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GENERALIZED MONOTONE ITERATIVE TECHNIQUES FOR CAPUTO FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS WITH INITIAL CONDITION

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ABSTRACT. The purpose of this work is to develop two Monotone Iterative Techniques for a nonlinear integro-differential initial value problem with Caputo derivative.

Before proving the main results we will define different types of coupled lower and upper solutions. In the first theorem we will construct two natural sequences which converge uniformly and monotonically to coupled minimal and maximal solutions. In the second theorem we will construct two intertwined sequences which converge uniformly and monotonically to coupled minimal and maximal solutions. We also establish conditions for uniqueness of the solution.

Finally, we present two examples that illustrate the results obtained.

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1. INTRODUCTION

Fractional Calculus is as old as the "conventional" Calculus, however the study of fractional differential equations was not a popular subject until the last few decades when it was discovered that differential equations involving fractional derivatives frequently provide more accurate models than those with integer derivatives, see the books [4, 5, 7, 16, 17] for more information. A well known technique in the theory of nonlinear ordinary differential equations with initial or boundary conditions is the method of upper and lower solutions, see [6] for further details. In recent years these methods have been applied to fractional differential equations, as it can be found in the book [7] and the papers [2, 3, 9, 10, 11, 12, 13, 18, 19, 20, 21].

On the other hand, the basic theory and several methods for integro-differential equations, including the study of upper and lower solutions, are provided in the book [8], as well as several applications. Moreover, a monotone method was first introduced in [8] for first order ordinary integro-differential equations with periodic boundary conditions and the result was extended in [22].

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In this paper we establish a comparison theorem similar to the one developed in [7] for a Caputo fractional integro-differential equation of order q, $0 < q < 1$, with initial condition. We will define and use coupled lower and upper solutions combined with a generalized monotone iterative technique to prove the existence of coupled minimal and maximal solutions. The results developed provide either natural or intertwined sequences which converge uniformly and monotonically to coupled minimal and maximal solutions of the integro-differential initial value problem.

2. PRELIMINARIES

In this section we state the definitions and results concerning the Riemann–Liouville and Caputo derivatives of fractional order that are required to prove our main result.

We start by stating the definition of the Mittag–Leffler function.

Definition 2.1. The two parameter Mittag–Leffler function is defined as

$$E_{\alpha,\beta}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha k + \beta)},$$

and the one parameter Mittag–Leffler function is defined as

$$E_{\alpha}(t) = E_{\alpha,1}(t).$$

In particular $E_1(t) = e^t$, and $E_{\alpha,\beta}(t)$ is also called the generalized exponential function.

Let $J = [a, b]$ be a finite interval in the real axis $\mathbb{R}$. The definition of Caputo and Riemann–Liouville fractional derivatives are given in [4, 5, 7, 17] as follows.

Definition 2.2. The Riemann-Liouville fractional derivative of order α, where $n-1 \leq \alpha < n$ and $n \in \mathbb{N}$, is denoted by D^{α} and defined by

$$D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^{n} \int_a^t (t-s)^{n-\alpha-1} f(s) ds.$$

Definition 2.3. The Caputo derivative of order $n-1 \leq \alpha < n$ for $t \in [a, b]$, denoted by ${}^{c}D^{\alpha}$, is defined as

$${}^{c}D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} f^{(n)}(s) ds.$$

Consider the nonlinear initial value problem of the form

$$\begin{aligned} {}^{c}D^{q} u(t) &= f\left(t, u(t)\right), \\ u(a) &= u_0. \end{aligned} \tag{2.1}$$

Throughout this work we will consider the Caputo derivative of order q, where $0 < q < 1$.

We recall the following definition.

Definition 2.4. Let $0 < q < 1$ and $p = 1 - q$. If G is an open set in $\mathbb{R}$, then we denote by $C_p([a,b],G)$ the function space

$$C_p([a,b],G) = \left\{ u \in C((a,b],G) \,\middle|\, (t-a)^p u(t) \in C([a,b],G) \right\}.$$

If $u \in C_p([a,b],G)$, then u is said to be C_p continuous in $[a,b]$.

Remark 2.5. In [5] it is shown that if $0 < q < 1$, G is an open set of $\mathbb{R}$, and $f : (a,b] \times G \to \mathbb{R}$ is such that for any $u \in G$, $f \in C_p([a,b],G)$, then u satisfies (2.1) if and only if it satisfies the Volterra fractional integral equation

$$u(t) = u_0 + \frac{1}{\Gamma(q)} \int_a^t (t-s)^{q-1} f(s,u(s))ds. \tag{2.2}$$

In particular, this relationship is true if $f : [a,b] \times G \to \mathbb{R}$ is continuous.

In [7], it was shown that the solution to (2.1) for $f(t,u(t)) = Mu(t) + f(t)$ where M is a real number and $f \in C([a,b],\mathbb{R})$, i.e., the solution of a non homogeneous linear fractional differential equation, is given by

$$u(t) = u_0 E_q\left(M(t-a)^q\right) + \int_a^t (t-s)^{q-1} E_{q,q}\left(M(t-s)^q\right) f(s)ds \quad t \in [a,b], \tag{2.3}$$

where $E_q(t)$ and $E_{q,q}(t)$ are the one parameter and two parameter Mittag–Leffler functions, respectively.

Suppose that $u \in C^1[J,\mathbb{R}]$, $Tu(t) = \int_a^t K(t,s)u(s)ds$, and $K \in C([a,b] \times [a,b],\mathbb{R})$ is a positive function. Since K is continuous, then Tu is continuous and Remark 2.5 can be generalized as follows:

Remark 2.6. The nonlinear integro-differential initial value problem

$$\begin{aligned} {}^cD^q u &= f\left(t,u(t),Tu(t)\right), \\ u(a) &= u_0, \end{aligned} \tag{2.4}$$

is equivalent to the Volterra fractional integral equation

$$u(t) = u_0 + \frac{1}{\Gamma(q)} \int_a^t (t-s)^{q-1} f\left(s,u(s),Tu(s)\right) ds. \tag{2.5}$$

That is, every solution of (2.4) is a solution of (2.5) and viceversa.

Now we are ready to state some comparison results relative to initial value problems with the Caputo derivative. First we state a lemma that was proven in [3] for Riemann–Liouville derivatives.

Lemma 2.7. *Let $m \in C_p([a,b],\mathbb{R})$ and for any $t_1 \in (a,b]$ we have that on (a,t_1), $m(t) \leq 0$, $m(t_1) = 0$ and $m(t)(t-a)^{1-q}|_{t=a} \leq 0$. Then $D^q m(t_1) \geq 0$.*

The above lemma allows us to prove an equivalent result for Caputo derivatives.

Lemma 2.8. *Let $m(t) \in C^1([a,b],\mathbb{R})$. If there exists $t_1 \in [a,b]$ such that $m(t_1) = 0$ and $m(t) \leq 0$ on $[a,t_1]$, then it follows that*

$$^cD^q m(t_1) \geq 0.$$

Proof. Let $t_1 \in [a,b]$, then using the relation between the Caputo derivative and Riemann–Liouville derivative for $n-1 < \alpha \leq n$ given by

$$^cD^\alpha u(t) = D^\alpha \left[u(s) - \sum_{k=0}^{n-1} \frac{u^{(k)}(a)}{k!}(s-a)^k \right](t),$$

we have for $0 < q \leq 1$ that

$$^cD^q m(t_1) = D^q \left[m(t_1) - m(a)\right] = D^q m(t_1) - \frac{m(a)}{\Gamma(1-q)}(t-a)^{-q} \geq D^q m(t_1).$$

From Lemma 2.7 we have that $D^q m(t_1) \geq 0$, therefore $^cD^q m(t_1) \geq 0$ and the proof is complete. $\square$

Remark 2.9. In [7] the authors proved the above result by assuming that $m(t)$ is Hölder continuous of order $\lambda > q$. Although the proof is correct, it is not useful in the monotone method or any iterative method because we will not be able to prove that each of those iterates are Hölder continuous of order $\lambda > q$.

We finish this section with a comparison theorem and an important consequence.

Theorem 2.10. *Let $J = [a,b]$, and suppose that there exist two functions $v^0(t), w^0(t) \in C[J,\mathbb{R}]$ with $v^0(t) < w^0(t)$ such that the following conditions hold*

(a) $f, g \in C\left(J \times [v^0(t), w^0(t)] \times [Tv^0(t), Tw^0(t)]\right)$,

(b) f is increasing in u and Tu, g is decreasing in u and Tu, and

(c) For $v(t), w(t) \in C^1[J,\mathbb{R}]$ such that $v^0(t) \leq v(t), w(t) \leq w^0(t)$ the following inequalities are true for $t \in [a,b]$,

$$\begin{aligned} ^cD^q v(t) &\leq f(t,v(t),Tv(t)) + g(t,w(t),Tw(t)),\ v(a) \leq u_0,\ \text{and} \\ ^cD^q w(t) &\geq f(t,w(t),Tw(t)) + g(t,v(t),Tv(t)),\ w(a) \geq u_0. \end{aligned} \tag{2.6}$$

Suppose further that $f(t,u,Tu)$ and $g(t,u,Tu)$ satisfy the following Lipschitz condition for $L_1, L_2 > 0$, $M_1, M_2 \geq 0$, and $x \geq y$,

$$\begin{aligned} f(t,x,Tx) - f(t,y,Ty) &\leq L_1(x-y) + M_1 T(x-y), \\ g(t,x,Tx) - g(t,y,Ty) &\geq -L_2(x-y) - M_2 T(x-y), \end{aligned} \tag{2.7}$$

then $v(a) \leq w(a)$ implies that

$$v(t) \leq w(t),\ \text{for } a \leq t \leq b.$$

Proof. Assume first without loss of generality that one of the inequalities in (2.6) is strict, say ${}^cD^qv(t) < f(t, v(t), Tv(t)) + g(t, w(t), Tw(t))$, and $v_0 < w_0$ where $v(a) = v_0$ and $w(a) = w_0$. We will show that $v(t) < w(t)$ for $t \in [a, b]$.

Suppose, to the contrary, that there exists t_1 such that $a < t_1 \leq b$ for which

$$v(t_1) = w(t_1), \text{ and } v(t) < w(t), \text{ for } t < t_1.$$

Setting $m(t) = v(t) - w(t)$ it follows that $m(t_1) = 0$ and $m(t) < 0$ for $a \leq t < t_1$. Also, if $a \leq s \leq t_1$ then $v(s) \leq w(s)$ and

$$Tv(t_1) = \int_a^{t_1} K(t_1, s)v(s)ds \leq \int_a^{t_1} K(t_1, s)w(s)ds = Tw(t_1).$$

Then by Lemma 2.8 we have that ${}^cD^qm(t_1) \geq 0$. Thus

$$\begin{aligned} &f(t_1, v(t_1), Tv(t_1)) + g(t_1, w(t_1), Tw(t_1)) \\ &\qquad > \; {}^cDv(t_1) \geq {}^cDw(t_1) \\ &\qquad \geq \; f(t_1, w(t_1), Tw(t_1)) + g(t_1, v(t_1), Tv(t_1)), \end{aligned}$$

which is a contradiction to the assumption $v(t_1) = w(t_1)$. Therefore $v(t) < w(t)$ for $t > a$.

Now assume that the inequalities in (2.6) are non strict. We will show that $v(t) \leq w(t)$.

Set $v_\varepsilon(t) = v(t) - \varepsilon E_q(\lambda(t-a)^q)$ and $w_\varepsilon(t) = w(t) + \varepsilon E_q(\lambda(t-a)^q)$ where $\varepsilon > 0$, and $\lambda > 1$ is a constant that will be determined later.

This implies that $v_\varepsilon(a) = v_0 - \epsilon < v_0$, $w_\varepsilon(a) = w_0 + \varepsilon > w_0$, $v_\varepsilon(t) < v(t)$, and $w_\varepsilon(t) > w(t)$ for $a < t \leq b$.

Hence,

$$Tv_\varepsilon(t) = \int_a^t K(t, s)v_\varepsilon(s)ds \leq \int_a^t K(t, s)v(s)ds = Tv(t),$$

and

$$Tw_\varepsilon(t) = \int_a^t K(t, s)w_\varepsilon(s)ds \geq \int_a^t K(t, s)w(s)ds = Tw(t),$$

for $t > a$.

Using (2.6) and the Lipschitz condition (2.7), we find for $t > a$ that

$$\begin{aligned} &{}^cD^qv_\varepsilon(t) \\ &= {}^cD^qv(t) - \varepsilon\lambda E_q(\lambda(t-a)^q) \\ &\leq f(t, v(t), Tv(t)) + g(t, w(t), Tw(t)) - \varepsilon\lambda E_q(\lambda(t-a)^q) \\ &= f(t, v(t), Tv(t)) + g(t, w(t), Tw(t)) - f(t, v_\varepsilon(t), Tv_\varepsilon(t)) - g(t, w_\varepsilon(t), Tw_\varepsilon(t)) \\ &\quad + f(t, v_\varepsilon(t), Tv_\varepsilon(t)) + g(t, w_\varepsilon(t), Tw_\varepsilon(t)) - \varepsilon\lambda E_q(\lambda(t-a)^q) \\ &\leq L_1(v(t) - v_\varepsilon(t)) + M_1T(v(t) - v_\varepsilon(t)) + L_2(w_\varepsilon(t) - w(t)) \end{aligned}$$

$$
\begin{aligned}
&\quad + M_2 T\left(w_\varepsilon(t) - w(t)\right) + f\left(t, v_\varepsilon(t), T v_\varepsilon(t)\right) + g\left(t, w_\varepsilon(t), T w_\varepsilon(t)\right) \\
&\quad - \varepsilon \lambda E_q\left(\lambda(t-a)^q\right) \\
&= \varepsilon L_1\left(E_q\left(\lambda(t-a)^q\right)\right) + \varepsilon M_1 T\left(E_q\left(\lambda(t-a)^q\right)\right) \\
&\quad + \varepsilon L_2\left(E_q\left(\lambda(t-a)^q\right)\right) + \varepsilon M_2 T\left(E_q\left(\lambda(t-a)^q\right)\right) \\
&\quad + f\left(t, v_\varepsilon(t), T v_\varepsilon(t)\right) + g\left(t, w_\varepsilon(t), T w_\varepsilon(t)\right) - \varepsilon \lambda E_q\left(\lambda(t-a)^q\right) \\
&= \varepsilon\left(L_1 + L_2\right)\left(E_q\left(\lambda(t-a)^q\right)\right) + \varepsilon\left(M_1 + M_2\right) T\left(E_q\left(\lambda(t-a)^q\right)\right) \\
&\quad + f\left(t, v_\varepsilon(t), T v_\varepsilon(t)\right) + g\left(t, w_\varepsilon(t), T w_\varepsilon(t)\right) - \varepsilon \lambda E_q\left(\lambda(t-a)^q\right).
\end{aligned}
$$

Now consider the expression

$$T\left(E_q\left(\lambda(t-a)^q\right)\right) = \int_a^t K(t,s) E_q\left(\lambda(s-a)^q\right) ds,$$

and let $K_0 = \max_{a \le s \le t \le b} \{\Gamma(q) K(t,s)(t-s)^{1-q}\}$. Clearly $K_0 > 0$.

Then,

$$
\begin{aligned}
T\left(E_q\left(\lambda(t-a)^q\right)\right) &= \int_a^t K(t,s) E_q\left(\lambda(s-a)^q\right)\left(\frac{\Gamma(q)(t-s)^{q-1}}{\Gamma(q)(t-s)^{q-1}}\right) ds \\
&\le \frac{K_0}{\Gamma(q)} \int_a^t (t-s)^{q-1} E_q\left(\lambda(s-a)^q\right) ds \\
&= \frac{K_0}{\lambda} E_q\left(\lambda(s-a)^q\right)\Big|_a^t \\
&= \frac{K_0}{\lambda}\left[E_q\left(\lambda(t-a)^q\right) - 1\right] \\
&\le \frac{K_0}{\lambda} E_q\left(\lambda(t-a)^q\right).
\end{aligned}
$$

We have now obtained that

$$
\begin{aligned}
{}^c D^q v_\varepsilon(t) &\le \varepsilon\left(L_1 + L_2\right)\left(E_q\left(\lambda(t-a)^q\right)\right) + \varepsilon\left\{\frac{K_0\left(M_1 + M_2\right)}{\lambda}\right\} E_q\left(\lambda(t-a)^q\right) \\
&\quad + f\left(t, v_\varepsilon(t), T v_\varepsilon(t)\right) + g\left(t, w_\varepsilon(t), T w_\varepsilon(t)\right) - \varepsilon \lambda E_q\left(\lambda(t-a)^q\right) \\
&\le \varepsilon\left(L_1 + L_2 + \frac{K_0\left(M_1 + M_2\right)}{\lambda} - \lambda\right)\left(E_q\left(\lambda(t-a)^q\right)\right) \\
&\quad + f\left(t, v_\varepsilon(t), T v_\varepsilon(t)\right) + g\left(t, w_\varepsilon(t), T w_\varepsilon(t)\right).
\end{aligned}
$$

Choose $\lambda = 2\left[(L_1 + L_2) + K_0(M_1 + M_2)\right] + 1$, then

$$L_1 + L_2 + \frac{K_0\left(M_1 + M_2\right)}{\lambda} - \lambda < 0,$$

and

$${}^c D^q v_\varepsilon(t) < f\left(t, v_\varepsilon(t), T v_\varepsilon(t)\right) + g\left(t, w_\varepsilon(t), T w_\varepsilon(t)\right).$$

By a similar argument we can show that

$${}^c D^q w_\varepsilon(t) > f\left(t, w_\varepsilon(t), T w_\varepsilon(t)\right) + g\left(t, v_\varepsilon(t), T v_\varepsilon(t)\right).$$

Applying now the result for strict inequalities to $v_\varepsilon(t), w_\varepsilon(t)$, we get that $v_\varepsilon(t) < w_\varepsilon(t)$ for $t \in J$ and for every $\varepsilon > 0$. That is $v(t) - \varepsilon E_q(\lambda(t-a)^q) < w(t) + \varepsilon E_q(\lambda(t-a)^q)$, or $v(t) < w(t) + 2\varepsilon E_q(\lambda(t-a)^q)$.

Consequently, making $\varepsilon \to 0$, we get that $v(t) \le w(t)$ for $t \in J$. □

We now state the following corollary that will be useful in our main results.

Corollary 2.11. *Let $m \in C^1[J, \mathbb{R}]$ be such that*

$$\begin{aligned} {}^cD^q m(t) &\le Lm(t) + MTm(t), \\ m(a) &\le 0, \end{aligned}$$

where $L > 0, M \ge 0$. Then we have from the previous theorem that

$$m(t) \le 0,$$

for $a \le t \le b$.

Similarly, if $m \in C^1[J, \mathbb{R}]$ is such that

$$\begin{aligned} {}^cD^q m(t) &\ge -Lm(t) - MTm(t), \\ m(a) &\ge 0, \end{aligned}$$

for $L > 0, M \ge 0$, then we have from the previous theorem that

$$m(t) \ge 0,$$

for $a \le t \le b$.

The result of Corollary 2.11 is still true even if $L = M = 0$, which we state separately.

Corollary 2.12. *Let ${}^cD^q m(t) \le 0$ on $[a, b]$. Then $m(t) \le 0$, if $m(a) \le 0$.*

3. MAIN RESULTS

In this section we will give the definition of coupled lower and upper solutions in order to develop two generalized monotone iterative techniques for the nonlinear integro-differential initial value problem (3.1), given below.

For that purpose consider the problem

$$\begin{aligned} {}^cD^q u(t) &= f(t, u(t), Tu(t)) + g(t, u(t), Tu(t)), \\ u(a) &= u_0, \end{aligned} \tag{3.1}$$

where $J = [a, b]$, $f, g \in C[J \times \mathbb{R} \times \mathbb{R}, \mathbb{R}]$, $u \in C^1[J \times \mathbb{R}]$, and $Tu(t) = \int_a^t K(t,s)u(s)ds$, where $K \in C(J \times J, \mathbb{R})$ is a positive function.

If $u \in C^1[a, b]$ satisfies the fractional differential equation

$${}^cD^q u(t) = f(t, u(t), Tu(t)) + g(t, u(t), Tu(t)),$$

and u is such that $u(a) = u_0$ for $t \in J$, then u is said to be a solution of (3.1).

Throughout the rest of this paper, we will assume that f is increasing in u and Tu, and g is decreasing in u and Tu for $t \in J$.

Here below we provide the definition of coupled lower and upper solutions of (3.1).

Definition 3.1. Let $v_0, w_0 \in C^1[J, \mathbb{R}]$. Then v_0 and w_0 are said to be,

- Natural lower and upper solutions of (3.1) if

$$\begin{aligned} {}^cD^q v_0(t) &\leq f(t, v_0(t), Tv_0(t)) + g(t, v_0(t), Tv_0(t)),\ v_0(a) \leq u_0, \\ {}^cD^q w_0(t) &\geq f(t, w_0(t), Tw_0(t)) + g(t, w_0(t), Tw_0(t)),\ w_0(a) \geq u_0. \end{aligned} \tag{3.2}$$

- Coupled lower and upper solutions of Type I of (3.1) if

$$\begin{aligned} {}^cD^q v_0(t) &\leq f(t, v_0(t), Tv_0(t)) + g(t, w_0(t), Tw_0(t)),\ v_0(a) \leq u_0, \\ {}^cD^q w_0(t) &\geq f(t, w_0(t), Tw_0(t)) + g(t, v_0(t), Tv_0(t)),\ w_0(a) \geq u_0. \end{aligned} \tag{3.3}$$

- Coupled lower and upper solutions of Type II of (3.1) if

$$\begin{aligned} {}^cD^q v_0(t) &\leq f(t, w_0(t), Tw_0(t)) + g(t, v_0(t), Tv_0(t)),\ v_0(a) \leq u_0, \\ {}^cD^q w_0(t) &\geq f(t, v_0(t), Tv_0(t)) + g(t, w_0(t), Tw_0(t)),\ w_0(a) \geq u_0. \end{aligned} \tag{3.4}$$

- Coupled lower and upper solutions of Type III of (3.1) if,

$$\begin{aligned} {}^cD^q v_0(t) &\leq f(t, w_0(t), Tw_0(t)) + g(t, w_0(t), Tw_0(t)),\ v_0(a) \leq u_0, \\ {}^cD^q w_0(t) &\geq f(t, v_0(t), Tv_0(t)) + g(t, v_0(t), Tv_0(t)),\ w_0(a) \geq u_0. \end{aligned} \tag{3.5}$$

We will state the following theorem related to coupled lower and upper solutions of the form (3.3). Next, we develop a generalized monotone iterative technique for the integro-differential initial value problem. Finally, we obtain natural sequences which converge uniformly and monotonically to coupled minimal and maximal solutions of (3.1).

Theorem 3.2. *Assume that*

(A1) v_0, w_0 are coupled lower and upper solutions of type I for (3.1) *with $v_0(t) \leq w_0(t)$ in J; and*

(A2) $f, g \in C\left(J \times [v_0(t), w_0(t)] \times [Tv_0(t), Tw_0(t)], \mathbb{R}\right)$, where $f\left(t, u(t), Tu(t)\right)$ is increasing in u and Tu and $g\left(t, u(t), Tu(t)\right)$ is decreasing in u and in Tu.

If $u(t)$ is a solution of (3.1) *such that $v_0(t) \leq u(t) \leq w_0(t)$ for all $t \in J$, then the sequences defined by*

$$\begin{aligned} {}^cD^q v_{n+1}(t) &= f\left(t, v_n(t), Tv_n(t)\right) + g\left(t, w_n(t), Tw_n(t)\right), \\ v_{n+1}(a) &= u_0, \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} {}^cD^q w_{n+1}(t) &= f\big(t, w_n(t), Tw_n(t)\big) + g\big(t, v_n(t), Tv_n(t)\big), \\ w_{n+1}(a) &= u_0, \end{aligned} \tag{3.7}$$

are such that

$$v_0 \le v_1 \le \cdots \le v_n \le v_{n+1} \le u \le w_{n+1} \le w_n \le \cdots \le w_1 \le w_0,$$

where $v_n(t) \to \rho(t)$ *and* $w_n(t) \to r(t)$ *uniformly and monotonically in* $C^1[J, \mathbb{R}]$*, and* ρ, r *are coupled minimal and maximal solutions of* (3.1)*, respectively; i.e.,* ρ *and* r *satisfy the coupled system*

$$\begin{aligned} {}^cD^q\rho(t) &= f(t, \rho(t), T\rho(t)) + g(t, r(t), Tr(t)), \\ \rho(a) &= u_0 \text{ on } J, \end{aligned}$$

and

$$\begin{aligned} {}^cD^q r(t) &= f(t, r(t), Tr(t)) + g(t, \rho(t), T\rho(t)), \\ r(a) &= u_0 \text{ on } J, \end{aligned}$$

with $\rho \le u \le r$*.*

Proof. By hypothesis, $v_0 \le u \le w_0$. We will show that $v_0 \le v_1 \le u \le w_1 \le w_0$.

It follows from (3.3) that

$$\begin{aligned} {}^cD^q v_0(t) &\le f(t, v_0(t), Tv_0(t)) + g(t, w_0(t), Tw_0(t)), \ v_0(a) \le u_0, \\ {}^cD^q w_0(t) &\ge f(t, w_0(t), Tw_0(t)) + g(t, v_0(t), Tv_0(t)), \ w_0(a) \ge u_0, \end{aligned}$$

and by (3.6), we get that

$$\begin{aligned} {}^cD^q v_1 &= f(t, v_0(t), Tv_0(t)) + g(t, w_0(t), Tw_0(t)), \\ v_1(a) &= u_0. \end{aligned}$$

Therefore, $v_0(a) \le u_0 = v_1(a)$. If we let $p = v_0 - v_1$, then $p(a) \le 0$ and,

$$\begin{aligned} {}^cD^q p &= {}^cD^q v_0 - {}^cD^q v_1 \\ &\le f(t, v_0, Tv_0) + g(t, w_0, Tw_0) - f(t, v_0, Tv_0) - g(t, w_0, Tw_0) \\ &= 0. \end{aligned}$$

Since ${}^cD^q p \le 0$ and $p(a) \le 0$, by an application of Corollary 2.12 we have that $p(t) \le 0$ and, consequently, $v_0(t) \le v_1(t)$ on J.

Suppose that u is a solution of (3.1) such that $v_0(t) \le u(t) \le w_0(t)$. In order to prove that $v_1(t) \le u(t)$, observe that since $v_0(t) \le u(t)$ for each t in $[a, b]$ and $K > 0$, then

$$Tv_0(t) = \int_a^t K(t, s)v_0(s)ds \le \int_a^t K(t, s)u(s)ds = Tu(t)$$

for each $t \in [a, b]$. Similarly we have that $Tu(t) \le Tw_0(t)$ for each $t \in [a, b]$.

Letting $p(t) = v_1(t) - u(t)$, we have that $p(a) = v_1(a) - u(a) = u_0 - u_0 = 0$. Moreover, by the increasing nature of f and the decreasing nature of g we have that

$$ {}^cD^q p = {}^cD^q v_1 - {}^cD^q u$$

$$\begin{aligned} &= f(t, v_0(t), Tv_0(t)) + g(t, w_0(t), Tw_0(t)) \\ &\quad - f(t, u(t), Tu(t)) - g(t, u(t), Tu(t)) \\ &\leq 0, \end{aligned}$$

and by Corollary 2.12 we have that $v_1(t) \leq u(t)$. By a similar argument we can show that $u(t) \leq w_1(t)$ and $w_1(t) \leq w_0(t)$. Thus, $v_0(t) \leq v_1(t) \leq u(t) \leq w_1(t) \leq w_0(t)$.

Now we will show that $v_k \leq v_{k+1}$ for $k \geq 1$.

Assume that

$$v_{k-1}(t) \leq v_k(t) \leq u(t) \leq w_k(t) \leq w_{k-1}(t),$$

for $k > 1$.

If $a \leq s \leq t \leq b$, we have that $x_1(s) \leq x_2(s)$ implies that

$$Tx_1(t) = \int_a^t K(t,s)x_1(s)ds \leq \int_a^t K(t,s)x_2(s)ds = Tx_2(t).$$

Thus

$$Tv_{k-1}(t) \leq Tv_k(t) \leq Tu(t) \leq Tw_k(t) \leq Tw_{k-1}(t).$$

Let $p = v_k - v_{k+1}$. Then

$$v_k(a) = u_0 = v_{k+1}(a),$$

so $p(a) = 0$. By the increasing nature of f and the decreasing nature of g it follows that

$$\begin{aligned} {}^cD^q p &= {}^cD^q v_k - {}^cD^q v_{k+1} \\ &= f(t, v_{k-1}, Tv_{k-1}) + g(t, w_{k-1}, Tw_{k-1}) - f(t, v_k, Tv_k) - g(t, w_k, Tw_k) \\ &\leq 0. \end{aligned}$$

Similarly, by Corollary 2.12 we have that $p(t) \leq 0$ and consequently $v_k(t) \leq v_{k+1}(t)$.

Using the hypothesis that $v_0(t) \leq u(t) \leq w_0(t)$ on J, the above argument and induction we can also show that $w_{k+1} \leq w_k$, $v_{k+1} \leq u$, and $u \leq w_{k+1}$. Therefore for $n > 0$,

$$v_0 \leq v_1 \leq v_2 \leq \cdots \leq v_n \leq u \leq w_n \leq \cdots \leq w_2 \leq w_1 \leq w_0.$$

Now we have to show that the sequences converge uniformly. We will use the Arzela-Ascoli Theorem by showing that the sequences are uniformly bounded and equicontinuous.

First we show uniform boundedness. By hypothesis both $v_0(t)$ and $w_0(t)$ are bounded on $[a, b]$, then there exists $M > 0$ such that for any $t \in [a, b]$, $|v_0(t)| \leq M$ and $|w_0(t)| \leq M$. Since $v_0(t) \leq v_n(t) \leq w_n(t) \leq w_0(t)$ for each $n > 0$, it follows that

$$0 \leq v_n(t) - v_0(t) \leq w_n(t) - v_0(t) \leq w_0(t) - v_0(t),$$

and consequently $\{v_n(t)\}$ and $\{w_n(t)\}$ are uniformly bounded.

To prove that $\{v_n(t)\}$ is equicontinuous, let $a \le t_1 \le t_2 \le b$. Then for $n > 0$,

$$
\begin{aligned}
&|v_n(t_1) - v_n(t_2)| = \\
= &\left| u_0 + \frac{1}{\Gamma(q)} \int_a^{t_1} (t_1 - s)^{q-1} \left[f\left(s, v_{n-1}(s), Tv_{n-1}(s)\right) + g\left(s, w_{n-1}(s), Tw_{n-1}(s)\right)\right] ds \right. \\
- &\left. u_0 - \frac{1}{\Gamma(q)} \int_a^{t_2} (t_2 - s)^{q-1} \left[f\left(s, v_{n-1}(s), Tv_{n-1}(s)\right) + g\left(s, w_{n-1}(s), Tw_{n-1}(s)\right)\right] ds \right| \\
= &\left| \frac{1}{\Gamma(q)} \int_a^{t_1} \left[(t_1 - s)^{q-1} - (t_2 - s)^{q-1}\right] \Big[f\left(s, v_{n-1}(s), Tv_{n-1}(s)\right) \right. \\
&+ g\left(s, w_{n-1}(s), Tw_{n-1}(s)\right) \Big] ds \\
&\left. - \frac{1}{\Gamma(q)} \int_{t_1}^{t_2} (t_2 - s)^{q-1} \left[f\left(s, v_{n-1}(s), Tv_{n-1}(s)\right) + g\left(t, w_{n-1}(s), Tw_{n-1}(s)\right)\right] ds \right| \\
\le &\frac{1}{\Gamma(q)} \int_a^{t_1} \Big| \left[(t_1 - s)^{q-1} - (t_2 - s)^{q-1}\right] \Big[f\left(s, v_{n-1}(s), Tv_{n-1}(s)\right) \\
&+ g\left(s, w_{n-1}(s), Tw_{n-1}(s)\right) \Big] \Big| ds \\
&+ \frac{1}{\Gamma(q)} \int_{t_1}^{t_2} (t_2 - s)^{q-1} \Big| \left[f\left(s, v_{n-1}(s), Tv_{n-1}(s)\right) + g\left(t, w_{n-1}(s), Tw_{n-1}(s)\right)\right] \Big| ds.
\end{aligned}
$$

Since $\{v_n(t)\}$ and $\{w_n(t)\}$ are uniformly bounded and $f\left(t, u(t), Tu(t)\right)$ and $g\left(t, u(t), Tu(t)\right)$ are continuous on $[a, b]$, there exists $\bar{M}$ independent of n such that

$$
\begin{aligned}
\left| f\left(t, v_n(t), Tv_n(t)\right) \right| &\le \bar{M}, \\
\left| f\left(t, w_n(t), Tw_n(t)\right) \right| &\le \bar{M}, \\
\left| g\left(t, v_n(t), Tv_n(t)\right) \right| &\le \bar{M}, \text{ and} \\
\left| g\left(t, w_n(t), Tw_n(t)\right) \right| &\le \bar{M}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\frac{1}{\Gamma(q)} \int_a^{t_1} \Big| \left[(t_1 - s)^{q-1} - (t_2 - s)^{q-1}\right] \Big[f\left(s, v_{n-1}(s), Tv_{n-1}(s)\right) \\
&+ g\left(s, w_{n-1}(s), Tw_{n-1}(s)\right) \Big] \Big| ds \\
&+ \frac{1}{\Gamma(q)} \int_{t_1}^{t_2} (t_2 - s)^{q-1} \Big| \left[f\left(s, v_{n-1}(s), Tv_{n-1}(s)\right) + g\left(t, w_{n-1}(s), Tw_{n-1}(s)\right)\right] \Big| ds \\
\le &\frac{\bar{M}}{\Gamma(q)} \int_a^{t_1} \left[(t_1 - s)^{q-1} - (t_2 - s)^{q-1}\right] ds + \frac{\bar{M}}{\Gamma(q)} \int_{t_1}^{t_2} (t_2 - s)^{q-1} ds \\
= &-\frac{\bar{M}}{q\Gamma(q)} (t_1 - s)^q \Big|_a^{t_1} + \frac{\bar{M}}{q\Gamma(q)} (t_2 - s)^q \Big|_a^{t_1} - \frac{\bar{M}}{q\Gamma(q)} (t_2 - s)^q \Big|_{t_1}^{t_2}
\end{aligned}
$$

$$\begin{aligned} &= \frac{\bar{M}}{\Gamma(q+1)}(t_1-a)^q + \frac{\bar{M}}{\Gamma(q+1)}(t_2-t_1)^q - \frac{\bar{M}}{\Gamma(q+1)}(t_2-a)^q + \frac{\bar{M}}{\Gamma(q+1)}(t_2-t_1)^q \\ &\leq \frac{2\bar{M}}{\Gamma(q+1)}(t_2-t_1)^q = \frac{2\bar{M}}{\Gamma(q+1)}|t_1-t_2|^q. \end{aligned}$$

Thus, for any $\varepsilon > 0$ there exists $\delta = \dfrac{\Gamma(q+1)}{2\bar{M}}\varepsilon^{1/q} > 0$ independent of n such that for each n,

$$|v_n(t_1) - v_n(t_2)| < \varepsilon,$$

provided that $|t_1 - t_2| < \delta$. This finishes the proof that $\{w_n(t)\}$ is equicontinuous.

Similarly we can prove that $\{w_n(t)\}$ is equicontinuous.

We have obtained that $\{v_n(t)\}$ and $\{w_n(t)\}$ are uniformly bounded and equicontinuous on $[a, b]$. Hence by the Arzela-Ascoli Theorem there exist subsequences $\{v_{n_k}(t)\}$ and $\{w_{n_k}(t)\}$ which converge uniformly to $\rho(t)$ and $r(t)$, respectively. Since the sequences are monotone, the entire sequences converge uniformly.

We have shown that the sequences converge in $C[a, b]$. In order to show that they converge in $C^1[a, b]$, observe that since each v_n is constructed as follows

$$\begin{aligned} {}^cD^q v_n &= f(t, v_{n-1}, Tv_{n-1}) + g(t, w_{n-1}, Tw_{n-1}), \\ v_n(a) &= u_0, \end{aligned}$$

and we get that

$$v_n(t) = u_0 + \frac{1}{\Gamma(q)}\int_a^t (t-s)^{q-1}[f(s, v_{n-1}(s), Tv_{n-1}(s)) + g(s, w_{n-1}(s), Tw_{n-1}(s))]ds.$$

Taking limits when $n \to \infty$, we obtain by the Lebesgue Dominated Convergence theorem that

$$\rho(t) = u_0 + \frac{1}{\Gamma(q)}\int_a^t (t-s)^{q-1}\left[f\left(s, \rho(s), T\rho(s)\right) + g\left(s, r(s), Tr(s)\right)\right] ds.$$

Hence $v_n(t) \to \rho(t)$ in $C^1[a, b]$. Furthermore, the above expression is equivalent to

$$\begin{aligned} {}^cD^q \rho &= f(t, \rho, T\rho) + g(t, r, Tr) \text{ on } J, \\ \rho(a) &= u_0. \end{aligned}$$

By a similar argument $w_n(t) \to r(t)$ in $C^1[a, b]$ and it can be shown that

$$\begin{aligned} {}^cD^q r &= f(t, r, Tr) + g(t, \rho, T\rho) \text{ on } J, \\ r(a) &= u_0. \end{aligned}$$

Since $v_n \leq u \leq w_n$ on $[a, b]$ for all n, we get that $\rho \leq u \leq r$ on $[a, b]$ which shows that ρ and r are coupled minimal and maximal solutions of (3.1), respectively. This completes the proof. $\square$

Finding coupled lower and upper solutions of Type I as in (3.3) can be itself a challenge, see the recent papers [1, 14, 15] for methods to construct lower wand upper solutions of the form (3.3) for different types of initial value problems. However, with an additional assumption on the first two iterates we can use coupled lower and upper solutions of Type II (3.4) to construct intertwined sequences that converge uniformly and monotonically to minimal and maximal solutions. Furthermore, these sequences converge to a unique solution. The proof is similar to the one in Theorem 3.2, so we state the result without a proof. We state the conditions for uniqueness separately.

Theorem 3.3. *Assume that*

(B1) v_0, w_0 *are coupled lower and upper solutions of type II for* (3.1) *with* $v_0(t) \leq w_0(t)$ *in* J*; and*

(B2) $f, g \in C\left(J \times [v_0(t), w_0(t)] \times [Tv_0(t), Tw_0(t)], \mathbb{R}\right)$*, where* $f\left(t, u(t), Tu(t)\right)$ *is increasing in* u *and* Tu *and* $g\left(t, u(t), Tu(t)\right)$ *is decreasing in* u *and* Tu*.*

Define the following sequences,

$$\begin{aligned} {}^cD^q v_{n+1}(t) &= f\left(t, w_n(t), Tw_n(t)\right) + g\left(t, v_n(t), Tv_n(t)\right), \\ v_{n+1}(a) &= u_0, \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} {}^cD^q w_{n+1}(t) &= f\big(t, v_n(t), Tv_n(t)\big) + g\big(t, w_n(t), Tw_n(t)\big), \\ w_{n+1}(a) &= u_0. \end{aligned} \tag{3.9}$$

If $u(t)$ *is a solution of* (3.1) *such that* $v_0(t) \leq w_1(t) \leq u(t) \leq v_1(t) \leq w_0(t)$*, then* (3.8) *and* (3.9) *provide intertwined sequences of the form*

$$\begin{aligned} &v_0 \leq w_1 \leq v_2 \leq \cdots \leq v_{2n} \leq w_{2n+1} \leq u \\ &\leq v_{2n+1} \leq w_{2n} \leq \cdots \leq w_2 \leq v_1 \leq w_0, \end{aligned}$$

where $\{v_{2n}(t), w_{2n+1}(t)\} \to \rho(t)$ *and* $\{w_{2n}(t), v_{2n+1}(t)\} \to r(t)$ *uniformly and monotonically in* $C^1[J, \mathbb{R}]$*, and* ρ, r *are coupled minimal and maximal solutions of* (3.1)*, respectively; i.e.,* ρ *and* r *satisfy the coupled system*

$$\begin{aligned} {}^cD^q \rho(t) &= f\left(t, \rho(t), T\rho(t)\right) + g\left(t, r(t), Tr(t)\right), \\ \rho(a) &= u_0 \text{ on } J, \end{aligned}$$

and

$$\begin{aligned} {}^cD^q r(t) &= f\left(t, r(t), Tr(t)\right) + g\left(t, \rho(t), T\rho(t)\right), \\ r(a) &= u_0 \text{ on } J, \end{aligned}$$

with $\rho \leq u \leq r$*.*

Remark 3.4. In addition to conditions (A1)–(A2) of Theorem 3.2 or (B1)-(B2) of Theorem 3.3, suppose that there exist positive constants M_1, M_2, and non negative constants N_1, N_2 such that f and g satisfy the following one–sided Lipschitz conditions for $x \geq y$,

$$\begin{aligned} f(t,x,Tx) - f(t,y,Ty) &\leq M_1(x-y) + N_1T(x-y), \\ g(t,x,Tx) - g(t,y,Ty) &\geq -M_2(x-y) - N_2T(x-y), \end{aligned} \tag{3.10}$$

then $\rho = r = u$; i.e., the sequences converge to a unique solution.

We already proved that $\rho \leq r$. In order to show that $r \leq \rho$, let $p(t) = r(t) - \rho(t)$. Clearly, $p(a) = r(a) - \rho(a) = u_0 - u_0 = 0$. Since $\rho \leq r$ we have from the conclusion of Theorem 3.2 and (3.10) that

$$\begin{aligned} {}^cD^q p &= {}^cD^q r - {}^cD^q \rho \\ &= f(t,r,Tr) + g(t,\rho,T\rho) - f(t,\rho,T\rho) - g(t,r,Tr) \\ &\leq M_1(r-\rho) + N_1T(r-\rho) + M_2(r-\rho) + N_2T(r-\rho) \\ &= (M_1+M_2)(r-\rho) + (N_1+N_2)T(r-\rho) \\ &= (M_1+M_2)p + (N_1+N_2)Tp. \end{aligned}$$

We obtain from Corollary 2.11 that $p(t) \leq 0$ and, consequently, $r(t) \leq \rho(t)$. Therefore $\rho(t) = r(t) = u(t)$, and the sequences converge to the same solution.

4. NUMERICAL RESULTS

In this section we present two examples that illustrate the result from Theorem 3.3.

Example 4.1. Consider the following integro-differential initial value problem of order $q = \frac{1}{2}$ on $J = [0,1]$,

$$\begin{aligned} {}^cD^{1/2}u &= \tfrac{1}{5}u(t) + \tfrac{1}{4}\left[\int_0^t (1+s^2)u(s)ds\right]^2 - \tfrac{1}{3}u^2(t) - \tfrac{1}{8}\int_0^t (1+s^2)u(s)ds, \\ u(0) &= \tfrac{1}{2}. \end{aligned} \tag{4.1}$$

Here
$$Tu(t) = \int_0^t (1+s^2)u(s)ds.$$

Then the function
$$f(t,u(t),Tu(t)) = \frac{1}{5}u(t) + \frac{1}{4}\left[Tu(t)\right]^2$$
is increasing in u and Tu, and
$$g(t,u(t),Tu(t)) = -\frac{1}{3}u^2(t) - \frac{1}{8}Tu(t)$$
is decreasing in u and Tu for all $t \in J$. We will show graphically that $v_0 \equiv 0$ and $w_0 \equiv 1$ are coupled lower and upper solutions of type II that satisfy (3.4) on the interval $J = [0,1]$. Clearly ${}^cD^{1/2}v_0(t) = {}^cD^{1/2}w_0(t) = 0$. In Figure 1 we show the

graph of $f\left(t, w_0(t), Tw_0(t)\right) + g\left(t, v_0(t), Tv_0(t)\right)$ and in Figure 2 we show the graph of $f\left(t, v_0(t), Tv_0(t)\right) + g\left(t, w_0(t), Tw_0(t)\right)$.

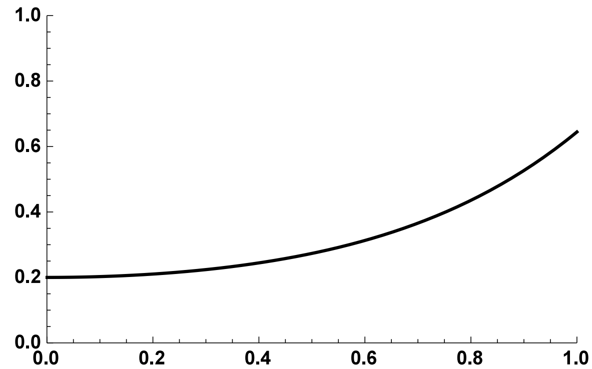

FIGURE 1. $0 = {}^cD^{1/2}v_0(t) \leq f\left(t, w_0(t), Tw_0(t)\right) + g\left(t, v_0(t), Tv_0(t)\right).$

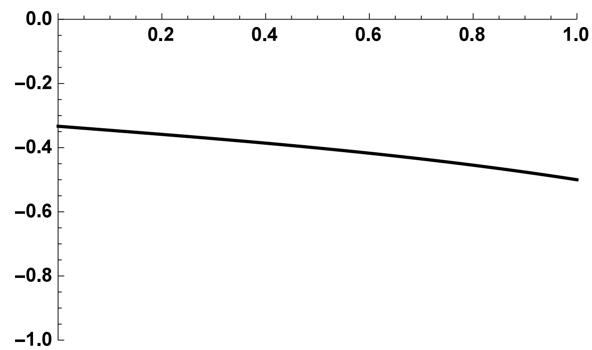

FIGURE 2. $0 = {}^cD^{1/2}w_0(t) \geq f\left(t, v_0(t), Tv_0(t)\right) + g\left(t, w_0(t), Tw_0(t)\right).$

We construct the sequences according to Theorem 3.3, in Figure 3 we show five iterates of $\{v_n\}$ and five iterates of $\{w_n\}$ on $[0, 1]$.

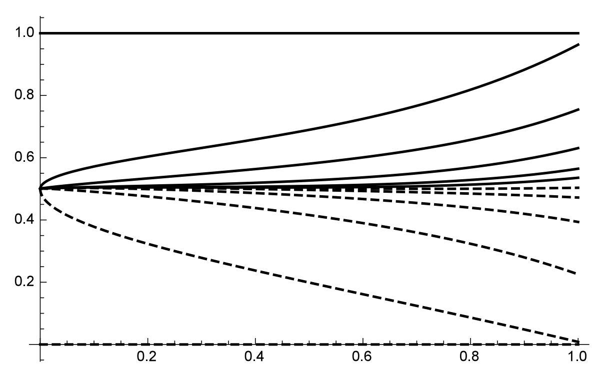

FIGURE 3. Dashed: $v_0 \leq w_1 \leq v_2 \leq w_3 \leq v_4 \leq w_5$. Solid: $v_5 \leq w_4 \leq v_3 \leq w_2 \leq v_1 \leq w_0$.

TABLE 1. Table of ten points in $[0,1]$ of $v_5(t)$ and $w_5(t)$ for equation (4.1).

t	$v_5(t)$	$w_5(t)$
0.0	0.500000	0.500000
0.1	0.504411	0.504369
0.2	0.504670	0.504420
0.3	0.504343	0.503619
0.4	0.504077	0.502500
0.5	0.504332	0.501368
0.6	0.505581	0.500474
0.7	0.508398	0.500046
0.8	0.513551	0.500303
0.9	0.522111	0.501422
1.0	0.535633	0.503454

Example 4.2. Consider the integro-differential initial value problem of order $q = \frac{1}{2}$ on $J = [0,1]$,

$$\begin{aligned} {}^cD^{1/2}u &= \tfrac{1}{5}u(t) + \tfrac{1}{5}\textstyle\int_0^t (1+s)u(s)ds - \tfrac{1}{10}u^2(t) - \tfrac{1}{10}\left[\int_0^t (1+s)u(s)ds\right]^2, \\ u(0) &= 1. \end{aligned} \tag{4.2}$$

Here

$$Tu(t) = \int_0^t (1+s)u(s)ds,$$

the function

$$f\left(t, u(t), Tu(t)\right) = \frac{1}{5}u(t) + \frac{1}{5}Tu(t)$$

is increasing in u and Tu, and

$$g\left(t, u(t), Tu(t)\right) = -\frac{1}{10}u^2(t) - \frac{1}{10}\left[Tu(t)\right]^2$$

is decreasing in u and Tu for all $t \in J$. We will show graphically that $v_0 \equiv 0$ and $w_0 \equiv 2$ are coupled lower and upper solutions of type II that satisfy (3.4) on the interval $J = [0,1]$. Observe that ${}^cD^{1/2}v_0(t) = {}^cD^{1/2}w_0(t) = 0$. In Figure 4 we show the graph of $f\left(t, w_0(t), Tw_0(t)\right) + g\left(t, v_0(t), Tv_0(t)\right)$ and in Figure 5 we show the graph of $f\left(t, v_0(t), Tv_0(t)\right) + g\left(t, w_0(t), Tw_0(t)\right)$.

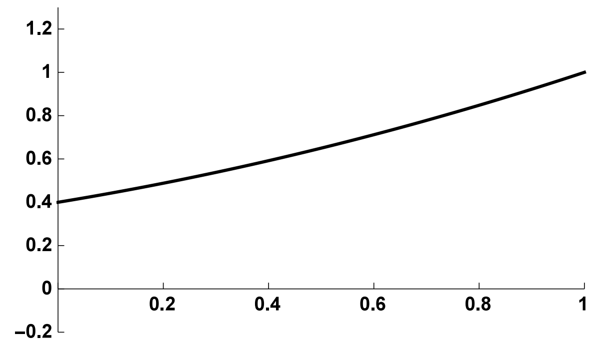

FIGURE 4. $0 = {}^{c}D^{1/2}v_0(t) \leq f\left(t, w_0(t), Tw_0(t)\right) + g\left(t, v_0(t), Tv_0(t)\right).$

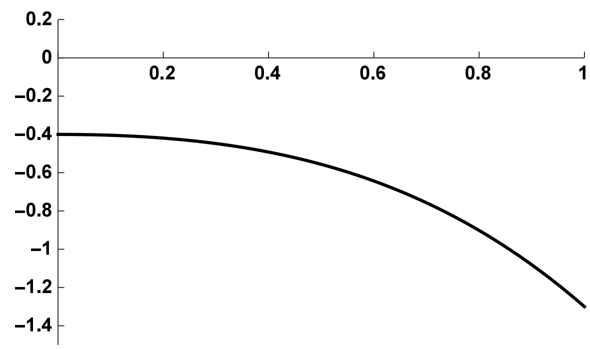

FIGURE 5. $0 = {}^{c}D^{1/2}w_0(t) \geq f\left(t, v_0(t), Tv_0(t)\right) + g\left(t, w_0(t), Tw_0(t)\right).$

In Figure 6 we show five iterates of $\{v_n\}$ and five iterates of $\{w_n\}$ on $[0, 1]$, which were constructed according to Theorem 3.3.

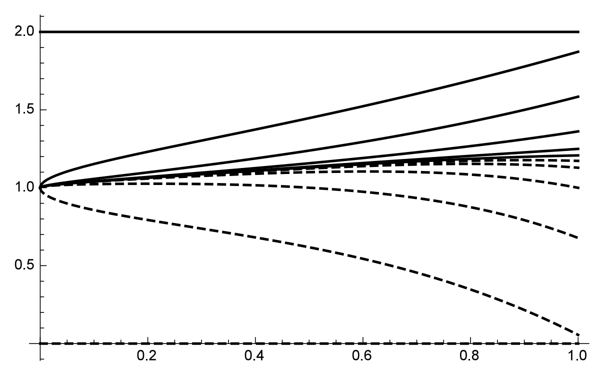

FIGURE 6. Dashed: $v_0 \leq w_1 \leq v_2 \leq w_3 \leq v_4 \leq w_5$. Solid: $v_5 \leq w_4 \leq v_3 \leq w_2 \leq v_1 \leq w_0$.

We have used Mathematica to compute the iterates, the graphs and the tables.

TABLE 2. Table of ten points in $[0,1]$ of $v_5(t)$ and $w_5(t)$ for equation (4.2).

t	$v_5(t)$	$w_5(t)$
0.0	1.00000	1.00000
0.1	1.04049	1.04048
0.2	1.06408	1.06394
0.3	1.08654	1.08609
0.4	1.10863	1.10758
0.5	1.13017	1.12799
0.6	1.15062	1.14649
0.7	1.16929	1.16192
0.8	1.18545	1.17274
0.9	1.19836	1.17702
1.0	1.2075	1.17235

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