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COEFFICIENT IDENTIFICATION IN EULER-BERNOULLI EQUATION FROM OVER-POSED DATA

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ABSTRACT. This is a study concerning the identification of the heterogeneous flexural rigidity of a beam governed by the steady-state Euler-Bernoulli fourth order ordinary differential equation. We use the method of Variational Imbedding (MVI) to deal with the inverse problem for the coefficient identification from over-posed data. The method is identifying the coefficient by approximating it with a piece-wise polynomial function. Several types of piece-wise polynomial functions are considered: piece-wise constant; linear spline; and cubic spline. It is observed in this study that the numerical solution of the variational problem coincides with the direct simulation of the original problem within the second order of approximation.

Keywords: Euler-Bernoulli beam equation; Inverse problem; Coefficient identification; Method of Variational Imbedding.

AMS (MOS) Subject Classification.34A55, 45Q05, 49N45, 65L09, 70F17.

1. Introduction

The inverse problem for identifying the flexural rigidity of a beam from over-posed data has been the subject of recent studies by the authors, [11, 12]; see also other works [8, 1].

For simplicity, we consider the steady-state Euler-Bernoulli beam equation in the following form

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\sigma(x)\frac{\mathrm{d}^2 u}{\mathrm{d}x^2}\right) = f(x). \tag{1.1}$$

Without loss of generality, we assume that $0 \le x \le 1$. The function $f(x)$ represents the transversely distributed load. The coefficient $\sigma(x)$, called the flexural rigidity, is the product of the modulus of the elasticity E and the moment of inertia I of the

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cross-section of the beam about an axis through its centroid at right angles to the cross-section.

We assume the solution satisfies the following boundary conditions

$$u(0) = \alpha_{0,0}, \quad u(1) = \alpha_{1,0}, \qquad u'(0) = \alpha_{0,1}, \quad u'(1) = \alpha_{1,1}. \tag{1.2}$$

If the coefficient $\sigma(x) > 0$ and the function $f(x) \geq 0$ in equation (1.1) are known, under proper boundary conditions such as (1.2), the problem possesses a unique solution, usually referred as a direct solution. In practice, there are lots of practical problems, in which the coefficient $\sigma(x)$ is not known. Thus, a new so-called *inverse* problem appear: to find simultaneously the solution $u(x)$ and the coefficient $\sigma(x)$ of the Euler-Bernoulli equations. According to [2], a problem is called inverse if the values of some model parameter(s) must be obtained from observed data. Identifying the flexural rigidity of a beam is one such problem.

In reality, Euler-Bernoulli equation models a tensioned beam. Under environmental loads, caused by environmental phenomena such as wind, waves, current, tides, earthquakes, temperature, ice, seabed movement, and marine growth, the structure of the ingredients of the beam is changing. Usually it is expensive, even not possible, to measure the changes of the properties of the materials directly. On the other hand, changes in the physical properties of the materials cause changes in the coefficient $\sigma(x)$ in equation (1.1) and, respectively, changes on the deflection $u(x)$.

The method for solving the inverse problem for coefficient identification in equation (1.1) used here is proposed by C. I. Christov in [3] and is called the Method of Variational Imbedding (MVI). The idea of the MVI is to replace the incorrect problem with a well-posed problem for minimization of a quadratic functional built from the original equations. In other words, we "embed" the original incorrect problem into a higher-order boundary value problem which is well-posed.

MVI has been successfully applied to various type of problems such as boundary-layer-thickness identification as inverse problem [4] and identification of heat-conduction coefficient [5]. Other works include coefficient identification in two-dimensional elliptic partial differential equation [10] as well as solitary-wave solutions identification of Boussinesq and Korteweg-de Vries equation [7]. Regardless of the fact that only an ordinary differential equation is considered here, the proposed method can be generalized and applied to the problem for identification of coefficient in partial differential equations. The identifications of a coefficient in parabolic partial differential equations (see [5]) and in elliptic partial differential equation (see [10]) use procedures similar to the one proposed here.

The paper is organized as follows. The inverse problem for identification of the unknown coefficient is formulated in the next section. Then, the application of the

MVI to the inverse problem is presented. The numerical scheme, followed by numerical examples, is described next. Finally, conclusions are given in the end.

2. **Inverse Problem Formulation**

Note that the direct problem of the Euler-Bernoulli equation requires the determination of the deflection $u(x)$ satisfying (1.1) provided that $\sigma(x) > 0$ and $f(x) \geq 0$ are given and four boundary conditions are prescribed. However, additional information is needed for identifying the unknown coefficient $\sigma(x)$ if the inverse problem for coefficient identification is considered.

In addition to the boundary conditions (1.2), we assume that the function $u(x)$ satisfies the following conditions

$$u''(0) = \alpha_{0,2}, \quad u''(1) = \alpha_{1,2}, \qquad u'''(0) = \alpha_{0,3}, \quad u'''(1) = \alpha_{1,3}, \tag{2.1}$$

$$u(\xi_i) = \gamma_i, \qquad i = 1, 2, \ldots, n-1. \tag{2.2}$$

Then the problem for obtaining $u(x)$ from the equation (1.1) is over-posed (if $\sigma(x)$ is known) because the number of the conditions is greater than the number of unknowns. Conditions (2.2) result from measuring the deflection $u(x)$ in $n-1$ internal points $\xi_i \in (0, 1)$. These $(n-1)+4 = n+3$ extra conditions will help identify the coefficient $\sigma(x)$.

In this section we formulate of the inverse problem and also present the additional conditions needed for identifying the coefficient.

Suppose the coefficient $\sigma(x)$ in the equation (1.1) is a piece-wise polynomial function. In the present study we consider three cases for the function $\sigma(x)$.

1. The coefficient $\sigma(x)$ is a piece-wise constant

$$\sigma(x) = \sigma_i(x) = c_i, \tag{2.3}$$

for $\xi_{i-1} < x < \xi_i$, where the points ξ_i are given, and the constants c_i, $i = 1, 2, \ldots, n$, are unknown. The number of the unknown constants is equal to n.

2. The coefficient $\sigma(x)$ is a linear spline

$$\sigma(x) = \sigma_i(x) = a_i + b_i(x - \xi_{i-1}), \tag{2.4}$$

for $\xi_{i-1} < x < \xi_i$, $i = 1, 2, \ldots, n$, where the constants a_i, b_i, are unknown ($\xi_0 = 0, \xi_n = 1$), or the number of unknown constants is $2n$.

In the case of a linear spline coefficient $\sigma(x)$, the following $n-1$ conditions for continuity must be added, i.e.,

$$\sigma_i(\xi_i) = \sigma_{i+1}(\xi_i), \tag{2.5}$$

$i = 1, 2, \ldots, n-1$.

The number of the additional conditions is sufficient for finding the unknown constants.

3. The coefficient $\sigma(x)$ is a cubic spline

$$\sigma(x) = \sigma_i(x) = b_{0i} + b_{1i}(x - \xi_{i-1}) + b_{2i}(x - \xi_{i-1})^2 + b_{3i}(x - \xi_{i-1})^3, \tag{2.6}$$

for $\xi_{i-1} < x < \xi_i$. The points ξ_i are given, whereas the constants b_{ki}, $k = 1, 2, 3, 4$, $i = 1, 2, \ldots, n$, are unknown. The number of the unknown constants is $4n$.

The conditions for continuity of the cubic spline function $\sigma(x)$ and its first and second derivatives read

$$\sigma_i(\xi_i) = \sigma_{i+1}(\xi_i), \quad \sigma_i'(\xi_i) = \sigma_{i+1}'(\xi_i),\ \sigma_i''(\xi_i) = \sigma_{i+1}''(\xi_i), \tag{2.7}$$

where $i = 1, 2, \ldots, n - 1$.

Or, the number of the conditions becomes $n + 3 + 3(n - 1)$, which is exactly equal to the number of unknown constants $4n$, namely sufficient for finding the constants.

For arbitrary values of the unknown parameters, there may be no solution satisfying all of the conditions. For this reason, we assume that the problem is posed correctly after Tikhonov, [13], i.e., it is known *a-priori* that a solution of the problem exists. In other words, we assume that the data in the conditions have "physical meaning" and, therefore, a solution exists. The problem is how to convert the additional information available for the deflection $u(x)$ to the missing information for the coefficient $\sigma(x)$.

3. **Variational Imbedding**

Following the idea of the MVI, we replace the original problem with the problem of minimization of the functional

$$\mathcal{I}(u, \sigma) = \int_0^1 \mathcal{A}^2(u, \sigma)\mathrm{d}x \longrightarrow \min\ ,\ \mathcal{A}(u, \sigma) = \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\sigma(x)\frac{\mathrm{d}^2 u}{\mathrm{d}x^2}\right) - f(x)\ , \tag{3.1}$$

where u satisfies the appropriate conditions, and $\sigma(x)$ is an unknown piece-wise polynomial function, defined in Section 2.

The functional $\mathcal{I}(u, \sigma)$ is a quadratic and homogeneous function of the function $\mathcal{A}(u, \sigma)$; hence, it attains its absolute minimum if and only if $\mathcal{A}(u, \sigma) \equiv 0$. In this sense there is one-to-one correspondence between the original equation (1.1) and the minimization problem (3.1).

Since $\sigma(x)$ is a piece-wise function, we can rewrite the functional $\mathcal{I}$ as

$$\mathcal{I}(u, \sigma) = \sum_{i=1}^{n} \int_{\xi_{i-1}}^{\xi_i} \left[\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\sigma_i \frac{\mathrm{d}^2 u}{\mathrm{d}x^2}\right) - f(x)\right]^2 \longrightarrow \min\ . \tag{3.2}$$

The necessary condition for minimization of the functional $\mathcal{I}$ is expressed by the Euler-Lagrange equations for the functions $u(x)$ and $\sigma(x)$.

3.1. **Euler-Lagrange Equation for the Deflection.** The Euler-Lagrange equation with respect to the function $u(x)$ reads

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\sigma\frac{\mathrm{d}^2}{\mathrm{d}x^2}\mathcal{A} = \frac{\mathrm{d}^2}{\mathrm{d}x^2}\sigma\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left[\frac{\mathrm{d}^2}{\mathrm{d}x^2}\sigma\frac{\mathrm{d}^2u}{\mathrm{d}x^2} - f(x)\right] = 0, \tag{3.3}$$

i.e.,

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\sigma\frac{\mathrm{d}^4}{\mathrm{d}x^4}\sigma\frac{\mathrm{d}^2u}{\mathrm{d}x^2} = \frac{\mathrm{d}^2}{\mathrm{d}x^2}\sigma\frac{\mathrm{d}^2}{\mathrm{d}x^2}f(x). \tag{3.4}$$

Therefore, in each interval $\xi_{i-1} < x < \xi_i$, the function $u(x)$ satisfies the equation

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\sigma_i\frac{\mathrm{d}^4}{\mathrm{d}x^4}\sigma_i\frac{\mathrm{d}^2u}{\mathrm{d}x^2} = \frac{\mathrm{d}^2}{\mathrm{d}x^2}\sigma_i\frac{\mathrm{d}^2}{\mathrm{d}x^2}f(x), \tag{3.5}$$

under the boundary conditions (1.2), (2.1), and (2.2).

Since the equation (3.5) is of the eight order, we need some additional boundary conditions.

3.1.1. *Piece-wise Constant Coefficient.* In this case, the equation (3.5) for the function $u(x)$ becomes

$$c_i\frac{\mathrm{d}^8u}{\mathrm{d}x^8} = \frac{\mathrm{d}^4}{\mathrm{d}x^4}f(x). \tag{3.6}$$

As mentioned already, we need some additional boundary conditions for the eight order equation. We derive them from the original equation (1.1), namely

$$c_i\frac{\mathrm{d}^4u}{\mathrm{d}x^4}\Big|_{\xi_i^-} = f(\xi_i^-), \quad c_{i+1}\frac{\mathrm{d}^4u}{\mathrm{d}x^4}\Big|_{\xi_i^+} = f(\xi_i^+), \tag{3.7}$$

$$c_i\frac{\mathrm{d}u}{\mathrm{d}x}\Big|_{\xi_i^-} = c_{i+1}\frac{\mathrm{d}u}{\mathrm{d}x}\Big|_{\xi_i^+}, \tag{3.8}$$

$$c_i\frac{\mathrm{d}^2u}{\mathrm{d}x^2}\Big|_{\xi_i^-} = c_{i+1}\frac{\mathrm{d}^2u}{\mathrm{d}x^2}\Big|_{\xi_i^+}, \tag{3.9}$$

$$c_i\frac{\mathrm{d}^3u}{\mathrm{d}x^3}\Big|_{\xi_i^-} = c_{i+1}\frac{\mathrm{d}^3u}{\mathrm{d}x^3}\Big|_{\xi_i^+}, \tag{3.10}$$

where $i = 0, 1, \ldots, n-1$, and ξ_i^- and ξ_i^+ are used for the left-hand and right-hand derivatives at ξ_i.

3.1.2. *Linear Spline Coefficient.* The additional boundary conditions for solving the variational problem come from the original equation (1.1), or

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\sigma_i\frac{\mathrm{d}^2u}{\mathrm{d}x^2}\Big|_{\xi_i^-} = f(\xi_i), \qquad \frac{\mathrm{d}^2}{\mathrm{d}x^2}\sigma_{i+1}\frac{\mathrm{d}^2u}{\mathrm{d}x^2}\Big|_{\xi_i^+} = f(\xi_i), \tag{3.11}$$

and

$$\frac{\mathrm{d}^4}{\mathrm{d}x^4}\sigma_i\frac{\mathrm{d}^2u}{\mathrm{d}x^2}\Big|_{\xi_i^-} = \frac{\mathrm{d}^2}{\mathrm{d}x^2}f(\xi_i), \qquad \frac{\mathrm{d}^4}{\mathrm{d}x^4}\sigma_{i+1}\frac{\mathrm{d}^2u}{\mathrm{d}x^2}\Big|_{\xi_i^+} = \frac{\mathrm{d}^2}{\mathrm{d}x^2}f(\xi_i), \tag{3.12}$$

$$\sigma_i \frac{\mathrm{d}^2 u}{\mathrm{d}x^2}\Big|_{\xi_i^-} = \sigma_{i+1}\frac{\mathrm{d}^2 u}{\mathrm{d}x^2}\Big|_{\xi_i^+}, \qquad \frac{\mathrm{d}}{\mathrm{d}x}\sigma_i \frac{\mathrm{d}^2 u}{\mathrm{d}x^2}\Big|_{\xi_i^-} = \frac{\mathrm{d}}{\mathrm{d}x}\sigma_{i+1}\frac{\mathrm{d}^2 u}{\mathrm{d}x^2}\Big|_{\xi_i^+}, \tag{3.13}$$

for $i = 1, 2, \dots, n-1$.

3.1.3. *Cubic Spline Coefficient.* Similarly to the case of a linear spline coefficient, the additional boundary conditions come from the original problem, or

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\sigma_i \frac{\mathrm{d}^2 u}{\mathrm{d}x^2}\bigg|_{\xi_i^-} = f(\xi_i^-), \qquad \frac{\mathrm{d}^2}{\mathrm{d}x^2}\sigma_i \frac{\mathrm{d}^2 u}{\mathrm{d}x^2}\bigg|_{\xi_i^+} = f(\xi_i^+), \tag{3.14}$$

where $i = 0, 1, \dots, n$.

The following conditions can be added as well,

$$\frac{\mathrm{d}^4}{\mathrm{d}x^4}\sigma_i \frac{\mathrm{d}^2 u}{\mathrm{d}x^2}\bigg|_{\xi_i^-} = \frac{\mathrm{d}^2}{\mathrm{d}x^2} f(\xi_i^-), \qquad \frac{\mathrm{d}^4}{\mathrm{d}x^4}\sigma_i \frac{\mathrm{d}^2 u}{\mathrm{d}x^2}\bigg|_{\xi_i^+} = \frac{\mathrm{d}^2}{\mathrm{d}x^2} f(\xi_i^+), \tag{3.15}$$

$$\sigma_i \frac{\mathrm{d}^2 u}{\mathrm{d}x^2}\bigg|_{\xi_i^-} = \sigma_{i+1}\frac{\mathrm{d}^2 u}{\mathrm{d}x^2}\bigg|_{\xi_i^+}, \qquad \frac{\mathrm{d}}{\mathrm{d}x}\sigma_i \frac{\mathrm{d}^2 u}{\mathrm{d}x^2}\bigg|_{\xi_i^-} = \frac{\mathrm{d}}{\mathrm{d}x}\sigma_{i+1}\frac{\mathrm{d}^2 u}{\mathrm{d}x^2}\bigg|_{\xi_i^+}, \tag{3.16}$$

where $i = 1, 2, \dots, n-1$.

3.2. **Euler-Lagrange Equation for the Coefficient.** First, we derive the equations in the general case, namely when the coefficient is a piece-wise cubic polynomial function. Next, we present the two special cases of piece-wise constant and piece-wise linear coefficients.

3.2.1. *General Case (piece-wise cubic polynomial coefficient).* Since $\sigma(x)$ is a piece-wise cubic polynomial function, the functional $\mathcal{I}$ becomes

$$\begin{aligned}
\mathcal{I}(u,\sigma) &= \sum_{i=1}^{n} \int\limits_{\xi_{i-1}}^{\xi_i} \left[\frac{\mathrm{d}^2}{\mathrm{d}x^2}\sigma_i \frac{\mathrm{d}^2 u}{\mathrm{d}x^2} - f(x)\right]^2 \mathrm{d}x \\
&= \sum_{i=1}^{n} \int\limits_{\xi_{i-1}}^{\xi_i} \left(\frac{\mathrm{d}^2}{\mathrm{d}x^2}\sum_{k=0}^{3}\left[b_{ki}(x-\xi_{i-1})^k\right]\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} - f(x)\right)^2 \mathrm{d}x \\
&= \sum_{i=1}^{n}\left(A_0^i + 2\sum_{k=0}^{3} A_{k+1}^i b_{ki} + \sum_{k=0}^{3}\sum_{l=0}^{3} A_{kl}^i b_{ki} b_{li}\right),
\end{aligned} \tag{3.17}$$

where the coefficients A_k^i and A_{kl}^i represent integrals from the function $u(x)$ and its derivatives.

After some algebraic manipulations we find A_k^i, $k = 0, 1, 2, 3, 4$, or

$$A_0^i = \int\limits_{\xi_{i-1}}^{\xi_i} f^2 \mathrm{d}x,$$

$$A_1^i = -\int\limits_{\xi_{i-1}}^{\xi_i} u^{iv} f \mathrm{d}x,$$

$$A_2^i = -\int\limits_{\xi_{i-1}}^{\xi_i} \left[(x-\xi_{i-1})u^{iv} + 2u'''\right] f \mathrm{d}x,$$

$$A_3^i = -\int\limits_{\xi_{i-1}}^{\xi_i} \left[(x-\xi_{i-1})^2 u^{iv} + 4(x-\xi_{i-1})u''' + 2u''\right] f \mathrm{d}x,$$

$$A_4^i = -\int\limits_{\xi_{i-1}}^{\xi_i} \left[(x-\xi_{i-1})^3 u^{iv} + 6(x-\xi_{i-1})^2 u''' + 6(x-\xi_{i-1})u''\right] f \mathrm{d}x.$$

The expressions for A_{kl}^i, $k,l = 0,1,2,3$, are

$$A_{00}^i = \int\limits_{\xi_{i-1}}^{\xi_i} \left(u^{iv}\right)^2 \mathrm{d}x,$$

$$A_{01}^i = 2\int\limits_{\xi_{i-1}}^{\xi_i} (x-\xi_{i-1})(u^{iv})^2 \mathrm{d}x + 2(u''')^2,$$

$$A_{02}^i = 2\int\limits_{\xi_{i-1}}^{\xi_i} \left[(x-\xi_{i-1})^2(u^{iv})^2 + 2(u'' + 2xu''')u^{iv}\right] \mathrm{d}x - 4\xi_{i-1}(u''')^2,$$

$$A_{03}^i = 2\int\limits_{\xi_{i-1}}^{\xi_i} \left[(x-\xi_{i-1})^3(u^{iv})^2 + 6x(x-2\xi_{i-1})u'''u^{iv} + 6(x-\xi_{i-1})u''u^{iv}\right] \mathrm{d}x + 6\xi_{i-1}^2(u''')^2,$$

$$A_{11}^i = \int\limits_{\xi_{i-1}}^{\xi_i} \left[(x-\xi_{i-1})^2(u^{iv})^2 + 4(xu^{iv} + u''')u'''\right] \mathrm{d}x - 2\xi_{i-1}(u''')^2,$$

$$A_{12}^i = 2\int\limits_{\xi_{i-1}}^{\xi_i} \left[(x-\xi_{i-1})^3(u^{iv})^2 + 6x(x-2\xi_{i-1})u'''u^{iv}\right.$$
$$\left. + 2(x-\xi_{i-1})u''u^{iv} + 8(x-\xi_{i-1})(u''')^2\ \right] \mathrm{d}x + 6\xi_{i-1}^2(u''')^2 + 4(u'')^2,$$

$$A_{13}^i = 2\int\limits_{\xi_{i-1}}^{\xi_i} [\ (x-\xi_{i-1})^4(u^{iv})^2 + 8[(x-\xi_{i-1})^3 + \xi_{i-1}^3]u'''u^{iv}$$
$$+ 6(x-\xi_{i-1})^2[2(u''')^2 + u''u^{iv}] + 12xu''u'''\ ] \mathrm{d}x - 8\xi_{i-1}^3(u''')^2 - 12\xi_{i-1}(u'')^2,$$

$$A_{22}^i = \int_{\xi_{i-1}}^{\xi_i} \Big[(x-\xi_{i-1})^4 (u^{iv})^2 + 8[(x-\xi_{i-1})^3 + \xi_{i-1}^3] u''' u^{iv} + 4(x-\xi_{i-1})^2$$
$$[4(u''')^2 + u'' u^{iv}] + 4(u'' + 4xu''')u''\Big] \, \mathrm{d}x - 4\xi_{i-1}^3 (u''')^2 - 8\xi_{i-1}(u'')^2,$$

$$A_{23}^i = 2\int_{\xi_{i-1}}^{\xi_i} \Big[(x-\xi_{i-1})^5 (u^{iv})^2 + 10[(x-\xi_{i-1})^4 - \xi_{i-1}^4] u''' u^{iv}$$
$$+ 8(x-\xi_{i-1})^3 [3(u''')^2 + u'' u^{iv}] + 36x(x-2\xi_{i-1})u''u'''\Big] \, \mathrm{d}x$$
$$+ 12(x-\xi_{i-1})(u'')^2 + 10\xi_{i-1}^4 (u''')^2 + 36\xi_{i-1}^2 (u'')^2,$$

$$A_{33}^i = \int_{\xi_{i-1}}^{\xi_i} \Big[(x-\xi_{i-1})^6 (u^{iv})^2 + 12[(x-\xi_{i-1})^5 + \xi_{i-1}^5] u''' u^{iv} + 36(x-\xi_{i-1})^4 (u''')^2$$
$$+72[(x-\xi_{i-1})^3 + \xi_{i-1}^3] u'' u''' + 36(x-\xi_{i-1})^2 (u'')^2\Big] \, \mathrm{d}x - 6\xi_{i-1}^5 (u''')^2 - 36\xi_{i-1}^3 (u'')^2.$$

We arrive to the problem for minimization of the function

$$\sum_{i=1}^{n} \left(A_0^i + 2\sum_{k=0}^{3} A_{k+1}^i b_{ki} + \sum_{k=0}^{3}\sum_{l=0}^{3} A_{kl}^i b_{ki} b_{li} \right) \tag{3.18}$$

with respect to b_{kj}, $k = 0, 1, 2, 3$, $j = 0, 1, \ldots, n$ under the continuity conditions (2.7), which we rewrite in the form

$$\begin{aligned} & b_{0i} + b_{1i}(\xi_i - \xi_{i-1}) + b_{2i}(\xi_i - \xi_{i-1})^2 + b_{3i}(\xi_i - \xi_{i-1})^3 - b_{0\ i+1} = 0, \\ & b_{1i} + 2b_{2i}(\xi_i - \xi_{i-1}) + 3b_{3i}(\xi_i - \xi_{i-1})^2 - b_{1\ i+1} = 0, \\ & b_{2i} + 3b_{3i}(\xi_i - \xi_{i-1}) - b_{2\ i+1} = 0. \end{aligned} \tag{3.19}$$

Using the standard way for minimization of the quadratic function under the constraints (3.19), we introduce Lagrange multipliers μ_{ki} and consider the following function

$$\begin{aligned} & Q(\mathbf{b}_0, \mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mu_1, \mu_2, \mu_3) \\ & = \sum_{i=1}^{n} \left(A_0^i + 2\sum_{k=0}^{3} A_{k+1}^i b_{ki} + \sum_{k=0}^{3}\sum_{l=0}^{3} A_{kl}^i b_{ki} b_{li} \right) \\ & + \sum_{i=1}^{n-1} \mu_{1i} \left[b_{0i} + b_{1i}(\xi_i - \xi_{i-1}) + b_{2i}(\xi_i - \xi_{i-1})^2 + b_{3i}(\xi_i - \xi_{i-1})^3 - b_{0\ i+1} \right] \\ & + \sum_{i=1}^{n-1} \mu_{2i} \left[b_{1i} + 2b_{2i}(\xi_i - \xi_{i-1}) + 3b_{3i}(\xi_i - \xi_{i-1})^2 - b_{1\ i+1} \right) \\ & + \sum_{i=1}^{n-1} \mu_{3i} \left[b_{2i} + 3b_{3i}(\xi_i - \xi_{i-1}) - b_{2\ i+1} \right], \end{aligned} \tag{3.20}$$

where $A_{kl} = A_{lk}$, $\mathbf{b}_k = (b_{k1}, b_{k2}, \ldots, b_{kn})^T$, $k = 0, 1, 2, 3$ and $\mu_l = (\mu_{l1}, \mu_{l2}, \ldots, \mu_{ln})^T$, $l = 1, 2, 3$.

The necessary conditions for minimization of (3.20) are

$$\frac{\partial Q}{\partial b_{0i}} = 0, \quad \frac{\partial Q}{\partial b_{1i}} = 0, \quad \frac{\partial Q}{\partial b_{2i}} = 0, \quad \frac{\partial Q}{\partial b_{3i}} = 0, \quad \frac{\partial Q}{\partial \mu_{li}} = 0. \tag{3.21}$$

The resulting system of equations (3.21) for b_{ki} and μ_{li} is a multi-diagonal system of linear equations

$$\begin{aligned}
&2A_1^i + 2\sum_{l=1}^{3} A_{0l}^i b_{li} + \mu_{1i} - \mu_{1i+1} = 0,\\
&2A_{21}^i + 2\sum_{l=1}^{3} A_{1l}^i b_{li} + \mu_{1i}(\xi_i - \xi_{i-1}) + \mu_{2i} - \mu_{2i+1} = 0,\\
&2A_3^i + 2\sum_{l=1}^{3} A_{2l}^i b_{li} + \mu_{1i}(\xi_i - \xi_{i-1})^2 + 2\mu_{2i}(\xi_i - \xi_{i-1}) + \mu_{3i} - \mu_{3i+1} = 0,\\
&2A_4^i + 2\sum_{l=1}^{3} A_{3l}^i b_{li} + \mu_{1i}(\xi_i - \xi_{i-1})^3 + 3\mu_{2i}(\xi_i - \xi_{i-1})^2 + 3\mu_{3i}(\xi_i - \xi_{i-1}) = 0,\\
&\sigma_i^{(l)}(\xi_i) - \sigma_{i+1}^{(l)}(\xi_i) = 0.
\end{aligned} \tag{3.22}$$

3.2.2. *Piece-wise Constant Coefficient ($b_{1i} = b_{2i} = b_{3i} = 0$).* Since $\sigma(x)$ is a piece-wise constant function; for the functional $\mathcal{I}$,

$$\begin{aligned}
\mathcal{I}(u,\sigma) &= \sum_{i=1}^{n} \int_{\xi_{i-1}}^{\xi_i} \left[c_i \frac{\mathrm{d}^4 u}{\mathrm{d}x^4} - f(x)\right]^2\\
&= \sum_{i=1}^{n}\left[c_i^2 \int_{\xi_{i-1}}^{\xi_i} (u^{iv})^2 \mathrm{d}x - 2c_i \int_{\xi_{i-1}}^{\xi_i} u^{iv} f \mathrm{d}x + \int_{\xi_{i-1}}^{\xi_i} f^2 \mathrm{d}x\right].
\end{aligned} \tag{3.23}$$

After fairly obvious manipulations the equation for the constant c_i, from the definition of $\sigma(x)$, namely equation (2.3), adopts the form:

$$c_i = \frac{\int_{\xi_i}^{\xi_{i+1}} u^{iv} f \mathrm{d}x}{\int_{\xi_i}^{\xi_{i+1}} (u^{iv})^2 \mathrm{d}x}, \tag{3.24}$$

$i = 1, \ldots, n$.

3.2.3. *Piece-wise Linear Coefficient ($b_{2i} = b_{3i} = 0$).* Since $\sigma(x)$ is a piece-wise linear function, for the functional $\mathcal{I}$ we have

$$\mathcal{I}(u,\sigma) = \sum_{i=1}^{n} \int_{\xi_{i-1}}^{\xi_i} \left[\frac{\mathrm{d}^2}{\mathrm{d}x^2} \sigma_i \frac{\mathrm{d}^2 u}{\mathrm{d}x^2} - f(x)\right]^2 \mathrm{d}x$$

$$= \sum_{i=1}^{n} \int\limits_{\xi_{i-1}}^{\xi_i} \left[\frac{\mathrm{d}^2}{\mathrm{d}x^2} \left(a_i + b_i(x - \xi_{i-1}) \right) \frac{\mathrm{d}^2 u}{\mathrm{d}x^2} - f(x) \right]^2 \mathrm{d}x$$

$$= \sum_{i=1}^{n} \int\limits_{\xi_{i-1}}^{\xi_i} \left[2b_i u''' + \left(a_i + b_i(x - \xi_{i-1}) \right) u^{iv} - f(x) \right]^2 \mathrm{d}x$$

$$= \sum_{i=1}^{n} \Bigg[\int\limits_{\xi_{i-1}}^{\xi_i} f^2 \mathrm{d}x - 2a_i \int\limits_{\xi_{i-1}}^{\xi_i} u^{iv} f \mathrm{d}x + b_i \int\limits_{\xi_{i-1}}^{\xi_i} \left(2\xi_{i-1} f u^{iv} - 4 f u''' - 2x f u^{iv} \right) \mathrm{d}x$$

$$+ a_i^2 \int\limits_{\xi_{i-1}}^{\xi_i} (u^{iv})^2 \mathrm{d}x + a_i b_i \int\limits_{\xi_{i-1}}^{\xi_i} \left(2x u^{iv} - 2\xi_i u^{iv} + 2u''' \right) \mathrm{d}x$$

$$+ b_i^2 \int\limits_{\xi_{i-1}}^{\xi_i} \left[4x u''' u^{(4)} + 4(u''')^2 + (x - \xi_{i-1})^2 (u^{iv})^2 - 2\xi_{i-1} (u''')^2 \right] \mathrm{d}x \Bigg] .$$

Introducing the notations

$$A_0^i = \int\limits_{\xi_{i-1}}^{\xi_i} f^2 \mathrm{d}x, \qquad A_1^i = -2 \int\limits_{\xi_{i-1}}^{\xi_i} u^{iv} f \mathrm{d}x, \tag{3.25}$$

$$A_2^i = \int\limits_{\xi_{i-1}}^{\xi_i} \left(2\xi_{i-1} f u^{iv} - 4 f u''' - 2x f u^{iv} \right) \mathrm{d}x, \tag{3.26}$$

and

$$A_{11}^i = \int\limits_{\xi_{i-1}}^{\xi_i} (u^{iv})^2 \mathrm{d}x, \qquad A_{12}^i = \sum_{i=1}^{n} \int\limits_{\xi_{i-1}}^{\xi_i} \left(x u^{iv} - x u^{iv} + u''' \right) \mathrm{d}x, \tag{3.27}$$

$$A_{22}^i = \sum_{i=1}^{n} \int\limits_{\xi_{i-1}}^{\xi_i} \left(4x u''' u^{iv} + 4(u''')^2 + (x - \xi_{i-1})^2 (u^{iv})^2 - 2\xi_{i-1} (u''')^2 \right) \mathrm{d}x, \tag{3.28}$$

one arrives at the problem for minimization of the function

$$q(a_1, \dots, a_n, b_1, \dots, b_n) = \sum_{i=1}^{n} (A_{11}^i a_i^2 + 2A_{12}^i a_i b_i + A_{22}^i b_i^2 + A_1^i a_i + A_2^i b_i + A_0^i), \tag{3.29}$$

with respect to $a_1, \dots, a_n, b_1, \dots, b_n$ under the continuity conditions (2.7) which we rewrite in the form

$$a_i + b_i(\xi_i - \xi_{i-1}) - a_{i+1} = 0. \tag{3.30}$$

Using the standard way for minimization the function q under the constraints (3.30) we introduce Lagrange multipliers μ_i and consider the following function

$$Q(a_1,\dots,a_n,b_1,\dots,b_n,\mu_1,\dots,\mu_n) = \sum_{i=1}^{n}\left(A_{11}^i a_i^2 + 2A_{12}^i a_i b_i + A_{22}^i b_i^2 + A_1^i a_i + A_2^i b_i + A_0^i\right) + \sum_{i=1}^{n-1}\mu_i\left(a_i + b_i(\xi_i-\xi_{i-1}) - a_{i+1}\right). \tag{3.31}$$

We obtain the following five diagonal system of linear equations for a_i, b_i and μ_i:

$$\frac{\partial Q}{\partial a_1} = 2A_{11}^1 a_1 + 2A_{12}^1 b_1 + A_1^1 + \mu_1 = 0, \tag{3.32}$$

$$\frac{\partial Q}{\partial b_1} = 2A_{12}^1 a_1 + 2A_{22}^1 b_1 + A_2^1 + \mu_1 = 0, \tag{3.33}$$

$$\frac{\partial Q}{\partial \mu_i} = a_1 + b_1(\xi_1-\xi_0) - a_2 = 0, \tag{3.34}$$

and

$$\frac{\partial Q}{\partial a_i} = 2A_{11}^i a_i + 2A_{12}^i b_i + A_1^i + \mu_i - \mu_{i-1} = 0, \tag{3.35}$$

$$\frac{\partial Q}{\partial b_i} = 2A_{12}^i a_i + 2A_{22}^i b_i + A_2^i + \mu_i = 0, \tag{3.36}$$

$$\frac{\partial Q}{\partial \mu_i} = a_i + b_i(\xi_i-\xi_{i-1}) - a_{i+1} = 0, \tag{3.37}$$

for $i = 2,\dots,n-1$, and

$$\frac{\partial Q}{\partial a_n} = 2A_{11}^n a_n + 2A_{12}^n b_n + A_1^n - \mu_{n-1} = 0, \tag{3.38}$$

$$\frac{\partial Q}{\partial b_n} = 2A_{12}^n a_n + 2A_{22}^n b_n + A_2^n + \mu_n = 0. \tag{3.39}$$

4. Existence and Uniqueness of the Weak Solution

4.1. **Uniqueness for u if σ is given.** In this section we establish the existence and uniqueness of the weak solution of the problem (3.4) under the conditions (1.2)–(2.2), (3.7)–(3.10) if the function $\sigma(x)$ is given. We only consider the case of a piece-wise constant coefficient.

Let us consider now the space $\mathcal{H}(0,1)$ comprised by the functions α, defined in the domain $[0,1]$, and satisfying the following conditions

$$\alpha(0) = \alpha'(0) = \alpha''(0) = \alpha'''(0) = 0 \tag{4.1}$$

$$\alpha(1) = \alpha'(1) = \alpha''(1) = \alpha'''(1) = \alpha(\xi_i) = 0\,, \tag{4.2}$$

for $i = 1,2,\dots,n-1$, and, for a given piecewise function $\sigma(x) > 0$, defined with equation (2.3), and

$$c_i\alpha'(\xi_i^-) = c_{i+1}\alpha'(\xi_i^+), \tag{4.3}$$

$$c_i\alpha''(\xi_i^-) = c_{i+1}\alpha''(\xi_i^+), \tag{4.4}$$

$$c_i\alpha'''(\xi_i^-) = c_{i+1}\alpha'''(\xi_i^+). \tag{4.5}$$

We expect that the functions under consideration are as many time differentiable as necessary. The following scalar product is introduced in $\mathcal{H}(0,1)$

$$[\alpha,\beta] = \int_0^1 \left(\frac{\mathrm{d}^2}{\mathrm{d}x^2}\sigma\frac{\mathrm{d}^2}{\mathrm{d}x^2}\alpha(x)\right)\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2}\sigma\frac{\mathrm{d}^2}{\mathrm{d}x^2}\beta(x)\right)\mathrm{d}x\,. \tag{4.6}$$

The equation (4.6) is a scalar product since for $\sigma(x) > 0$, the equation

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\sigma\frac{\mathrm{d}^2}{\mathrm{d}x^2}\alpha(x) = 0 \tag{4.7}$$

with the homogeneous boundary conditions (4.1)–(4.5) has only a trivial solution, i.e. $[\alpha,\alpha] = 0$ is true only when $\alpha(x,y) \equiv 0$ in $\mathcal{D}$. The space $\mathcal{H}(\mathcal{D})$, with the scalar product (4.6) is a Hilbert space.

Let us introduce a sufficiently differentiable function $\chi(x)$, defined in $(0,1)$, and satisfying the respective conditions (1.2)–(2.2), (3.7)–(3.10). Let us now define the functional

$$F(\Phi) \stackrel{\text{def}}{=} \int_0^1 \left(\frac{\mathrm{d}^2}{\mathrm{d}x^2}\sigma\frac{\mathrm{d}^2}{\mathrm{d}x^2}\chi(x)\right)\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2}\sigma\frac{\mathrm{d}^2}{\mathrm{d}x^2}\Phi(x)\right)\mathrm{d}x - \sum_{i=1}^{n}\int_{\xi_{i-1}}^{\xi_i} c_i f(x)\left(\frac{\mathrm{d}^4}{\mathrm{d}x^4}\Phi(x)\right)\mathrm{d}x, \tag{4.8}$$

where $\Phi \in \mathcal{H}(0,1)$. Following the Riesz Representation Theorem, for the continuous linear functional F on the Hilbert space $\mathcal{H}$, there is a unique $v \in \mathcal{H}$ such that

$$F(\Phi) = -[v,\Phi] \text{ for all } \Phi \in \mathcal{H}(0,1). \tag{4.9}$$

Definition 1. *A generalized (weak) solution of the problem* (3.5), (1.2)–(2.2), (3.7)–(3.10), *is defined as the function* $u := v + \chi$.

Therefore, for the weak solution u the following expression holds true

$$\begin{aligned}\int_0^1 &\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2}\sigma\frac{\mathrm{d}^2}{\mathrm{d}x^2}u(x)\right)\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2}\sigma\frac{\mathrm{d}^2}{\mathrm{d}x^2}\Phi(x)\right)\mathrm{d}x \\ &= \int_0^1 \left(\frac{\mathrm{d}^2}{\mathrm{d}x^2}\sigma\frac{\mathrm{d}^2}{\mathrm{d}x^2}(v(x)+\chi(x))\right)\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2}\sigma\frac{\mathrm{d}^2}{\mathrm{d}x^2}\Phi(x)\right)\mathrm{d}x \\ &= \sum_{i=1}^{n}\int_{\xi_{i-1}}^{\xi_i} c_i f(x)\left(\frac{\mathrm{d}^4}{\mathrm{d}x^4}\Phi(x)\right)\mathrm{d}x\,, \text{ for all } \Phi \in \mathcal{H}(\mathcal{D}).\end{aligned} \tag{4.10}$$

If the classical solution of (3.4), (1.2)–(2.2), (3.7)–(3.10) exists, it is also a weak solution. We multiply the equation (3.4) by $\Phi \in \mathcal{H}$, and integrate over the domain

$(0,1)$ to obtain

$$\int_0^1 \left(\frac{\mathrm{d}^2}{\mathrm{d}x^2}\sigma\frac{\mathrm{d}^4}{\mathrm{d}x^4}\sigma\frac{\mathrm{d}^2 u}{\mathrm{d}x^2}\right)\Phi(x)\mathrm{d}x = \int_0^1 \left(\frac{\mathrm{d}^2}{\mathrm{d}x^2}\sigma\frac{\mathrm{d}^2}{\mathrm{d}x^2}f(x)\right)\Phi(x)\mathrm{d}x. \tag{4.11}$$

Integrating by parts over the intervals $\xi_{i-1} < x < \xi_i$, and acknowledging the conditions for u, (1.2)–(2.2), (3.7)–(3.10), and Φ, (4.1), (4.5), this becomes

$$\int_0^1 \left(\frac{\mathrm{d}^2}{\mathrm{d}x^2}\sigma\frac{\mathrm{d}^2}{\mathrm{d}x^2}u(x)\right)\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2}\sigma\frac{\mathrm{d}^2}{\mathrm{d}x^2}\Phi(x)\right)\mathrm{d}x = \sum_{i=1}^{n}\int_{\xi_{i-1}}^{\xi_i} c_i f(x)\left(\frac{\mathrm{d}^4}{\mathrm{d}x^4}\Phi(x)\right)\mathrm{d}x\,. \tag{4.12}$$

Theorem 1. *The weak solution of the problem* (3.5), (1.2)–(2.2), (3.7)–(3.10) *is unique.*

In order to prove the uniqueness, we consider the difference $\hat{u} = u_1 - u_2$ between two supposed solutions u_1 and u_2. It is obvious that $\hat{u} \in \mathcal{H}(0,1)$. On the other hand, from equation (4.10), we obtain that

$$\int_0^1 \left(\frac{\mathrm{d}^2}{\mathrm{d}x^2}\sigma\frac{\mathrm{d}^2}{\mathrm{d}x^2}\hat{u}(x)\right)\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2}\sigma\frac{\mathrm{d}^2}{\mathrm{d}x^2}\Phi(x)\right)\mathrm{d}x = 0. \tag{4.13}$$

holds for $\hat{u}$. Then, simply taking $\Phi \equiv \hat{u}$, we have $[\hat{u},\hat{u}] = 0$ and then $\hat{u} \equiv 0$.

Thus, we have shown that the Euler-Lagrange equation (3.4) possesses a unique solution under the boundary conditions (1.2)–(2.2), (3.7)–(3.10) provided that the coefficient $\sigma(x) > 0$ is given.

4.2. **Correctness of the problem for σ.** Since equation (3.24) is an explicit expression for the coefficients, it provides a unique solution for $\sigma(x)$ when the function $u(x)$ is thought of as known.

4.3. **Existence of a Solution to the Full MVI Problem.** Up to this point, we have shown that the two Euler-Lagrange equations (3.4) and (3.24) for $u(x)$ and $\sigma(x)$ possess unique solutions, provided that in each of them the other function is thought of as known. This allows one to construct a procedure for finding a solution to the full nonlinear problem by means of iterations replacing $\sigma(x)$ (when calculating u), or $u(x)$ (when calculating σ) with their values calculated at the previous iteration.

If the iterations converge, then they will give one of the possible solutions of the problem. Thus, the existence of the solution to the identification problem can be established *a-posteriori*. In the light of what has been shown above in this section, one can conclude that divergence of the global iteration will necessarily mean that there exists no solution to the identification problem.

The convergence of the iterations, however, secure only the existence of the solution. It may not be unique, and the iterations can converge to different solutions depending on the initial guess for the functions $u(x,y)$ and $\sigma(x)$. This reflects the physical nature of the problem since one cannot expect to recover the exact shape of an object behind a translucent screen using the shapes seen on the screen. There can exist objects that differ from each other in constitution, but throw similar shadow on the screen. Regardless to this limitation, the approach based on the MVI is a very useful tool, that allows one to find at least one possible coefficient that is consistent with the over-posed data. In order to limit the uncertainty of the coefficients estimation, it is possible to incorporate additional restrictions on $\sigma(x)$ based on additional physical information, but they go beyond the framework of the present study.

5. **Difference Scheme**

We solve the formulated eight-order boundary value problem using finite differences. It is convenient for the numerical treatment to rewrite the eight order equation (3.4) as a system of two fourth order equations. In each of the subintervals $[\xi_{i-1}, \xi_i]$, $i = 1, 2, \ldots, n$ we solve the following system of two equations

$$\frac{\partial^2}{\partial x^2}\left(\sigma_i \frac{\partial^2 u}{\partial x^2}\right) = v, \qquad \frac{\partial^2}{\partial x^2}\left(\sigma_i \frac{\partial^2 v}{\partial x^2}\right) = \frac{\partial^2}{\partial x^2}\left(\sigma_i \frac{\partial^2 f}{\partial x^2}\right). \tag{5.1}$$

5.1. **Grid and approximations.** We introduce a regular mesh with step h_i (see Fig. 1) in each of the subintervals $[\xi_{i-1}, \xi_i]$, $i = 1, 2, \ldots, n$, allowing to approximate all operators with standard central differences with second order of approximation.

FIGURE 1. The mesh used in our numerical experiments.

For the grid spacing in the interval $[\xi_{i-1}, \xi_i]$ we have $h_i \equiv \frac{1}{n_i - 2}$, where n_i is the total number of grid points in the i-th interval. Then, the grid points are defined as follows: $x_j^i = (j - 1.5)h_i$ for $j = 1, 2, \ldots, n_i$. Let us introduce the notation $u_j^i = u(x_j^i)$ for $i = 1, 2, \ldots, n$, and $j = 1, \ldots, n_i$. We employ symmetric central differences for approximating the differential operators as follows:

$$\left.\frac{\mathrm{d}^2 u}{\mathrm{d}x^2}\right|_{x = x_j^i} = \frac{u_{j-1}^i - 2u_j^i + u_{j+1}^i}{h_i^2} + O(h^2), \tag{5.2}$$

for $i = 1, 2, \ldots n$ and $j = 2, \ldots n_i - 1$. We approximate the differential operators in the boundary conditions by second order formula using central differences and half sums.

5.2. **Algorithm.**

(I): With the obtained experimentally observed values of $\alpha_{k,l}$, (for $k = 0, 1$ and $l = 0, 1, 2, 3$), and γ_i, (for $i = 1, 2, \dots, n-1$) the eight-order boundary value problem (3.3), (1.2)–(2.2), (3.7)–(3.10) is solved for the function u with an initial guess for the function σ.

(II): The current iteration for the function $\sigma(x)$ is calculated from the system (3.21). If the difference between the new and the old field for σ is less than ε_0 then the calculations are terminated. Otherwise, the algorithm returns to **(I)** with the new calculated $\sigma(x)$.

6. **Numerical Experiments**

The accuracy of the difference scheme developed here is validated with tests involving various grid spacing h. We conducted a number of calculations with different values of the mesh parameters and verified the practical convergence and the $O(h^2)$ approximation of the difference scheme.

To illustrate the numerical implementation of the MVI we present here several coefficient identification problems.

6.1. **Constant coefficient as a constant function.** Consider the case when $\sigma(x) \equiv 1$ and

$$f(x) = \exp(x) + x^2 + x + 1. \tag{6.1}$$

Then, under proper boundary conditions, the exact solution is

$$u(x) = \exp(x) + \frac{x^6}{360} + \frac{x^5}{120} + \frac{x^4}{24}. \tag{6.2}$$

For this test we keep the number of intervals n in the definition of σ (2.3), equals to 1, i.e. $n = 1$. In other word, we know *a-priori* that the coefficient is constant. The goal of this test is to confirm second order of approximation of the proposed scheme.

The values of the identified coefficient σ with four different steps h are given in Table 1. The rate of convergence, calculated as

$$\text{rate} = \log_2 \left| \frac{\sigma_h - \sigma_{\text{exact}}}{\sigma_{2h} - \sigma_{\text{exact}}} \right| \tag{6.3}$$

is also shown in Table 1.

Similar results for the l^2 norm of the difference between the exact and the numerical values of the function u are given in Table 2.

This test clearly confirm the second order of convergence of the numerical solution to the exact one.

TABLE 1. Obtained values of the constant σ and the rate of convergence for four different values of the mesh spacing.

h	σ	$\lvert\sigma - \sigma_{\text{exact}}\rvert$	rate
exact	1.0	—	—
0.05	0.99999123093	8.769069934522E-06	—
0.025	0.99999804742	1.952580153119E-06	2.16704
0.0125	0.99999954187	4.581293466810E-07	2.09155
0.00625	0.99999988922	1.107772538145E-07	2.04809

TABLE 2. l^2 norm of the difference $u - u_{\text{exact}}$ and the rate of convergence for four different values of the mesh spacing.

h	$\lvert\lvert u - u_{\text{exact}}\rvert\rvert_{l^2}$	rate
0.05	7.282717992259E-04	—
0.025	1.782538799647E-04	2.03054
0.0125	4.408739248318E-05	2.0155
0.00625	1.096238335563E-05	2.0078

6.2. **Constant coefficient as a piece-wise constant function.** Consider the same solution (6.2) but now we do not assume *a-priori* that the coefficient is a constant in the whole interval. We identify the coefficient as a piecewise function, as defined in (2.3) for $n = 10$. In each subinterval, the expected value of sigma is 1. For this test we performed a number of calculations with different spacings h.

The l^2 norm of the difference between the exact and the numerical values of the functions u and σ, and the rate of convergence, calculated using the norm of the difference, for four different steps h, are given in Table 3.

TABLE 3. l^2 norm of the differences $u - u_{\text{exact}}$ and $\sigma - \sigma_{\text{exact}}$ and the rate of convergence for four different values of the mesh spacing.

h	$\lvert\lvert\sigma - \sigma_{\text{exact}}\rvert\rvert_{l^2}$	rate	$\lvert\lvert u - u_{\text{exact}}\rvert\rvert_{l^2}$	rate
0.1	5.055440424643E-09	—	2.714973635993E-05	—
0.05	1.051200604992E-09	2.2658	6.629752554572E-06	2.03391
0.025	2.346843878445E-10	2.16324	1.637387943545E-06	2.01756
0.0125	5.362505644613E-11	2.12974	4.068185363751E-07	2.00894

The fact that the numerical solution approximate the analytical one with $O(h^2)$ is well seen from the Table 3.

6.3. **Smooth coefficient as a piecewise constant function.** In this example we consider the approximation of a smooth coefficient with a piecewise one. If the coefficient σ is

$$\sigma(x) = x^2 + 1, \tag{6.4}$$

and

$$f(x) = (x^2 + 4x + 3)\exp(x), \tag{6.5}$$

the exact solution, under respective boundary conditions is

$$u(x) = \exp(x). \tag{6.6}$$

Then, with the "experimentally observed" values of $\alpha_{k,l}$ and γ_i obtained from the numerical solution of the direct problem, we solve the inverse problem, restricting σ to a piecewise function. One example of the piecewise coefficient for $n = 5$ is given in figure 2. When the length of the subintervals (ξ_{i-1}, ξ_i) tends to zero (i.e., the number of subintervals n tends to infinity), we expect $O(1/n)$ approximation of the smooth coefficient (6.4), i.e.

$$c_i \to \left(\frac{\xi_{i-1} + \xi_i}{2}\right)^2 + 1 \qquad \text{when} \qquad n \to \infty, \tag{6.7}$$

for $i = 1, 2, 3 \ldots, n$.

For this test we use a fixed number of grid points for each subintervals, i.e. $n_i = 10$, and we vary the number of subintervals n and, respectively, the mesh size $h = 1/(10n)$.

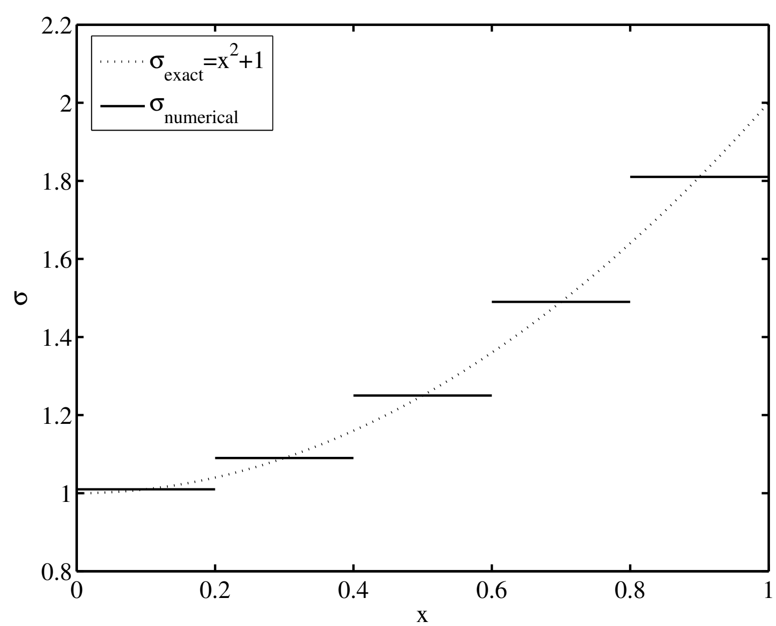

FIGURE 2. The difference between numerical and exact values of solution u for for steps h with ten subintervals.

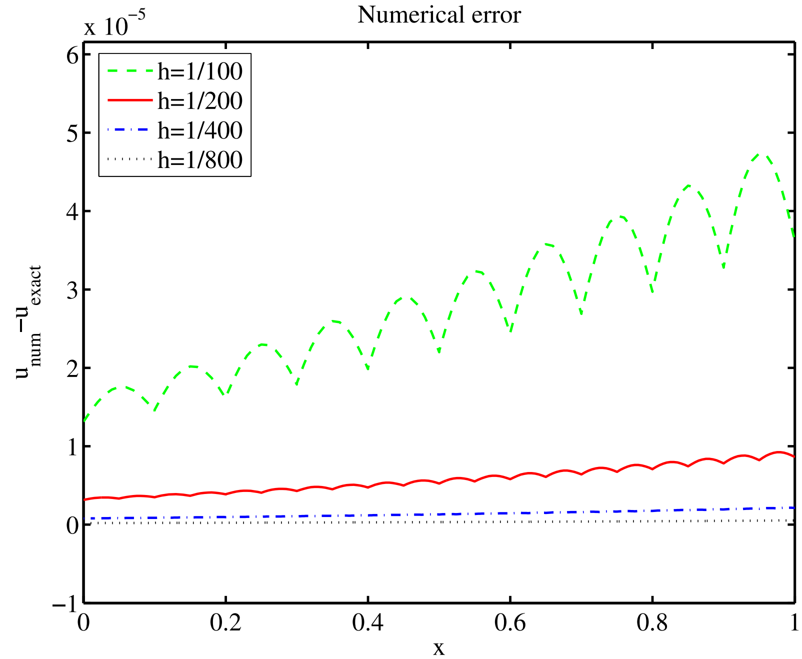

FIGURE 3. The difference between numerical and exact values of solution u for for steps h with ten subintervals.

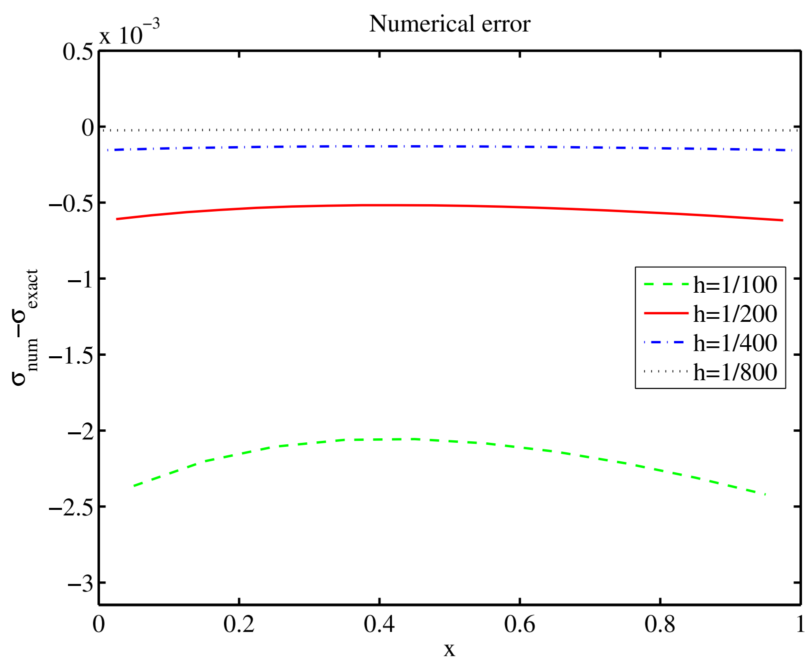

FIGURE 4. The difference between numerical and exact values of solution λ for for steps h with ten subintervals.

The differences between numerical and exact values of the function u for four different spacings: $(h = 1/100; 1/200; 1/400; 1/800)$, are given at Figure 3. The differences between identified values of σ and the exact values of the coefficient are given at figure 4.

The l^2 norm of the difference between exact and numeric values of σ, calculated as

$$||\sigma - \sigma_{\text{exact}}||_{l^2} = \left[\sum_{i=1}^{n} (\xi_i - \xi_{i-1}) \left(\sigma_i - \sigma_{\text{exact}}(\frac{\xi_{i-1} + \xi_i}{2}) \right)^2 \right]^{1/2} \tag{6.8}$$

and the l^2 norm of the difference between exact and numeric values of u are given in Table 4.

TABLE 4. l^2 norm of the differences $u - u_{\text{exact}}$ and $\sigma - \sigma_{\text{exact}}$ and the rate of convergence for four different values of the mesh spacing.

h	$\xi_i - \xi_{i-1}$	$\|\|\sigma - \sigma_{\text{exact}}\|\|_{l^2}$	rate	$\|\|u - u_{\text{exact}}\|\|_{l^2}$	rate
0.01	0.1	2.1992831E-03	—	3.0664219E-05	—
0.005	0.05	5.5359663E-04	1.9901	6.3011971E-06	2.2828
0.0025	0.025	1.3863657E-04	1.9975	1.4920286E-06	2.0783
0.00125	0.0125	3.4674002E-05	1.9993	3.6784289E-07	2.0201

Clearly, a second order of approximation is present in this case, too. This is because of the symmetry with respect to the mid-point $\frac{\xi_{i-1} + \xi_i}{2}$ of the subinterval $[\xi_{i-1}, \xi_i]$.

TABLE 5. Obtained values of the coefficients a and b, and the rate of convergence for four different values of the mesh spacing.

h	a	$\|a - a_{\text{exact}}\|$	rate	b	$\|b - b_{\text{exact}}\|$	rate
exact	1.0	—	—	1.0	—	—
0.1	0.99667	3.3287E-03	—	0.99833	1.6651E-03	—
0.05	0.99917	8.3304E-04	1.998	0.99958	4.1657E-04	1.999
0.025	0.99979	2.0831E-04	1.999	0.99989	1.0416E-04	1.999
0.0125	0.99995	5.2088E-05	1.999	0.99997	2.6038E-05	2.000

6.4. **Linear coefficient as a linear function.** Consider the case when $\sigma(x) = 1+x$ and $f(x) = (3 + x)\exp(x)$. for which under proper boundary conditions the exact solution is

$$u(x) = \exp(x). \tag{6.9}$$

For this test we let the number of intervals n in the definition (2.4) of σ equal to 1, i.e., $n = 1$. In other words, we know *a-priori* that the coefficient is a linear function. The goal of this test is to confirm the second order of approximation of the proposed scheme.

The values of the identified coefficient $\sigma = a + bx$ with four different steps h are given in Table 5. The rates of convergence, calculated as

$$\text{rate} = \log_2 \left| \frac{a_{2h} - a_{\text{exact}}}{a_h - a_{\text{exact}}} \right|, \qquad \text{rate} = \log_2 \left| \frac{b_{2h} - b_{\text{exact}}}{b_h - b_{\text{exact}}} \right|, \tag{6.10}$$

are also shown in Table 5.

TABLE 6. l^2 norm of the difference $u - u_{\text{exact}}$ and the rate of convergence for four different values of the mesh spacing.

h	$\lvert\lvert u - u_{\text{exact}} \rvert\rvert_{l^2}$	rate
0.1	1.018490804573E-04	—
0.05	2.367271717866E-05	2.10514
0.025	5.686319062981E-06	2.05766
0.0125	1.392183281145E-06	2.03015

Similar results for the l^2 norm of the difference between the exact and the numerical values of the function u are presented in Table 6. This test clearly confirms the second order of convergence of the numerical solution to the exact one.

6.5. **Linear coefficient as a linear spline.** Consider again the solution (6.9) but now we do not assume *a-priori* that the coefficient is the same function in the whole interval. We identify the coefficient as a piecewise linear function, as defined in (2.4), for $n = 10$. In each subinterval, the expected values of the coefficient σ are $a_i = 1$ and $b_i = 1$. For this test we performed a number of calculations with different spacings h.

The l^2 norm of the difference between the exact and the numerical values of the functions u and σ, and the rate of convergence, calculated using the norm of the difference, for four different steps h, are given in Table 7. The fact that the numerical solution approximates the analytical one with $O(h^2)$ is clearly seen from the Table 7.

TABLE 7. l^2 norm of the differences $u - u_{\text{exact}}$ and $\sigma - \sigma_{\text{exact}}$ and the rate of convergence for four different values of the mesh spacing.

h	$\lvert\lvert \sigma - \sigma_{\text{exact}} \rvert\rvert_{l^2}$	rate	$\lvert\lvert u - u_{\text{exact}} \rvert\rvert_{l^2}$	rate
0.01	1.0691879709E-08	—	4.5525812486E-05	—
0.005	2.5849565115E-09	2.0483	1.1385994369E-05	1.99942
0.0025	6.3798062289E-10	2.01856	9.6146829421E-07	3.56588
0.00125	1.5860000355E-10	2.00812	1.6248717134E-07	2.56491

The accuracy of the difference scheme developed here is checked with tests involving different grid spacing h. We conducted a number of calculations with different values of the mesh parameters and verified the practical convergence and the $O(h^2)$ approximation of the difference scheme.

6.6. **Cubic polynomial coefficient as a cubic polynomial.** Consider the case when $\sigma(x) = 1 + \dfrac{x^3}{6}$ and

$$f(x) = (1 + x + x^2 + \frac{x^3}{6}) \exp(x). \tag{6.11}$$

Then, under proper boundary conditions, the exact solution is

$$u(x) = \exp(x). \tag{6.12}$$

For this test we keep the number of intervals n, in the definition (2.6) of $\sigma(x)$, equals to 1. In other words, we know *a-priori* that the coefficient is a cubic function. The goal of this test is to confirm the second order of approximation of the proposed scheme. The l^2 norm of the difference between the identified coefficient $\sigma(x) = a + bx + cx^2 + dx^3$ and the exact one with four different steps h are given in Table 8.

The rate of convergence, calculated as

$$\text{rate} = \log_2 \frac{||\sigma_h - \sigma_{\text{exact}}||}{||\sigma_{2h} - \sigma_{\text{exact}}||}, \tag{6.13}$$

is also shown in Table 8. Similar results for the l^2 norm of the difference between the exact and the numerical values of the function u are also given in Table 8.

TABLE 8. l^2 norm of the differences $u - u_{\text{exact}}$ and $\sigma - \sigma_{\text{exact}}$ and the rate of convergence for four different values of the mesh spacing.

h	$\|\|\sigma - \sigma_{\text{exact}}\|\|_{l^2}$	rate	$\|\|u - u_{\text{exact}}\|\|_{l^2}$	rate
1/20	4.590357927325E-06	—	9.933506967349E-06	—
1/40	8.978949246507E-07	2.3540	2.353914510277E-06	2.0772
1/80	1.944168621994E-07	2.2074	5.730802221184E-07	2.0383
1/160	4.493541631371E-08	2.1132	1.413518057249E-07	2.0194

The calculated values of the coefficients of the cubic polynomial and the rate of convergence for each coefficient are given in Tables 9 and 10. The point-wise error for calculations above for the solution u is given at Figure 5, and for the coefficient σ at Figure 6, respectively.

This test clearly confirms the second order of convergence of the numerical solution to the exact one.

6.7. **Cubic polynomial coefficient as a cubic spline.** Consider the same solution (6.12) but now we do not assume *a-priori* that the coefficient is a the same function in the whole interval. We identify the coefficient as a piece-wise function, as defined

TABLE 9. The calculated values of the coefficients and the rate of convergence for four different values of the mesh spacing.

h	b_0	rate	b_1	rate
exact	1	—	0	—
1/20	1.00000321241	—	1.3176E-06	—
1/40	1.00000065309	2.2983	2.6429E-07	2.3178
1/80	1.00000014434	2.1778	5.7847E-08	2.1918
1/160	1.00000003372	2.0978	1.3435E-08	2.1062

TABLE 10. The calculated values of the coefficients and the rate of convergence for four different values of the mesh spacing.

h	b_2	rate	b_3	rate
exact	0	—	1/6	—
1/20	7.5951E-07	—	0.166667238350	—
1/40	1.4699E-07	2.3693	0.166666769207	2.4790
1/80	3.1463E-08	2.2240	0.166666687682	2.2867
1/160	7.2155E-09	2.1245	0.166666671373	2.1588

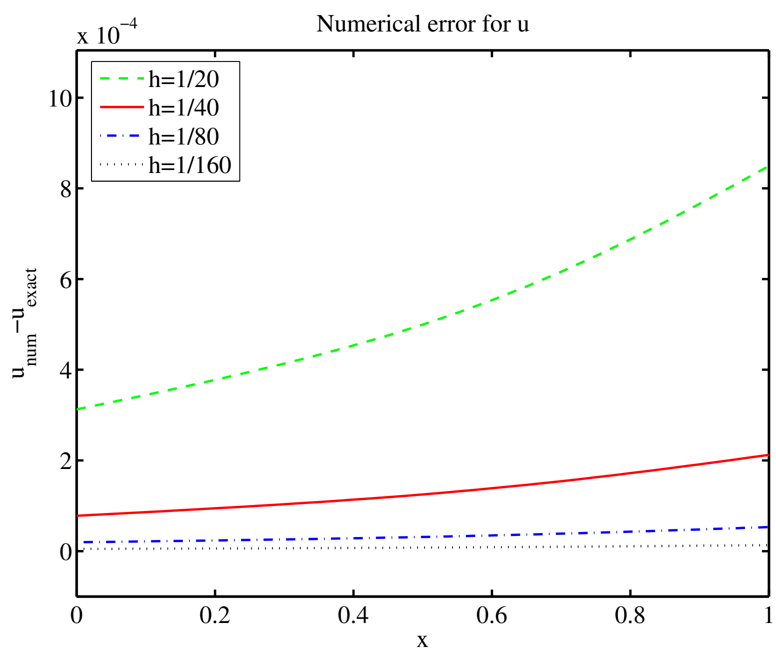

FIGURE 5. The difference between numerical and exact values of solution u for four steps h with one subinterval.

in (2.6) for $n = 10$. In each subinterval, the expected values of the coefficients of the

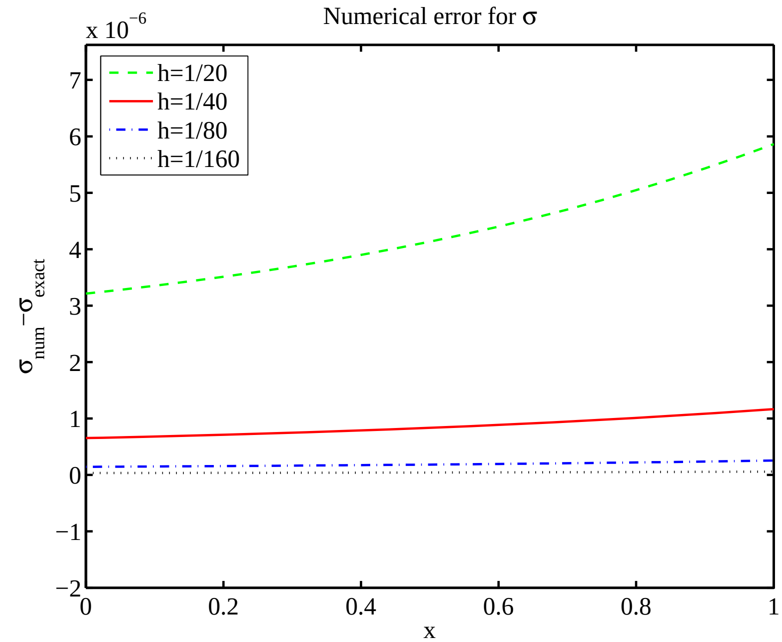

FIGURE 6. The difference between numerical and exact values of solution σ for four steps h with one subinterval.

spline for σ are

$$b_{0i} = 1 + \frac{\xi_{i-1}^3}{6}, \qquad b_{0i} = \frac{\xi_{i-1}^2}{2}, \qquad b_{0i} = \frac{\xi_{i-1}}{2}, \qquad b_{0i} = \frac{1}{6}, \tag{6.14}$$

$i = 1, 2, \dots, 10$, and $\xi_0 = 0$. For this test we performed a number of calculations with different spacings h. The l^2 norm of the difference between the exact and the numerical values of the functions u and σ, and the rate of convergence, calculated using the norm of the difference, for four different steps h, are given in Table 11. The distribution of the numerical error is given at Figure 7 for u, and Figure 8 for σ, respectively. The fact that the numerical solution approximate the analytical one with $O(h^2)$ is clearly seen from the Table 11.

TABLE 11. l^2 norm of the differences $u - u_{\text{exact}}$ and $\sigma - \sigma_{\text{exact}}$ and the rate of convergence for four different values of the mesh spacing with ten subintervals.

n	h	$\|\|\sigma - \sigma_{\text{exact}}\|\|_{l^2}$	rate	$\|\|u - u_{\text{exact}}\|\|_{l^2}$	rate
10	1/100	8.873084982813E-05	—	4.164182330705E-09	—
10	1/200	3.133601202514E-06	4.8235	9.120884527221E-10	2.1908
10	1/400	6.548193482471E-07	2.2587	2.133763950427E-10	2.0958
10	1/800	1.454290193124E-07	2.1708	5.138292772069E-11	2.0540

6.8. **How the number of points ξ_i influence on the accuracy of the numerical solution.** In this section we want to illustrate how the number of points ξ_i influence

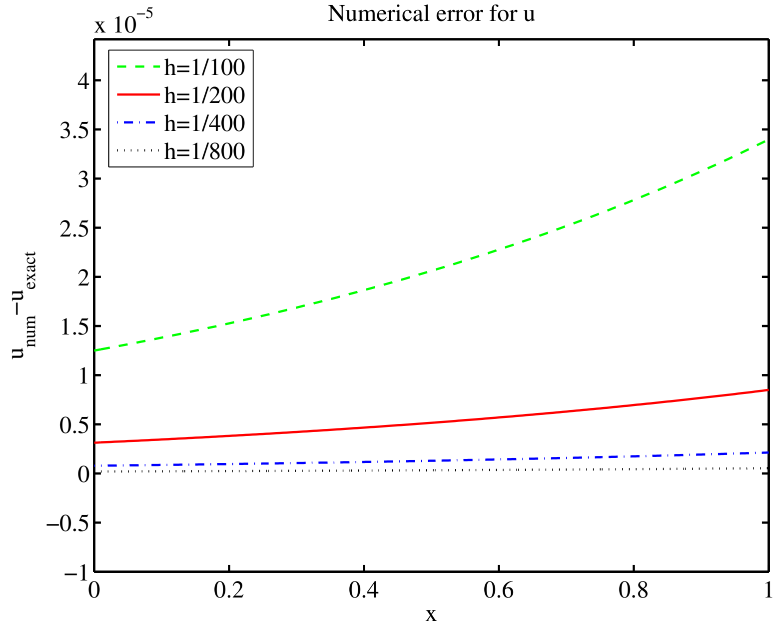

FIGURE 7. The difference between numerical and exact values of solution u for four steps h with ten subinterval.

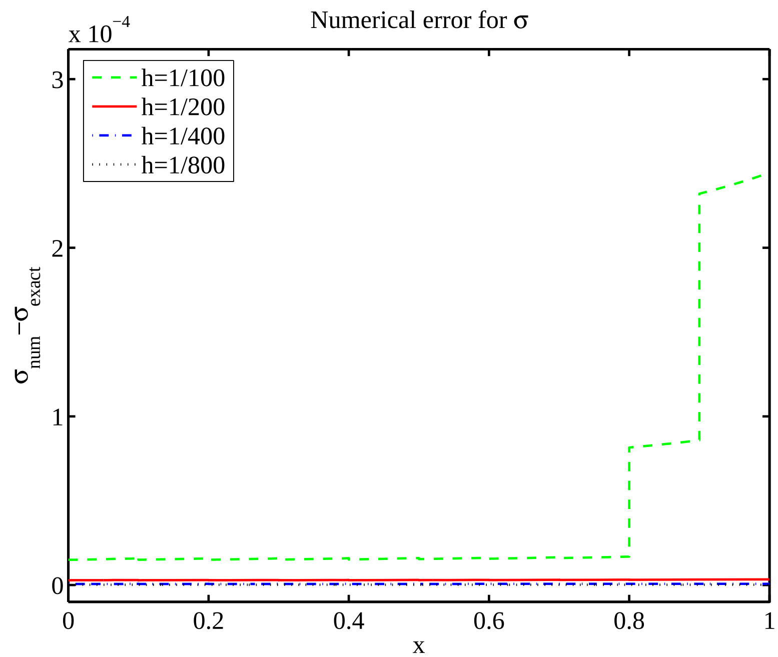

FIGURE 8. The difference between numerical and exact values of solution σ for four steps h with ten subinterval.

on the accuracy of the numerical solution. For this reason we conduct a number of calculations with different number of subintervals n, keeping the number of the nodes inside of each subinterval a constant.

The results from this experiment are given in Table 12. Since we keep the same number of nodes inside the subintervals, the order of approximation of the coefficient does not change during these calculations. On the other hand, because the number of boundary condition increasing along with the number of subinterval, we observe a supper-convergence of the solution u, in fact $O(h^4)$.

TABLE 12. l^2 norm of the differences $u - u_{\text{exact}}$ and $\sigma - \sigma_{\text{exact}}$ and the rate of convergence for four different values of the mesh spacing with ten subintervals.

n	h	$\|\|\sigma - \sigma_{\text{exact}}\|\|_{l^2}$	rate	$\|\|u - u_{\text{exact}}\|\|_{l^2}$	rate
1	1/10	2.677795296187E-05	—	4.439618196535E-05	—
2	1/20	2.011338694503E-05	0.4129	2.651886866020E-06	4.0653
4	1/40	1.831893902020E-05	0.1348	1.634312205571E-07	4.0203
8	1/80	1.760825150581E-05	0.0571	1.017661888639E-08	4.0054

7. Conclusion

To summarize, in the present paper we have displayed the performance of the Method of Variational Imbedding for solving the inverse problem of coefficient identification in Euler-Bernoulli equation from over-posed data. Examples are elaborated numerically through solving the direct problem with given coefficient and preparing the over-posed boundary data for the imbedding problem. The numerical results confirm that the solution of the imbedding problem coincides with the direct simulation of the original problem within the order of approximation error $O(h^2)$.

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