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# GEOMETRIC MESH EXPONENTIAL FINITE DIFFERENCE METHOD FOR THE NUMERICAL SOLUTIONS OF NON-LINEAR GENERAL TWO POINT BOUNDARY VALUE PROBLEMS

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**ABSTRACT.**In this article, we have presented non uniform geometric step size exponential finite difference method for the numerical solution of general two point boundary value problems with Dirichlet's boundary conditions. Under appropriate condition, we have discussed the local truncation error and the convergence of the proposed method. Numerical experiments approves the use and computational efficiency of the method in model problems. Numerical results showed that the proposed method is convergent and has at least second order of accuracy which is in good agreement with the theoretically established order of the method.

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**Keywords.** Exponential method, Finite difference method, Geometric mesh, General two-point boundary value problem, Nonlinear linear problem, Variable Mesh.

## 1. Introduction

In this article we consider the following general two-point boundary value problem of the form

$$y''(x) = f(x, y, y'), \quad a < x < b, \tag{1.1}$$

subject to the boundary conditions

$$y(a) = \alpha \quad \text{and} \quad y(b) = \xi,$$

where $\alpha$ and $\xi$ are real constants and $f$ is continuous on $(x, y)$ for all $x \in [a, b]$ and $y, y' \in \Re$.

The general two-point boundary value problems are of common occurrence in many areas of sciences and engineering. This class of problems has gained importance in the literature for the variety of their applications. In most cases it is impossible to obtain solutions of these problems using analytical methods which exactly satisfy the given boundary conditions. In these cases we resort to approximate solution of the problems and the last few decades have seen substantial progress in the development

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of approximate solutions of these problems. In the literature, there are many different methods and approaches such as method of integration and discretization that are used to derive the approximate solutions of these problems [1, 2, 3, 4].

The existence and uniqueness of the solution to problem (1.1) is assumed. We further assumed that problem (1.1) is well posed with continuous derivatives and that the solution depends differentially on the boundary conditions. The specific assumption on $f(x, y, y')$ to ensure existence and uniqueness will not be considered in this article [3, 4, 5].

Over the last few decades for the numerical solution this class of problems, the higher order finite difference methods [6, 7, 8, 9] and references therein have generated renewed interest. In recent years, variety of specialized techniques [12, 13] for the numerical solution of boundary value problems in ODEs have been reported in the literature. An exponential finite difference method uniform step size [14] and variable step size [15] for the numerical solution of linear two-point boundary value problems were proposed and generated impressive numerical results. Hence, the purpose of this article is to propose an exponential finite difference method with non-uniform geometric step size for problem (1.1).

The development of numerical method for the approximate solution of general two-point boundary value problems with a small parameter affecting the highest derivative of the differential equation invites special attention. It is a well known fact that these boundary value problems possess a small interval in which the solution varies rapidly. This small interval is known as the boundary layer in the literature and a variable mesh method is well suited for solving boundary layer problem [10, 11]. Our proposed geometric step size exponential difference method for the solution of general two-point boundary value problems is efficient in solving such boundary layer problems without any difficulty. We hope that others may find the proposed method as an improvement and appealing to those existing finite difference methods for general two-point boundary value problems.

A new method of at least quadratic order is proposed for the numerical solution of boundary value problems (1.1). Our idea is to apply the exponential finite difference method to discretize equation (1.1) in order to get a system of algebraic equations. The simplicity of the proposed method lies in its three point discretization. In addition, if we apply a linearization technique, the method results in a tri-diagonal matrix of the nodal values at central and two adjacent nodes. The elements of this tri-diagonal matrix depends on the source function i.e. right-hand side of the ordinary differential equation as well as on its partial derivatives with respect to the dependent variable and its first-order derivative. To the best of our knowledge, no similar method for the numerical solution of problem (1.1) has been discussed in the literature so far.

We have presented our work in this article as follows. In the next section we presented variable mesh size exponential finite difference method. In Section 3, we have presented derivation of the proposed method. In Section 4, local truncation error and in Section 5, convergence of the new method are discussed. The application of the proposed method to the problems in (1.1) has been presented and illustrative numerical results have been produced to show the efficiency of the new method in Section 6. Discussion and conclusion on the performance of the new method are presented in Section 7.

## 2. **Exponential Difference Method**

We define $N$ finite numbers of nodal points of the domain $[a, b]$, in which the solution of the problem (1.1) is desired, as $a \le x_0 < x_1 < x_2 < \cdots < x_N < x_{N+1} = b$, using nonuniform step length $h$ such that $x_{i+1} = x_i + h_{i+1}$, $i = 0, 1, 2, \ldots, N$ and $r_i = \frac{h_{i+1}}{h_i}$. Suppose that we wish to determine the numerical approximation of the theoretical solution $y(x)$ of the problem (1.1) at the nodal point $x_i$, $i = 1, 2, \ldots, N$. We denote the numerical approximation of $y(x)$ at node $x = x_i$ as $y_i$ . Let us denote $f_i$ as the approximation of the theoretical value of the source function $f(x, y(x), y'(x))$ at node $x = x_i$, $i = 0, 1, 2, \ldots, N + 1$. We can define other notations used in this article i.e. $f_{i\pm1}$, and $y_{i\pm1}$, in the similar way. Let us define following approximations

$$\overline{y}'_{i-1} = \frac{(-r_i(r_i + 2)y_{i-1} + (r_i + 1)^2 y_i - r_i y_{i+1})}{h_i r_i(r_i + 1)} \tag{2.1}$$

$$\overline{y}'_{i} = \frac{(y_{i+1} + (r_i^2 - 1)^2 y_i - r_i(r_i + 1)y_{i-1})}{h_i r_i(r_i + 1)} \tag{2.2}$$

$$\overline{y}'_{i+1} = \frac{((2r_i + 1)y_{i+1} - (r_i + 1)^2 y_i + r_i^2 y_{i-1})}{h_i r_i(r_i + 1)} \tag{2.3}$$

$$\overline{f}'_{i-1} = f(x_{i-1}, y_{i-1}, \overline{y}'_{i-1}) \tag{2.4}$$

$$\overline{f}'_{i} = f(x_i, y_i, \overline{y}'_i) \tag{2.5}$$

$$\overline{f}'_{i+1} = f(x_{i+1}, y_{i+1}, \overline{y}'_{i+1}). \tag{2.6}$$

Then at each internal mesh point $x_i$ to discretize problem (1.1) and following the idea in [14], we propose exponential finite difference method for an approximation to the theoretical solution $y(x_i)$ of the problem (1.1) as,

$$y_{i+1} - (1 + r_i)y_i + r_i y_{i-1} = \frac{h_i^2 r_i(r_i + 1)}{2}\overline{f}_i \exp\left(\frac{(r_i - 1)h_i \overline{f}'_i}{3\overline{f}_i}\right). \tag{2.7}$$

For each nodal point $x_i$, we will obtain the nonlinear system of equations given by (2.7) or a linear system of equations if the source function is $f(x)$. In the exponential method (2.7), the exponential function $\exp\left(\frac{(r_i-1)h_i\overline{f}'_i}{3\overline{f}_i}\right)$ has the argument $\frac{(r_i-1)h_i\overline{f}'_i}{3\overline{f}_i}$. If $\overline{f}_i$ in the denominator of the argument becomes zero in the domain of the solution, we take the series expansion of the function $\exp\left(\frac{(r_i-1)h_i\overline{f}'_i}{3\overline{f}_i}\right)$ and neglecting the second and higher order terms. Therefore method (2.7) becomes

$$y_{i+1} - (1+r_i)y_i + r_i y_{i-1} = h_i^2 r_i(r_1+1)\left(\overline{f}_i + \frac{h_i(r_i-1)}{3}\overline{f}'_i\right). \tag{2.8}$$

For the computational purpose in Section 6, we have used the following second order finite difference approximation in place of $h_i\overline{f}'_i$ in (2.7) and in (2.8):

$$h_i\overline{f}'_i = \frac{\overline{f}_{i+1} + (r_i^2-1)\overline{f}_i - r_i^2\overline{f}_{i-1}}{r_i(r_i+1)}. \tag{2.9}$$

## 3. **Derivation of the method**

In this section we outline the derivation of the method (2.7). From (2.1), expand $\overline{y}'_{i-1}$ in a Taylor series about the mesh point $x_{i-1}$ and simplify the expansion, we have

$$\overline{y}'_{i-1} = y_{i-1} + O(h_i^2). \tag{3.1}$$

Thus $\overline{y}'_{i-1}$ provides an $O(h_i^2)$ for $y_{i-1}$. Similarly from (2.2) and (2.3), we have

$$\overline{y}'_i = y_i + O(h_i^2), \tag{3.2}$$

$$\overline{y}'_{i+1} = y_{i+1} + O(r_i^2 h_i^2). \tag{3.3}$$

So from (2.4) and (3.1),

$$\overline{f}'_{i-1} = f(x_{i-1}, y_{i-1}, y_{i-1} + O(h_i^2)) = f_{i-1} + O(h_i^2). \tag{3.4}$$

Thus $\overline{f}_{i-1}$ provides an $O(h_i^2)$ for $f_{i-1}$. Similarly from (2.5–3.2) and (2.6–3.3), we have

$$\overline{f}'_i = f_i + O(h_i^2), \tag{3.5}$$

$$\overline{f}'_{i+1} = f_{i+1} + O(h_i^2). \tag{3.6}$$

Thus following the idea in [15], neglecting the remainders in (3.4), (3.5) and (3.6),we have

$$\begin{aligned} y_{i+1} - (1+r_i)y_i + r_i y_{i-1} &= \frac{h_i^2 r_i(r_i+1)}{2} f_i \exp\left(\frac{h_i(r_i-1)f'_i}{3f_i}\right) \\ &\equiv \frac{h_i^2 r_i(r_i+1)}{2}\overline{f}_i \exp\left(\frac{(r_i-1)h_i\overline{f}'_i}{3\overline{f}_i}\right). \end{aligned} \tag{3.7}$$

which is the proposed second order exponential method for the numerical solution of the problem (1.1).

## 4. **Local Truncation Error**

We can write the following expression for the term in (3.7) with the help of (2.9):

$$\exp\left(\frac{(r_i-1)h_i\overline{f}'_i}{3\overline{f}_i}\right)=\exp\Big(\frac{(r_i-1)(\overline{f}_{i+1}+(r_i^2-1)\overline{f}_i-r_i^2\overline{f}_{i-1})}{3r_i(r_i+1)\overline{f}_i}\Big). \tag{4.1}$$

Write the expansion for the exponential function in the (3.7) by neglecting the third and higher order terms, so we will obtain,

(4.2)

$$\exp\left(\frac{(r_i-1)h_i\overline{f}'_i}{3\overline{f}_i}\right)\equiv 1+\frac{(r_i-1)(\overline{f}_{i+1}+(r_i^2-1)\overline{f}_i-r_i^2\overline{f}_{i-1})}{3r_i(r_i+1)\overline{f}_i}+\frac{1}{2}\left(\frac{(r_i-1)h_i\overline{f}'_i}{3\overline{f}_i}\right)^2.$$

From (3.7) and (4.2), the truncation error $T_i$ at the nodal point $x=x_i$ may be written as [17, 18, 19],

$$\begin{aligned}T_i&=y_{i+1}-(1+r_i)y_i+r_iy_{i-1}\\&-\frac{h_i^2}{2}(r_i^2+r_i)\overline{f}_i\left(1+\frac{(r_i-1)(\overline{f}_{i+1}+(r_i^2-1)\overline{f}_i-r_i^2\overline{f}_{i-1})}{3r_i(r_i+1)\overline{f}_i}+\frac{1}{2}\left(\frac{h_i(r_i-1)\overline{f}'_i}{3\overline{f}_i}\right)^2\right).\end{aligned}$$

By the Taylor series expansion of $y$ at nodal point $x=x_i$ and application of approximations (3.4), (3.5) and (3.6) then we have $y''_i=f_i$, $y_i^{(3)}=f'_i$ and etc. Thus we obtained

$$T_i=\left(\frac{h_{i+1}^4}{24}+\frac{r_ih_i^4}{24}\right)y_i^{(4)}-\frac{r_i(r_i+1)}{36}\frac{(h_i^2(r_i-1)y_i^{(3)})^2}{f_i}+O(h_i^5). \tag{4.3}$$

(4.3) can be simplified and written as:

$$T_i=\frac{r_i(r_i+1)h_i^4}{72}\left\{3(r_i^2-r_i+1)y_i^{(4)}-\frac{2}{f_i}((r_i-1)y_i^{(3)})^2\right\}+O(h_i^5), \tag{4.4}$$

Thus we have obtained a truncation error at each internal mesh point $x_i$ of $O(h_i^4)$.

## 5. **Convergence of the Method**

Let us substitute (4.2) into (3.7) and then simplify (3.7), we have

$$\begin{aligned}y_{i+1}-(1+r_i)y_i+r_iy_{i-1}&=\frac{h_i^2}{2}(r_i^2+r_i)\overline{f}_i\left(1+\frac{(r_i-1)(\overline{f}_{i+1}+(r_i^2-1)\overline{f}_i-r_i^2\overline{f}_{i-1})}{3r_i(r_i+1)\overline{f}_i}\right)\\&=\frac{h_i^2}{6}\{3r_i(r_i+1)\overline{f}_i+(r_i-1)(\overline{f}_{i+1}+(r_i^2-1)\overline{f}_i-r_i^2\overline{f}_{i-1})\}.\end{aligned}$$

Thus

$$-y_{i+1}+(1+r_i)y_i-r_iy_{i-1}+\frac{h_i^2}{6}(\alpha_i\overline{f}_i+\gamma_i\overline{f}_{i+1}+\beta_i\overline{f}_{i-1})=0, \tag{5.1}$$

where $\alpha_i=(r_i+1)(r_i^2+r_i+1)$, $\beta_i=-r_i^2(r_i-1)$ and $\gamma_i=r_i-1$.

Let us define

$$\phi_1 = \frac{h_1^2}{6}(\alpha_1 f(x_1, y_1, \overline{y}'_1) + \gamma_1 f(x_2, y_2, \overline{y}'_2) + \beta_1 f(x_0, y_0, \overline{y}'_0)) + r_1 y_0, \quad i = 1$$

$$\phi_i = \frac{h_i^2}{6}(\alpha_i f(x_i, y_i, \overline{y}'_i) + \gamma_i f(x_{i+1}, y_{i+1}, \overline{y}'_{i+1}) + \beta_i f(x_{i-1}, y_{i-1}, \overline{y}'_{i-1})),$$
$$2 \le i \le N-1$$

$$\phi_N = \frac{h_N^2}{6}(\alpha_N f(x_N, y_N, \overline{y}'_N) + \beta_N f(x_{N-1}, y_{N-1}, \overline{y}'_{N-1})$$
$$+ \gamma_N f(x_{N+1}, y_{N+1}, \overline{y}'_{N+1})) + y_{N+1} \quad i = N$$

Let us define column matrix $\boldsymbol{\phi}_{N\times 1}$ and $\mathbf{y}_{N\times 1}$ as

$$\boldsymbol{\phi} = [\phi_1, \phi_2, \ldots, \phi_N]'_{1\times N}, \quad \mathbf{y} = [y_1, y_2, \ldots, y_N]'_{1\times N},$$

where $[\ldots]'$ is the transpose of a column matrix.

The difference method (5.1) represents a system of nonlinear equations in unknown $y_i, i = 1, 2, \ldots, N$. Let us write (5.1) in matrix form as,

$$\mathbf{D}\mathbf{y} + \boldsymbol{\phi}(\mathbf{y}) = \mathbf{0}, \tag{5.2}$$

where

$$\mathbf{D} = \begin{pmatrix} 1+r_1 & -1 & & & 0 \\ -r_2 & 1+r_2 & -1 & & \\ & -r_3 & 1+r_3 & -1 & \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & & & -r_N & 1+r_N \end{pmatrix}_{N\times N}$$

is a tridiagonal matrix. Let $\mathbf{Y}$ be the exact solution of (5.1), so it will satisfy the matrix equation

$$\mathbf{D}\mathbf{Y} + \boldsymbol{\phi}(\mathbf{Y}) + \mathbf{T} = \mathbf{0}, \tag{5.3}$$

where $\mathbf{Y}$ is a column matrix of order $N \times 1$ which can be obtained by replacing $y$ with $Y$ in matrix $\mathbf{y}$ and $\mathbf{T}$ is a truncation error matrix in which each element has $O(h_i^4)$.

Let us define

$$\overline{F}_{i+1} = f(x_{i+1}, Y_{i+1}, \overline{Y}'_{i+1}), \quad \overline{f}_{i+1} = f(x_{i+1}, y_{i+1}, \overline{y}'_{i+1}), \quad \overline{F}_i = f(x_i, Y_i, \overline{Y}'_i),$$

$$\overline{f}_i = f(x_i, y_i, \overline{y}'_i), \quad \overline{F}_{i-1} = f(x_{i-1}, Y_{i-1}, \overline{Y}'_{i-1}) \text{ and } \overline{f}_{i-1} = f(x_{i-1}, y_{i-1}, \overline{y}'_{i-1}).$$

After linearization of $\overline{f}_{i+1}$, we have

$$\overline{f}_{i+1} = \overline{F}_{i+1} + (y_{i+1} - Y_{i+1})G_{i+1} + (\overline{y}'_{i+1} - \overline{Y}'_{i+1})H_{i+1},$$

where $G_{i+1} = (\frac{\partial f}{\partial Y})_{i+1}$ and $H_{i+1} = (\frac{\partial f}{\partial \overline{Y}'})_{i+1}$. Thus

$$\overline{f}_{i+1} - \overline{F}_{i+1} = (y_{i+1} - Y_{i+1})G_{i+1} + (\overline{y}'_{i+1} - \overline{Y}'_{i+1})H_{i+1}. \tag{5.4}$$

Similarly, we can linearize $\overline{f}_{i-1}$, and $\overline{f}_i$ , to obtain the following results :

$$\overline{f}_{i-1} - \overline{F}_{i-1} = (y_{i-1} - Y_{i-1})G_{i-1} + (\overline{y}'_{i-1} - \overline{Y}'_{i-1})H_{i-1}. \tag{5.5}$$

$$\overline{f}_i - \overline{F}_i = (y_i - Y_i)G_i + (\overline{y}'_i - \overline{Y}'_i)H_i. \tag{5.6}$$

By taking the Taylor series expansion of $G_{i\pm 1}$ about $x = x_i$, and from the difference of (5.2) and (5.3), we can write

$$\boldsymbol{\phi}(\mathbf{y}) - \boldsymbol{\phi}(\mathbf{Y}) = \mathbf{PE}, \tag{5.7}$$

where $\mathbf{P} = (P_{lm})_{N\times N}$ is a tri-diagonal matrix defined as

$$\begin{aligned} P_{lm} = \frac{h_i^2}{6}\Bigg(&\alpha_i G_i - \frac{(r_i+1)}{h_i r_i}(\alpha_i(r_i-1)^2 + \gamma_i - \beta_i)H_i \\ &- \frac{(r_i\gamma_i + \beta_i)(r_i+1)}{r_i}(\frac{\partial H}{\partial x})_i\Bigg), \quad i = l = m, \quad l = 1, 2, \ldots, N, \end{aligned}$$

$$\begin{aligned} P_{lm} = \frac{h_i^2}{6}(\gamma_i G_i &+ \frac{(\alpha_i + \gamma_i(2r_i+1) - \beta_i r_i)}{h_i r_i(1+r_i)}H_i + \frac{\gamma_i(2r_i+1)+\beta_i}{r_i+1}(\frac{\partial H}{\partial x})_i \\ &+ \gamma_i h_{i+1}(\frac{\partial G}{\partial x})_i), \quad m = l+1, \quad i = l = 1, 2, \ldots, N-1, \end{aligned}$$

$$\begin{aligned} P_{lm} = \frac{h_i^2}{6}(\beta_i G_i &+ \frac{\gamma_i r_i - \beta_i(r_i+2) - \alpha_i(r_i+1)}{h_i(r_i+1)}H_i + \frac{\gamma_i r_i^2 + \beta_i(r_i+2)}{(r_i+1)}\left(\frac{\partial H}{\partial x}\right)_i \\ &- \beta_i h_i(\frac{\partial G}{\partial x})_i), \quad i = l = m+1, \quad m = 1, 2, \ldots, N-1, \end{aligned}$$

and $\mathbf{E} = [E_1, E_2, \ldots\ldots, E_N]'_{1\times N}$ , where $E_i = (y_i - Y_i), i = 1, 2, \ldots, N$ .

Let us assume that the solution of difference equation (3.7) has no roundoff error. So from (5.2), (5.3) and (5.7) we have

$$(\mathbf{D} + \mathbf{P})\mathbf{E} = \mathbf{JE} = \mathbf{T}. \tag{5.8}$$

Let us define $G^0 = \{G_i : i = 1, 2, \ldots, N\}$,

$$G_* = \min_{x\in[a,b]} \frac{\partial f}{\partial Y}, and \quad G^* = \max_{x\in[a,b]} \frac{\partial f}{\partial Y},$$

such that

$$0 \le G_* \le t \le G^*, \quad \forall t \in G^0.$$

and $\qquad H^0 = \{H_i : i = 1, 2, \ldots, N\}$,

$$H_* = \min_{x\in[a,b]} \frac{\partial f}{\partial \overline{Y}'}, and \quad H^* = \max_{x\in[a,b]} \frac{\partial f}{\partial \overline{Y}'},$$

such that

$$0 \le H_* \le t_0 \le H^*, \quad \forall t_0 \in H^0.$$

We further define

$$G_x^0 = \left\{ \left(\frac{\partial G}{\partial x}\right)_i, \quad i = 1, 2, \ldots, N \right\} \quad \text{and} \quad H_x^0 = \left\{ \left(\frac{\partial H}{\partial x}\right)_i, \quad i = 1, 2, \ldots, N \right\}.$$

Let there exist some positive constant $W$ and $W'$ such that $|t^0| \le W \ \forall \ t^0 \in G_x^0$ and $|t'| \le W'$, $\forall \ t' \in H_x^0$. So it is possible for very small $h_i, \forall \ i = 1, 2, \ldots, N$,

$$|P_{lm}| \le 1 + r_i, \quad \forall \quad i = l = m \quad l = 1, 2, \ldots, N,$$

$$|P_{lm}| \le 1, \quad \forall \quad m = l + 1, \quad i = l = 1, 2, \ldots, N - 1,$$

$$|P_{lm}| \le r_i, \quad \forall \quad i = l = m + 1, \quad m = 1, 2, \ldots, N - 2.$$

Let $\mathbf{R} = [R_1, R_2, \ldots, R_N]'_{1\times N}$, denote the row sum of the matrix $\mathbf{J} = (J_{lm})_{N\times N}$ where

$$R_1 = r_1 + \frac{h_1^2}{6}\Bigg((\alpha_1 + \gamma_1)G_1 + \frac{\alpha_1 r_1^2(2 - r_1^2) - r_1^2\gamma_1 - \beta_1(r_1^2 + 3r_1 + 1)}{h_1 r_1(1 + r_1)} H_1 + \frac{\gamma_1 r_1^2 - \beta_1}{r_1(r_1 + 1)}\left(\frac{\partial H}{\partial x}\right)_1 + \gamma_1 h_2 \left(\frac{\partial G}{\partial x}\right)_i\Bigg), \quad l = i = 1,$$

$$R_l = \frac{h_i^2}{6}\Bigg((\alpha_i + \gamma_i + \beta_i)G_i + \frac{\alpha_i(r_i^2 - r_i - r_i^4) + \beta_i}{h_i r_i(1 + r_i)} H_i + \frac{\beta_i}{r_i(r_1 + 1)}\left(\frac{\partial H}{\partial x}\right)_i + h_i(\gamma_i r_i - \beta_i)\left(\frac{\partial G}{\partial x}\right)_i\Bigg), \quad l = i = k, and \quad 2 \le k \le N - 1,$$

$$R_N = 1 + \frac{h_N^2}{6}\Bigg((\alpha_N + \beta_N)G_N + \frac{\alpha_N(r_N^2 - r_N^4 - r_N - 1) - \gamma_N(r_N^2 + 3r_N + 1) + \beta_N}{h_N r_N(1 + r_N)} H_N - \frac{\gamma_N r_N(2r_N + 1) + \beta_N}{r_N(r_N + 1)}\left(\frac{\partial H}{\partial x}\right)_N - \beta_N h_N \left(\frac{\partial G}{\partial x}\right)_N\Bigg), \quad l = i = N.$$

On neglecting the higher order terms i.e. $O(h_i^2)$ in $R_i$ then it is easy to see that $\mathbf{J}$ is irreducible [17]. By the row sum criterion and for sufficiently small $h_i, \forall i = 1, 2, \ldots, N$, $\mathbf{J}$ is monotone [19]. Thus $\mathbf{J}^{-1}$ exist and $\mathbf{J}^{-1} \ge 0$. For the bound of $\mathbf{J}$, we define [20, 21]

$$d_l(\mathbf{J}) = |J_{ll}| - \sum_{l\ne m}^{N} |J_{lm}|, \quad l = 1, 2, \ldots, N,$$

where

$$d_1(\mathbf{J}) = r_1 + \frac{h_1^2}{6}\Bigg((\alpha_1+\gamma_1)G_1 + \frac{\alpha_1 r_1^2(2-r_1^2) - r_1^2\gamma_1 + \beta_1(r_1^2+r_1+1)}{h_1 r_1(1+r_1)}H_1$$
$$- \frac{\gamma_1 r_1^3 + \beta_1 r_1(1+r_1)}{r_1(r_1+1)}\left(\frac{\partial H}{\partial x}\right)_1 + \gamma_1 h_2 \left(\frac{\partial G}{\partial x}\right)_1\Bigg), \quad l = i = 1,$$

$$d_l(\mathbf{J}) = \frac{h_l^2}{6}\Bigg((\alpha_l+\beta_l+\gamma_l)G_l + \frac{\alpha_l(r_l^2 - r_l^4 - r_l) + \beta_l(1-r_l)}{h_l r_l(1+r_l)}H_l$$
$$+ \frac{\gamma_l(2r_l^3 - 2r_l^2 - r_l) + \beta_l(r_l-1)}{r_l(r_l+1)}\left(\frac{\partial H}{\partial x}\right)_l + (\gamma_l h_{l+1} - \beta_l h_l)\left(\frac{\partial G}{\partial x}\right)_i\Bigg),$$
$$l = i = k, and \quad 2 \le k \le N-1,$$

$$d_N(\mathbf{J}) =$$
$$1 + \frac{h_N^2}{6}\Bigg((\alpha_N+\beta_N)G_N + \frac{\alpha_N(r_N^2 - r_N^4 - r_N - 1) - \gamma_N(2r_N+1) + \beta_N}{h_N r_N(1+r_N)}H_N$$
$$- \frac{\gamma_N r_N(2r_N+1) + \beta_N}{h_N(r_N+1)}\left(\frac{\partial H}{\partial x}\right)_N - \beta_N h_N\left(\frac{\partial G}{\partial x}\right)_N\Bigg), \quad l = i = N.$$

We observe the presence of the higher order terms i.e. $O(h_i^3)$ in the above expressions. Let $d_l(\mathbf{J}) \ge 0, \quad \forall \quad l$ and

$$d_*(\mathbf{J}) = \min_{1\le l\le N} d_l(\mathbf{J}).$$

Then

$$\|\mathbf{J}^{-1}\| \le \frac{1}{d_*(\mathbf{J})}. \tag{5.9}$$

Thus from (5.8) and (5.9), we have

$$\|\mathbf{E}\| \le \frac{1}{d_*(\mathbf{J})}\|\mathbf{T}\|. \tag{5.10}$$

It follows from (4.4) and (5.10) that $\|\mathbf{E}\| \to 0$ as $h_i \to 0$. Thus we conclude that method (3.7) converges and the order of the convergence of method (3.7) is at least quadratic.

## 6. Numerical Results

To illustrate our method and demonstrate its computational efficiency, we considered some model problems. In each model problem, we took non uniform step size $h_i$. In Table 1 - Table 5, we have shown the maximum absolute error (MAE) and root mean square error (RMSE), computed for different values of $N$ and is defined as

$$MAE = \max_{1\le i\le N} |y(x_i) - y_i|$$

$$RMSE = \sqrt{\frac{\Sigma_{i=1}^{N}(y(x_i) - y_i)^2}{N}}$$

The starting value of the step length $h_1$ is calculated by formula

$$h_1 = \begin{cases} \frac{(b-a)(r-1)}{r^N-1} & \text{if} \quad r > 1 \\ \frac{(b-a)(1-r)}{1-r^N} & \text{if} \quad r < 1 \end{cases}$$

where $r = r_i, \quad \forall \quad i = 1, 2, \ldots, N$ in computation. In case of uniform mesh $r = 1$ , the above formula for computation of step length becomes $h = \frac{b-a}{N}$. The order of the convergence $(O_N)$ of the method (3.7) is estimated by the formula

$$(O_N) = \log_m \left( \frac{MAE_N}{MAE_{mN}} \right),$$

where m can be estimated by considering the ratio of $N$.

We have used Newton-Raphson iteration method to solve the system of nonlinear equations arise from equation (3.7). All computations were performed on a MS Window 2007 professional operating system in the GNU FORTRAN environment version 99 compiler (2.95 of gcc) on Intel Duo Core 2.20 Ghz PC. The solutions are computed on $N$ nodes and iteration is continued until either the maximum difference between two successive iterates is less than $10^{-10}$ or the number of iteration reached $10^3$.

**Problem 1.** The first model problem is linear given by

$$y''(x) = \lambda y', \quad y(0) = 1, \quad y(1) = 0, \quad x \in [0, 1].$$

The analytical solution is $y(x) = \frac{1-\exp(\lambda(x-1))}{1-\exp(-\lambda)}$. The MAE and RMSE computed by method (3.7) for different values of $N$, $r_i$ and $\lambda$ are presented in Table 1 and Table 2.

**Problem 2.** The second model problem is a nonlinear problem given by

$$y''(x) = \beta(y(x) - A)y'(x), \quad y(0) = A, \quad y(1) = A\left(1 - \tanh\left(\frac{\beta A}{2}\right)\right), \quad x \in [0, 1].$$

The analytical solution is $y(x) = A(1 - \tanh(\frac{\beta A x}{2}))$. The MAE and RMSE computed by method (3.7) for different values of $N$, $\beta$, $r_i$ and $A = .5$ are presented in Table 3 and Table 4.

**Problem 3.** The third model problem is a nonlinear problem given by

$$y''(x) = y^3(x) - y(x)y'(x), \quad y(1) = \frac{1}{2}, \quad y(2) = \frac{1}{3}, \quad x \in [1, 2],$$

where $f(x)$ is calculated so that $y(x) = \frac{1}{1+x}$ is the analytical solution. The MAE and RMSE computed by method (3.7) for different values of $N$ and $r_i$ are presented in Table 5.

TABLE 1. Maximum absolute and root mean square errors (Problem 1).

| $r_i$ | $N$ | Error $\lambda = \frac{1}{5}$ *MAE* | Error $\lambda = \frac{1}{5}$ *RMSE* | Error $\lambda = \frac{1}{2}$ *MAE* | Error $\lambda = \frac{1}{2}$ *RMSE* |
|---|---|---|---|---|---|
| 1.0 | 4 | .51856041(-5) | .43565792(-5) | .80227852(-4) | .67814253(-4) |
| | 8 | .11324883(-5) | .88156457(-6) | .19848347(-4) | .15570105(-4) |
| | 16 | .59604645(-7) | .24934412(-7) | .45299530(-5) | .33803201(-5) |
| | 32 | .29802322(-7) | .59844503(-8) | .11920929(-6) | .50656173(-7) |
| | 64 | .29802322(-7) | .42044808(-8) | .59604645(-7) | .20565556(-7) |
| 1.06 | 4 | .52452087(-5) | .44867757(-5) | .80227852(-4) | .68902875(-4) |
| | 8 | .12814999(-5) | .94384478(-6) | .21666288(-4) | .16474240(-4) |
| | 16 | .11920929(-6) | .52893874(-7) | .56028366(-5) | .37998784(-5) |
| | 32 | .59604645(-7) | .29832350(-7) | .15497208(-5) | .76230248(-6) |
| | 64 | .59604645(-7) | .24657236(-7) | .71525574(-6) | .23258180(-6) |

TABLE 2. Maximum absolute and root mean square errors (Problem 1).

| $r_i$ | $N$ | Error $\lambda = 2$ *MAE* | Error $\lambda = 2$ *RMSE* | Error $\lambda = 5$ *MAE* | Error $\lambda = 5$ *RMSE* |
|---|---|---|---|---|---|
| 1.0 | 4 | .45045912(-2) | .38031123(-2) | .53083181(-1) | .34218341(-1) |
| | 8 | .11489391(-2) | .86355786(-3) | .11424065(-1) | .73574879(-2) |
| | 16 | .28461218(-3) | .20671070(-3) | .28715730(-2) | .17305844(-2) |
| | 32 | .69081783(-4) | .49064751(-4) | .70977211(-3) | .42099375(-3) |
| | 64 | .57816505(-5) | .32492633(-5) | .16701221(-3) | .96714764(-4) |
| | 128 | .59604645(-7) | .21272125(-7) | .17851591(-4) | .26822090(-5) |
| 1.06 | 4 | .49441755(-2) | .39151483(-2) | .57824194(-1) | .36301401(-1) |
| | 8 | .13208389(-2) | .93901559(-3) | .15277922(-1) | .86171133(-2) |
| | 16 | .40006638(-3) | .25886981(-3) | .48831105(-2) | .24562438(-2) |
| | 32 | .16370416(-3) | .89868263(-4) | .21458864(-2) | .91063540(-3) |
| | 64 | .10362267(-3) | .41623971(-4) | .15016794(-2) | .47723393(-3) |
| | 128 | .96023083(-4) | .27330037(-4) | .14093518(-2) | .31848496(-3) |

TABLE 3. Maximum absolute and root mean square errors for $A = 0.5$ (Problem 2).

| $r_i$ | $N$ | *Error* $\beta = 10$ *MAE* | $\beta = 10$ *RMSE* | $\beta = 20$ *MAE* | $\beta = 20$ *RMSE* |
|---|---|---|---|---|---|
| 1.0 | 10 | .40723383(-2) | .26633504(-2) | .19166492(-1) | .10014023(-1) |
| | 20 | .99445879(-3) | .63442264(-3) | .47840476(-2) | .22599460(-2) |
| | 40 | .24610758(-3) | .15507486(-3) | .11704564(-2) | .54811692(-3) |
| | 80 | .57488680(-4) | .36033434(-4) | .29011071(-3) | .13514442(-3) |
| 1.1 | 10 | .40757805(-2) | .28871789(-2) | .13311431(-1) | .77447360(-2) |
| | 20 | .13484955(-2) | .91870409(-3) | .30689538(-2) | .18740752(-2) |
| | 40 | .79128146(-3) | .46120372(-3) | .13497695(-2) | .78995252(-3) |
| | 80 | .71993470(-3) | .30763954(-3) | .11662990(-2) | .51750842(-3) |

TABLE 4. Maximum absolute and root mean square errors for $A = 0.5$ (Problem 2).

| $r_i$ | $N$ | *Error* $\beta = \frac{1}{10}$ *MAE* | $\beta = \frac{1}{10}$ *RMSE* | $\beta = \frac{1}{20}$ *MAE* | $\beta = \frac{1}{20}$ *RMSE* |
|---|---|---|---|---|---|
| 1.0 | 10 | .14613867(-2) | .11299137(-2) | .73072314(-3) | .56498824(-3) |
| | 20 | .28319657(-2) | .21121933(-2) | .14156699(-2) | .10558445(-2) |
| | 40 | .50392747(-2) | .36413758(-2) | .25184751(-2) | .18198452(-2) |
| | 80 | .75264573(-2) | .52004759(-2) | .37604868(-2) | .25984163(-2) |
| 1.1 | 10 | .29802322(-7) | .14048950(-7) | ***** | ***** |
| | 20 | .29802322(-7) | .96691499(-8) | .29802322(-7) | .13674243(-7) |
| | 40 | .29802322(-7) | .10670943(-7) | .29802322(-7) | .95443813(-8) |
| | 80 | .29802322(-7) | .10059070(-7) | .29802322(-7) | .11615213(-7) |

*****: Computational results either overflow or exact.

TABLE 5. Maximum absolute and root mean square errors (Problem 3).

| $N$ | *Error* $r_i = 1.0$ *MAE* | *RMSE* | $r_i = 1.01$ *MAE* | *RMSE* | $r_i = 1.06$ *MAE* | *RMSE* |
|---|---|---|---|---|---|---|
| 4 | .80442427(-4) | .69733680(-4) | .80807979(-4) | .69732094(-4) | .82333179(-4) | .70011636(-4) |
| 8 | .21132984(-4) | .16370834(-4) | .20745851(-4) | .16212709(-4) | .20314470(-4) | .15892289(-4) |
| 16 | .50099275(-5) | .37548225(-5) | .48018824(-5) | .36072279(-5) | .49892456(-5) | .37118384(-5) |
| 32 | .27685925(-6) | .14303454(-6) | .58990622(-7) | .29212892(-7) | .77840133(-6) | .37706067(-6) |
| 64 | .20193923(-7) | .88874659(-8) | .27518832(-7) | .11322014(-7) | .23545212(-6) | .70097087(-7) |
| 128 | .29415279(-7) | .98746957(-8) | .40079282(-7) | .13548161(-7) | .13728150(-6) | .24566976(-7) |

We have described a numerical method for solving general two-point boundary value problems. Linear and non-linear model problems considered to demonstrate the computational performance of the proposed method. Numerical results for problem 1 which is presented in table 1 and table 2, for different values of $r_i$, $N$ and $\lambda$ show if we consider uniform and non-uniform step size, maximum absolute error and root mean square error in our method increased as $\lambda$ increases. On the other hand both maximum absolute error and root mean square errors decreases with increase in $N$. The numerical results for problem 2 show in both *MAE* and *RMSE* less accurate in uniform than non-uniform mesh size. The results for problem 3 are uniformly accurate in both uniform and nonuniform mesh size. Over all method (3.7) is convergent and the convergence of the method depends on choice of mesh ratio $r_i$.

## 7. **Conclusion**

A method to find the numerical solution of general two point boundary value problems has been developed. The decision to use a certain difference scheme depends on computational efficiency of the method for the accurate solution and complexity of the problem. Thus is obvious that special method required for some special problem where the solution is not regular and varies rapidly or presence of more parameters in a problem. But on the other hand, the proposed method produces good numerical approximate solutions for variety of model problems without any modification either in method or in problem and its rate of convergence is quadratic. The numerical results of the model problems showed that the proposed method is computationally efficient and plays an important role to obtain accurate numerical solutions. The idea presented in this article leads to the possibility to develop difference methods to solve higher order boundary value problems in ordinary differential equations. Works in these directions are in progress.

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