6}, \frac{7}{6}, \frac{7}{9}\right]$ is the normalized invariant density of T. The L^1-norm $\parallel f_n - f^* \parallel_1$ is measures to estimate the convergence of the piecewise least square approximate density f_n to the actual density f^*.

Number of intervals	$\parallel f_n - f^* \parallel_1$
8	$5.2164005690616 \times 10^{-2}$
16	$3.1612758612959 \times 10^{-2}$
32	$1.7428820839961 \times 10^{-2}$
64	$8.9988737246294 \times 10^{-3}$

Example 5.2. Consider the random map $T = \{\tau_1(x), \tau_2(x); p_1, p_2\}$ where $\tau_1, \tau_2 : [0,1] \to [0,1]$ are defined by

$$\tau_1(x) = \begin{cases} 2x, & 0 \le x \le \frac{1}{2}, \\ x, & \frac{1}{2} < x \le 1. \end{cases}$$

$$\tau_2(x) = \begin{cases} x + \frac{1}{2}, & 0 \le x \le \frac{1}{2}, \\ 2x - 1, & \frac{1}{2} < x \le 1 \end{cases}$$

and $p_1, p_2 : [0,1] \to [0,1]$ defined by $p_1(x) = \frac{1}{2}$ and $p_2(x) = 1 - p_1(x)$. τ_1 and τ_2 are piecewise linear and Markov transformations. It can be easily check that the random map T satisfies conditions of Theorem 3.2 and thus T has an invariant density f^*. In fact, $f^* = \left[\frac{1}{2}, \frac{3}{2}\right]$ is the normalized invariant density of T. The L^1-norm $\parallel f_n - f^* \parallel_1$ is measures to estimate the convergence of the piecewise least square approximate density f_n to the actual density f^*.

Number of intervals	$\parallel f_n - f^* \parallel_1$
8	$1.7427239880681 \times 10^{-1}$
16	$1.0616413091152 \times 10^{-1}$
32	$6.2294654693028 \times 10^{-2}$
64	$3.5702810209757 \times 10^{-2}$

.

Remark 5.3. In the above examples, we have considered simple random maps where the actual densities of these random maps are known. However, our method can be applied to more general random maps satisfying conditions of Theorem 3.2 where the invariant density f^* is not necessarily of function of bounded variation.

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REFERENCES

[1] Barnsley, M., *Fractals everywhere*, Academic Press, London, 1998.

[2] Boyarsky, A. and Góra, P., *Laws of Chaos*: invariant measures and dynamical systems in one dimension Birkhauser, 1997.

[3] Boyarsky, A. and Góra, P., *A dynamical model for interference effects and two slit experiment of quantum physics*, Phys, Lett. A 168, 103–112, 1992.

[4] Ding, J., *The Projection Method for a class of Frobenius -Perron operators* , Applied mathematics and Computations, 217, p. 3257–3262, 1999.

[5] Ding, J. and Rhee, N. H. , *Piecewise linear least square approximations of Frobenius-Perron operators via interpolation* , Applied mathematics Letters, 12, p. 71–74, 2010.

[6] Froyland, G., *Ulam's method for random interval maps*, Nonlinearity, 12, 1029–1052, 1999.

[7] Góra P. and Boyarsky, A., *Absolutely continuous invariant measures for random maps with position dependent probabilities*, Math. Anal. and Appl. 278, 225–242, 2003.

[8] Islam, M. S., Góra, P. and Boyarsky, A., *A generalization of Straubeś Theorem: Existence of absolutely continuous invariant measures for random maps*, J. Appl. Math. Stoch. Anal., no. 2, 133–141, 2005.

[9] Islam, M. Shafiqul and Góra, Pawel, *Smoothness of invariant densities for random maps*, Dynamics of Continuous, Discrete and Impulsive Systems, Series A: Mathematical Analysis 17, 249–262, 2010.

[10] Li, T-Y,*Finite approximation for the Frobenius-Perron operator: A solution to Ulam's conjecture*, J. Approx. Theory. 17 , 177–186, 1976.

[11] Lasota, A., Mackey, M. C., *Chaos, fractals, and noise. Stochastic aspects of dynamics*, Applied Mathematical Sciences 97, Springer-Verlag, New York, 1994.

[12] Lasota, A. and Yorke, J.A., *On the existence of invariant measures for piecewise monotonic transformations*, Trans. Amer. Math. Soc. 186 ,481–488.

[13] Pelikan, S., *Invariant densities for random maps of the interval*, Proc. Amer. Math. Soc. 281, 813–825, 1984.

[14] Schenk-Hoppe, K. R., *Random Dynamical Systems in Economics*, working paper series, ISSN 1424-0459, Institute of empirical research in economics, University of Zurich, 2000.

[15] Swishchuk, A.V. and Islam, M. S., *The Geometric Markov Renewal Processes with applications to Finance*, Stochastic Analysis and applications, Volume 29, No. 4 (2011), 684–705..

[16] Slomczynski,W., Kwapien, J. and Zyczkowski, K., *Entropy computing via integration over fractal measures*, Chaos 10, 180–188, 2000.

[17] Ulam, S. M., *A collection of Mathematical problems Interscience Tracts in pure and Applied Math.*, vol 8, Interscience, New York, 1960.