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# QUENCHING BEHAVIOR OF INITIAL-BOUNDARY VALUE PROBLEM FOR A GENERALIZED EULER-POISSON-DARBOUX EQUATION

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**ABSTRACT.** We consider a generalization of the Euler-Poisson-Darboux equation with initial and boundary conditions. Criteria under which week solutions of the initial-boundary value problem for the generalized Euler-Poisson-Darboux equation quench in finite time are obtained. Also, criteria under which the first derivative of a weak solution of the problem blows up in finite time are given. Furthermore, numerical results for one-dimensional problems are discussed.

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## 1. INTRODUCTION

The concept of quenching was introduced in 1975 by Kawarada [5] through a first initial-boundary value problem for a semilinear heat equation. Chang and Levine [4] extended the concept to hyperbolic equations through a first initial-boundary value problem for a semilinear wave equation in 1981. For the one-dimensional semilinear Euler-Poisson-Darboux equations, Chan and Nip [1, 2] studied the critical length, and the blow-up of the second derivative of the solution with respect to time at quenching. Chan and Zhu [3] furthered the study of quenching for an initial-boundary value problem involving the $n$-dimensional semilinear Euler-Poisson-Darboux equations. Let $a$ and $b$ be any real numbers with $b$ greater than 0, $\Delta$ be the $n$-dimensional Laplace operator, $f : (-\infty, c) \to (0, \infty)$ for some positive constant $c$ such that $f$ is convex, and $\lim_{s\to c^-} f(s) = \infty$, $\Omega$ be a bounded domain in $\mathbb{R}^{\ltimes}$ with a piecewise smooth boundary $\partial\Omega$, and $u_0(x) \in C(\Omega)$. Let us consider the initial-boundary problem,

$$u_{tt} + \left(a + \frac{b}{t}\right) u_t - \Delta u = f(u), \quad (x,t) \in \Omega \times (0,T), \tag{1.1}$$

$$u(x,0) = u_0(x), u_t(x,0) = 0, \quad x \in \Omega, \tag{1.2}$$

$$u(x,t) = 0, \quad (x,t) \in \partial\Omega \times [0,T). \tag{1.3}$$

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In this paper, we give criteria under which a weak solution of the problem (1.1)–(1.3) quenches. For $a = 0$, the case when $b = 0$, $u_0(x) = 0$ and $f' \geq 0$ was discussed by Chang and Levine [4] , the case when $b \leq 1$, $u_0(x) \geq 0$, and $f' \geq 0$ was investigated by Chan and Nip [1] for one spatial dimension; and for multi-dimensional case, it was studied by Chan and Zhu [3]. Furthermore, we obtain criteria under which the first derivative of a weak solution of the problem (1.1)–(1.3) blows up in finite time and numerical results for one-dimensional initial-boundary value problems.

## 2. QUENCHING RESULTS FOR SOLUTIONS

**Definition 2.1** The function $u : \Omega \times [0, T) \to \mathbb{R}$ is called a weak solution of the problem (1.1)–(1.3) if $u$ and $u_t$ are continuous in $\Omega \times [0, T)$, $u$ satisfies (1.2) and (1.3), and for all $\varphi : \Omega \times [0, T) \to \mathbb{R}$ which are twice continuously differentiable in $t$, satisfy (1.3) and

$$\int_0^T \int_\Omega \frac{|\varphi(x,t)|}{t^2} dxdt + \int_0^T \int_\Omega \frac{|\varphi_t(x,t)|}{t} dxdt < \infty, \tag{2.1}$$

we have

$$\int_\Omega u_t \varphi dx + (a + \frac{k}{t}) \int_\Omega u\varphi dx = \int_0^t \int_\Omega \left[ \varphi_\eta u_\eta + u\Delta\varphi + \varphi f + \frac{b}{\eta}\varphi_\eta u + (\frac{2a}{\eta} - \frac{b}{\eta^2})\varphi u \right] dxd\eta. \tag{2.2}$$

Let $\lambda$ be the first eigenvalue of the eigenvalue problem,

$$\Delta\phi + \lambda\phi = 0,\ x \in \Omega;\ \phi = 0, x \in \partial\Omega,$$

and $\phi(x)$ denote the corresponding eigenfunction. Then, $\lambda > 0$ , and $\phi(x) > 0$, $x \in \Omega$ with $\int_\Omega \phi(x)dx = 1$. Let

$$w\,(t) = \int_\Omega \phi(x)u(x,t)dx, \tag{2.3}$$

where $u$ is a weak solution of the problem (1.1)–(1.3).

**Lemma 2.1.** *Assume that*

*(i)* $f(s) - \lambda s \geq 0$ *for* $s \in (0, c)$,
*(ii)* $f\left(\int_\Omega \phi(x)u_0\,(x)\,dx\right) > \lambda \int_\Omega \phi(x)u_0\,(x)\,dx$, *and*
*(iii)* $w\,(t) < c$ *for* $t > 0$.

*Then* $w'(t) > 0$ *for* $t > 0$.

*Proof.* From (2.3), we have

$$w'\,(t) = \int_\Omega \phi(x)u_t(x,t)dx.$$

Let $\varphi(x,t)=t^2\phi(x)$. It is obvious that (2.1) is satisfied with the definition of $\phi(x)$. From (2.2), we have

$$\begin{aligned} t^2w'(t)+at^2w(t)+btw(t) &= \int_0^t \Big[2\eta^2w'(\eta)+\left(b-\lambda\eta^2\right)w(\eta) \\ &\quad +\eta^2\int_\Omega \phi(x)f(u(x,\eta))\,dx\Big]\,d\eta. \end{aligned}$$

Therefore,

$$w''(t)+\left(a+\frac{b}{t}\right)w'(t)=-\lambda w(t)+\int_\Omega \phi(x)f(u)\,dx.$$

Since $f$ is positive and convex, by Jensen's inequality (Wheeden and Zygmund [6]), we have the following inequality

$$w''(t)+\left(a+\frac{b}{t}\right)w'(t)\geq f(w(t))-\lambda w(t). \tag{2.4}$$

Define

$$g(t)=t^be^{at},t\geq 0,$$

then, $g(t)$ is continuous on $[0,\infty)$ and differentiable on $(0,\infty)$.

Since

$$\frac{dg(t)w'(t)}{dt}=g'(t)w'(t)+g(t)w''(t),$$

it follows from $f(s)-\lambda s\geq 0$ on $(0,c)$, $g(t)>0$ for $t>0$, $f(s)>0$ on $(-\infty,c)$, and $\lambda>0$ that

$$\frac{dg(t)w'(t)}{dt}=g(t)\left[w''(t)+\left(a+\frac{k}{t}\right)w'(t)\right]\geq g(t)\left[f(w(t))-\lambda w(t)\right],t>0. \tag{2.5}$$

It follows that $f\left(\int_\Omega \phi(x)u_0(x)\,dx\right)>\lambda\int_\Omega \phi(x)u_0(x)\,dx$ and $g(t)>0$ for $t>0$ that there exists a positive constant $\varepsilon$ such that

$$\frac{dg(t)w'(t)}{dt}>0,t\in(0,\varepsilon).$$

Since $g(0)w'(0)=0$, then $g(t)w'(t)>0$ for $t\in(0,\varepsilon)$. It follows from $g(t)>0$ for $t>0$ that $w'(t)>0$ for $t\in(0,\varepsilon)$.

To prove that $w'(t)>0$ for $t>0$, we suppose that there exists some $\tau\geq\varepsilon$ such that $w'(\tau)=0$ and $w'(t)>0$ for $t\in(0,\tau)$. Define

$$P(t)=\left(w'(t)\right)^2,$$

then

$$P(\tau)=0,$$

and

$$P(t)>0,t\in(0,\tau).$$

Define

$$Q(t)=t^{2b}e^{2at},t\geq 0,$$

then, $Q(t) > 0$ for $t > 0$.

By the definitions of $P(t)$ and $Q(t)$, we know that for $t \in (0, \tau)$

$$\frac{dP(t)Q(t)}{dt} = \left(2a + \frac{2b}{t}\right) Q(t) P(t) + Q(t) P'(t) . \tag{2.6}$$

It follows from (2.4) and $w'(t) > 0$ for $t \in (0, \tau)$ that

$$P'(t) \geq \left(-a - \frac{2b}{t}\right) P(t) + 2w'(t) \left(f(w(t)) - \lambda w(t)\right). \tag{2.7}$$

From (2.6) and (2.7), we have

$$\begin{aligned} \frac{d\left(P(t)Q(t)\right)}{dt} &\geq Q(t)\left[\left(-a - \frac{2b}{t}\right) P(t) + 2w'(t)\left(f(w(t)) - \lambda w(t)\right)\right] \\ &\quad + \left(2a + \frac{2b}{t}\right) P(t) Q(t) \\ &\geq 2Q(t) w'(t) \left(f(w(t)) - \lambda w(t)\right). \end{aligned}$$

Therefore,

$$\frac{d\left(P(t)Q(t)\right)}{dt} \geq 2Q(t) w'(t) \left(f(w(t)) - \lambda w(t)\right). \tag{2.8}$$

By integrating (2.8) from $t_0$ to $\tau$ for some constant $t_0, 0 < t_0 < \tau$, we have

$$P(\tau) Q(\tau) \geq 2 \int_{t_0}^{\tau} Q(t) w'(t) \left(f(w(t)) - \lambda w(t)\right) dt + P(t_0) Q(t_0).$$

It follows from $Q(t) > 0, w'(t) > 0, f(w(t)) - \lambda w(t) \geq 0$, and $P(t_0) Q(t_0) > 0$ that $P(\tau) Q(\tau) > 0$. Since $Q(\tau) > 0$, then $P(\tau) > 0$, which is a contradiction to $P(\tau) = 0$. Therefore, $w'(t) > 0$ for $t > 0$.

**Theorem 2.1.** *For the problem* (1.1)–(1.3), *assume that*

*(i)* $f(s) - \lambda s \geq 0$ *for* $s \in (0, c)$,
*(ii)* $f\left(\int_\Omega \phi(x) u_0(x)\, dx\right) > \lambda \int_\Omega \phi(x) u_0(x)\, dx$, *and*
*(iii)* $a \leq 0$ *and* $b \leq 1$.

*Then a weak solution of the problem* (1.1)–(1.3) *must quench in finite time.*

*Proof.* Suppose that $u(x,t) < c$ for $(x,t) \in \Omega \times [0, \infty)$. By the definition of $w(t)$, we know that $w(t) < c$ for $t > 0$. Since all conditions in Lemma 2.1 are satisfied, then $w'(t) > 0$ for $t > 0$. It follows from (2.4) and $f(s) - \lambda s \geq 0$ in $(0, c)$ that

$$w''(t) + \frac{b}{t} w'(t) \geq 0.$$

Integrating this over $(t_1, t)$, $t_1 > 0$, we have

$$w'(t) \geq \frac{t_1^b w'(t_1)}{t^b}.$$

Integrating this over $(t_1, t)$, we have

$$w(t) \geq \begin{cases} w(t_1) + \dfrac{t_1^b w'(t_1)}{1-k}\left(t^{1-b} - t_1^{1-b}\right), \ 0 < b < 1, \\ w(t_1) + t_1 w'(t_1) \ln \frac{t}{t_1}, \ b = 1. \end{cases}$$

Since $w'(t_1) > 0$, there exists some $t_2 > t_1$ such that $w(t) \geq c$ for $t > t_2$. This contradiction shows that $u$ must quench in finite time.

**Theorem 2.2.** *For the problem* (1.1)–(1.3), *assume that*

*(i)* $f(s) - (1+\alpha)\lambda s \geq 0$ *in* $(0,c)$ *for any positive constant* $\alpha$,
*(ii)* $f'\left(\int_\Omega \phi(x)u_0(x)dx\right) \geq 0$, *and*
*(iii)* $a \leq 0$ *and* $b > 1$.

*Then a weak solution of the problem* (1.1)–(1.3) *must quench in finite time.*

*Proof.* Suppose that $u(x,t) < c$ for $(x,t) \in \Omega \times [0,\infty)$. By the definition of $w(t)$, we know that $w(t) < c$ for $t > 0$. It follows from $f(s) - (1+\alpha)\lambda s \geq 0$ *in* $(0,c)$ that $f(s) - \lambda s > 0$ *in* $(0,c)$. Since all conditions in Lemma 2.1 are satisfied, then $w'(t) > 0$ for $t > 0$.

Assume that $w(0) = \int_D \phi(x)u_0(x)dx > 0$. Then $0 < w(t) < c$, and $f(s) - \lambda s \geq \alpha\lambda s$ in $(0,c)$. Since $w'(t) > 0$, it follows from $w(t) > w(0)$, and (2.4) that

$$\frac{d}{dt}\left(t^b w'(t)\right) \geq (\alpha\lambda w(0))\, t^b, \ t > 0.$$

Integrating this from 0 to $t$, we have

$$w'(t) \geq \frac{\alpha\lambda w(0)}{b+1} t.$$

Another integration gives

$$w(t) \geq \frac{\alpha\lambda w(0)}{2(b+1)} t^2 + w(0).$$

It follows from $\alpha > 0$, and $w(0) > 0$ that there exists some $t_2 > 0$ such that $w(t_2) \geq c$. This contradiction proves that $u$ must quench in finite time.

Assume that $w(0) = \int_D \phi(x)u_0(x)dx \leq 0$. We claim that there exists some $t_3 > 0$ such that $w(t_3) \geq 0$. Suppose this is not true. Then, $w(t) < 0$ for $t > 0$. Since $w'(t) > 0$, and $-b/t > -1$ for any $t > b > 1$. It follows from (2.4) that,

$$w''(t) > -w'(t) + f(w(t)) - \lambda w(t).$$

Since $w(t) < 0$, $f\prime\left(\int_\Omega \phi(x)u_0(x)dx\right) \geq 0$, $f > 0$ is convex and $w'(t) > 0$, we have

$$f(w(t)) - \lambda w(t) > f(w(t)) \geq f(w(0)) > 0.$$

Hence,

$$w''(t) > -w'(t) + f(w(0)).$$

Integrating this inequality twice over $(b,t)$, it follows from $w(t)<0$ that

$$w(t) \geq \frac{1}{2} f(w(0))\, t^2 + + [w'(b) + w(b) - bf(w(0))]\, t + w(b).$$

Since $f(w(0)) > 0$, then there exists $t_4 > 0$ such that $w(t_4) \geq 0$. This proves our claim.

Since $w(t_4) \geq 0$ and $w'(t) > 0$, we have $w(t) > 0$ for $t > t_4$. For any $t_5 > t_4$ such that $w(t_5) < c$, it follows from (7) and $f(s) - \lambda s \geq \alpha\lambda s$ in $(0,c)$ that

$$\frac{d}{dt}\left(t^b w'(t)\right) \geqslant \alpha\lambda w(t)\, t^b \geqslant \alpha\lambda w(t_5)\, t^b,$$

where $w(t_5) > 0$. An argument similar to the above case in which $w(0) > 0$ shows that $u$ must quench in finite time.

**Theorem 2.3.** *For the problem* (1.1)–(1.3), *assume that*

*(i)* $f(s) - (1+\alpha)\lambda s \geq 0$ *in* $(0,c)$ *for any positive constant* $\alpha$,
*(ii)* $\int_\Omega \phi(x) u_0(x) dx > 0$, *and*
*(iii)* $a > 0$.

*Then a weak solution of the problem* (1.1)–(1.3) *must quench in finite time.*

*Proof.* Suppose that $u(x,t) < c$ for $(x,t) \in \Omega \times [0,\infty)$. By the definition of $w(t)$, we know that $w(t) < c$ for $t > 0$. It follows from $f(s) - (1+\alpha)\lambda s \geq 0$ *in* $(0,c)$ that $f(s) - \lambda s > 0$ *in* $(0,c)$. Since all conditions in Lemma 2.1 are satisfied, then $w'(t) > 0$ for $t > 0$.

Since $w'(t) > 0$, it follows from (2.5) and $g(t) > 0$ that

$$\frac{d\left(g(t)\, w'(t)\right)}{dt} \geq (\alpha\lambda w(0))\, g(t),\ t > 0. \tag{2.9}$$

It follows from (2.9), $g(t) > 0, g'(t) > 0$ for $t > 0$ and $g(t) \to \infty$ as $t \to \infty$ that $w'(t) \geq \alpha\lambda\beta w(0)$ for some positive constant $\beta$. Integrating the inequality over $[0,t]$ gives

$$w(t) \geq \alpha\lambda\beta w(0)\, t + w(0). \tag{2.10}$$

It follows from (2.10), $\alpha > 0$, and $w(0) > 0$ that there exists some $t_6 > 0$ such that $w(t_6) = c$. This contradiction proves that $u$ must quench in finite time.

**Theorem 2.4.** *For the problem* (1.1)–(1.3), *assume that*

*(i)* $f(s) - (1+\alpha)\lambda s \geq 0$ *in* $(0,c)$ *for any positive constant* $\alpha$,
*(ii)* $f'\left(\int_\Omega \phi(x) u_0(x) dx\right) \geq 0$,
*(iii)* $\int_\Omega \phi(x) u_0(x) dx \leq 0$, *and*
*(iv)* $a > 0$.

*Then a weak solution of the problem* (1.1)–(1.3) *must quench in finite time.*

*Proof.* Suppose that $u(x,t) < c$ for $(x,t) \in \Omega \times [0,\infty)$. By the definition of $w(t)$, we know that $w(t) < c$ for $t > 0$. We claim that there exists some $t_6 > 0$ such that

$w(t_6) \geq 0$. Suppose this is not true. Then, $w(t) < 0$ for $t > 0$. It follows from $f(s) - (1+\alpha)\lambda s \geq 0$ *in* $(0,c)$ that $f(s) - \lambda s > 0$ *in* $(0,c)$. Since all conditions in Lemma 2.1 are satisfied, then $w'(t) > 0$ for $t > 0$. Since $w'(t) > 0$, and $-b/t > -1$ for any $t > b > 1$. It follows (2.4) that,

$$w''(t) > -(1+a)\,w'(t) + f(w(t)) - \lambda w(t).$$

Since $w(t) < 0$, $f'\left(\int_\Omega \phi(x)u_0(x)dx\right) \geq 0$, $f > 0$ is convex and $w'(t) > 0$, we have

$$f(w(t)) - \lambda w(t) > f(w(t)) \geq f(w(0)) > 0.$$

Hence,

$$w''(t) > -(1+a)\,w'(t) + f(w(0)).$$

Multiplying this inequality by $e^{(1+a)t}$, and integrating twice over $(b,t)$, we have

$$w(t) \geq w(b) + \frac{1}{1+a}\left[w'(b) - f(w(0))\right]\left(1 - e^{(1+a)(b-t)}\right) + f(w(0))(t-b).$$

It follows from $0 < 1 - e^{(1+a)(b-t)} < 1$ for $t > b$ and $a > 0$ that

$$w(t) \geq w(b) - \frac{1}{1+a}\left|w'(b) - f(w(0))\right| + f(w(0))(t-b).$$

It follows from $f(w(0)) > 0$ that $w(t_6) \geq 0$ for some $t_6 > 0$. This proves our claim.

Since $w'(t) > 0$, we have $w(t) > 0$ for $t > t_6$. For any $t_7 > t_6$ such that $w(t_7) < c$, it follows from (2.5) and $f(s) - \lambda s \geq \alpha\lambda s$ in $(0,c)$ that

$$\frac{d}{dt}\left(g(t)\,w'(t)\right) \geq \alpha\lambda w(t)\,g(t) \geq \alpha\lambda w(t_7)\,g(t), \tag{2.11}$$

where $w(t_7) > 0$. It follows from (2.11), $g(t) > 0$, $g'(t) > 0$ for $t > 0$ and $g(t) \to \infty$ as $t \to \infty$ that $w'(t) \geq \alpha\lambda\beta w(0)$ for some positive constant $\beta$. Integrating the inequality over $[t_7, t]$ gives

$$w(t) \geq \alpha\lambda\beta w(t_7)\,t + w(t_7). \tag{2.12}$$

It follows from (2.12), $\alpha > 0$, and $w(t_7) > 0$ that there exists some $t_8 > 0$ such that $w(t_8) = c$. This contradiction proves that $u$ must quench in finite time.

## 3. BLOW-UP RESULTS FOR DERIVATIVES OF SOLUTIONS

We have the following conclusions.

**Theorem 3.1**. *For the problem* (1.1)–(1.3), *assume that*

*(i)* $f(s) - \lambda s \geq 0$ *for* $s \in (0,c)$,
*(ii)* $f\left(\int_\Omega \phi(x)u_0(x)\,dx\right) > \lambda \int_\Omega \phi(x)u_0(x)\,dx$,
*(iii)* $a \leq 0$ *and* $b \leq 1$, *and*
*(iv)* $\int_0^c f(u)\,du = \infty$.

*Then the first derivative of a weak solution of the problem* (1.1)–(1.3) *blows up in finite time.*

*Proof.* Since all conditions in Theorem 2.1 are satisfied, we know that there exists a positive number $t^*$ such that $\lim_{t\to t^*} w(t) = c$, and $w'(t) > 0$ for $t > 0$.

By the definition of $P(t)$ we know that $P(t) > 0$ for $t > 0$. Similar to (2.8), we have

$$\frac{d(P(t)Q(t))}{dt} \geq 2Q(t)\, w'(t)\,(f(w(t)) - \lambda w(t)),\, t > 0. \tag{3.1}$$

Integrating (3.1) over $(\gamma, t)$ for some $\gamma > 0$, we have

$$\begin{aligned} P(t)\,Q(t) &\geq P(\gamma)\,Q(\gamma) + 2\int_\gamma^t Q(\tau)\,(f(w(\tau)) - \lambda w(\tau))\,w'(\tau)\,d\tau \\ &= Q(\gamma)\,(w'(\gamma))^2 + 2\int_\gamma^t Q(\tau)\,(f(w(\tau)) - \lambda w(\tau))\,w'(\tau)\,d\tau. \end{aligned} \tag{3.2}$$

From (3.2) and $Q(t) > 0$, we have

$$P(t) \geq \frac{Q(\gamma)\,(w'(\gamma))^2}{Q(t)} + \frac{2}{Q(t)}\int_\gamma^t Q(\tau)\,(f(w(\tau)) - \lambda w(\tau))\,w'(\tau)\,d\tau. \tag{3.3}$$

Applying the fact that $0 < Q(t) \leq t^{2b}$, $Q'(t) > 0$ and $a \leq 0$ to (3.3), we obtain

$$P(t) \geq \frac{\gamma^{2b}\,(w'(\gamma))^2}{t^{2b}} + \frac{2\gamma^{2b}}{t^{2b}}\int_\gamma^t (f(w(\tau)) - \lambda w(\tau))\,w'(\tau)\,d\tau. \tag{3.4}$$

It follows from (3.4), $w'(\gamma) > 0$, $P(t) = (w'(t))^2$ and $t > \gamma > 0$ that

$$w'(t) > \frac{\sqrt{2}\gamma^b}{t^b}\left[\int_{w(\gamma)}^{w(t)} (f(s) - \lambda s)\,ds\right]^{\frac{1}{2}}.$$

From $\int_0^c f(u)\,du = \infty$, we have $\lim_{t\to t^*} w'(t) = \infty$. Since

$$w'(t) = \int_\Omega \phi(x) u_t(x,t) dx \leq \max_{x\in\Omega} u_t(x,t),$$

it follows that $u_t$ blows up in finite time.

**Theorem 3.2**. *For the problem* (1.1)–(1.3), *assume that*

*(i)* $f(s) - (1+\alpha)\lambda s \geq 0$ *in* $(0, c)$ *for any positive constant* $\alpha$,
*(ii)* $f'\left(\int_\Omega \phi(x) u_0(x) dx\right) \geq 0$,
*(iii)* $a \leq 0$ *and* $b > 1$, *and*
*(iiv)* $\int_0^c f(u)\,du = \infty$.

*Then the first derivative of a weak solution of the problem* (1.1)–(1.3) *blows up in finite time.*

*Proof.* Since all conditions in Theorem 2.2 are satisfied, we know that there exists a positive number $t^*$ such that $\lim_{t\to t^*} w(t) = c$, and $w'(t) > 0$ for $t > 0$. Similar to Theorem 3.1, upon integrating (3.1) over $(\delta, t)$ for some $\delta > 0$, we have

$$w'(t) > \frac{\sqrt{2}\delta^b}{t^b}\left[\int_{w(\delta)}^{w(t)} (f(s) - \lambda s)\, ds\right]^{\frac{1}{2}}.$$

From $\int_0^c f(u)\, du = \infty$, we have $\lim_{t\to t^*} w'(t) = \infty$. Since

$$w'(t) = \int_\Omega \phi(x) u_t(x,t) dx \le \max_{x\in\Omega} u_t(x,t),$$

it follows that $u_t$ blows up in finite time.

**Theorem 3.3**. *For the problem* (1.1)–(1.3), *assume that*

*(i)* $f(s) - (1+\alpha)\lambda s \ge 0$ *in* $(0, c)$ *for any positive constant* $\alpha$,
*(ii)* $\int_\Omega \phi(x) u_0(x) dx > 0$,
*(iii)* $a > 0$, *and*
*(iv)* $\int_0^c f(u)\, du = \infty$.

*Then the first derivative of a weak solution of the problem* (1.1)–(1.3) *blows up in finite time.*

*Proof.* Since all conditions in Theorem 2.3 are satisfied, we know that there exists a positive number $t^*$ such that $\lim_{t\to t^*} w(t) = c$, and $w'(t) > 0$ for $t > 0$.

By the definition of $P(t)$, we know that $P(t) > 0$ for $t > 0$. Similar to (2.8), we have

$$\frac{d(P(t)Q(t))}{dt} \ge 2Q(t)w'(t)(f(w(t)) - \lambda w(t)), t > 0. \tag{3.5}$$

Integrating (3.5) over $(\eta, t)$ for some $\eta > 0$, we have

$$\begin{aligned} P(t)Q(t) &\ge P(\eta)Q(\eta) + 2\int_\eta^t Q(\tau)(f(w(\tau)) - \lambda w(\tau))w'(\tau)\, d\tau \\ &= Q(\eta)(w'(\eta))^2 + 2\int_\eta^t Q(\tau)(f(w(\tau)) - \lambda w(\tau))w'(\tau)\, d\tau. \end{aligned} \tag{3.6}$$

From (3.6) and $Q(t) > 0$, we have

$$P(t) \ge \frac{Q(\eta)(w'(\eta))^2}{Q(t)} + \frac{2}{Q(t)}\int_\eta^t Q(\tau)(f(w(\tau)) - \lambda w(\tau))w'(\tau)\, d\tau. \tag{3.7}$$

Applying the fact that $Q(t) > 0$, and $Q'(t) > 0$ to (3.7), we obtain

$$P(t) \ge \frac{Q(\eta)(w'(\eta))^2}{Q(t)} + \frac{2Q(\eta)}{Q(t)}\int_\eta^t (f(w(\tau)) - \lambda w(\tau))w'(\tau)\, d\tau. \tag{3.8}$$

It follows from (3.8), $w'(\eta) > 0$, $P(t) = (w'(t))^2$ and $t > \eta > 0$ that

$$w'(t) > \frac{\sqrt{2}Q(\eta)}{Q(t)}\left[\int_{w(\gamma)}^{w(t)} (f(s) - \lambda s)\, ds\right]^{\frac{1}{2}}.$$

From $\int_0^c f(u)\,du = \infty$ and $Q(t) < \infty$ on $(0, t^*)$, we have $\lim_{t\to t^*} w'(t) = \infty$. Since

$$w'(t) = \int_\Omega \phi(x) u_t(x,t)dx \le \max_{x\in\Omega} u_t(x,t),$$

it follows that $u_t$ blows up in finite time.

**Theorem 3.4**. *For the problem* (1.1)–(1.3), *assume that*

*(i)* $f(s) - (1+\alpha)\lambda s \ge 0$ *in* $(0,c)$ *for any positive constant* $\alpha$,
*(ii)* $f'\left(\int_\Omega \phi(x) u_0(x) dx\right) \ge 0$,
*(iii)* $\int_\Omega \phi(x) u_0(x) dx \le 0$,
*(iv)* $a > 0$, *and*
*(v)* $\int_0^c f(u)\,du = \infty$.

*Then the first derivative of a weak solution of the problem* (1.1)–(1.3) *blows up in finite time.*

*Proof.* Since all conditions in Theorem 2.4 are satisfied, we know that there exists a positive number $t^*$ such that $\lim_{t\to t^*} w(t) = c$, and $w'(t) > 0$ for $t > 0$.

By the definition of $P(t)$ we know that $P(t) > 0$ for $t > 0$. Similar to Theorem 3.3, we have

$$\frac{d\left(P(t)\,Q(t)\right)}{dt} \ge 2Q(t)\,w'(t)\left(f(w(t)) - \lambda w(t)\right), t > 0. \tag{3.9}$$

Upon integrating (3.9) over $(\rho, t)$ for some $\rho > 0$, we have

$$w'(t) > \frac{\sqrt{2}Q(\rho)}{Q(t)}\left[\int_{w(\rho)}^{w(t)} \left(f(s) - \lambda s\right) ds\right]^{\frac{1}{2}}.$$

From $\int_0^c f(u)\,du = \infty$ and $Q(t) < \infty$ on $(0, t^*)$, we have $\lim_{t\to t^*} w'(t) = \infty$. Since

$$w'(t) = \int_\Omega \phi(x) u_t(x,t)dx \le \max_{x\in\Omega} u_t(x,t),$$

it follows that $u_t$ blows up in finite time.

## 4. NUMERICAL RESULTS

In this section, quenching phenomena of various initial-boundary value problems for the generalized Euler-Poisson-Darboux equation are discussed with application of theorems stated in Section 2. The numerical solutions of these problems before the quenching time are obtained. Let us consider the one-dimensional initial-boundary problem,

$$u_{tt} + \left(a + \frac{b}{t}\right) u_t - \Delta u = f(u), \quad (x,t) \in (0,L) \times (0,T), \tag{4.1}$$

$$u(x,0) = u_0(x), u_t(x,0) = 0, \quad x \in (0,L), \tag{4.2}$$

$$u(0,t) = 0,\ u(L,t) = 0, \quad t \in [0,T). \tag{4.3}$$

Considering the problem (4.1)–(4.3) with $a = -1, b = 0.5, L = 2, f(s) = \frac{1}{1-s}$, and the initial condition $u_0(x) = 0$. With $a < 0$, $b \leq 1$, and $u_0(x) = 0$, it is obvious that conditions (ii) and (iii) are satisfied, and if $L \geq \frac{\pi}{2}$, then the condition (i) is satisfied. According to the Theorem 2.1, a solution of the one-dimensional initial-boundary problem (4.1)–(4.3) quenches in finite time. Figure 1.1 shows the behavior of the solution $u(x,t)$, $x \in [0,2]$ before the quenching time; Figure 1.2 shows the solution at different times $t = 0.0$, 0.2, 0.4, 0.8, 1.0, 1.2, and 1.31. The numerical results indicate that the quenching time is approximately 1.311.

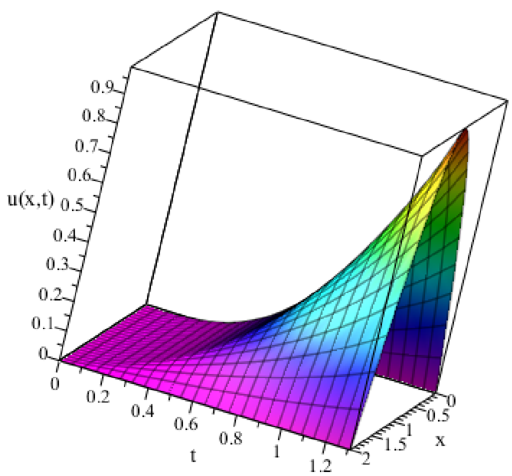

Figure 1.1. Solution u(x,t) in 3-dimensional space

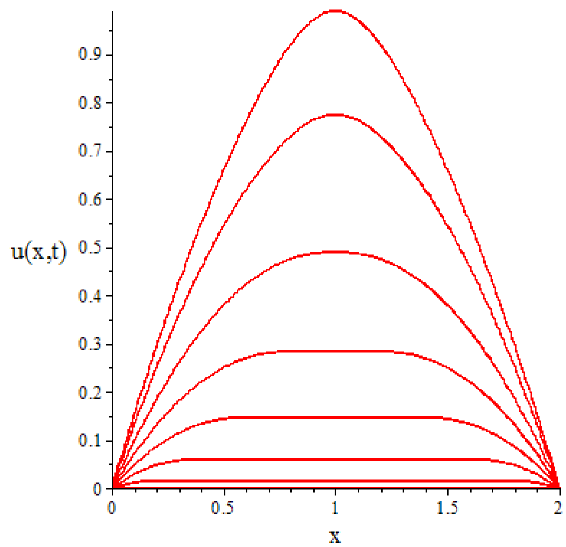

Figure 1.2. Solutions at t = 0.0, 0.2, 0.4, 0.8, 1.0, 1.2, 1.31

For the problem (4.1)–(4.3) with given $a = 1$, $b = 1$, $L = 2, f(s) = \frac{1}{1-s}$, and the initial condition $u_0(x) = \frac{x(2-x)}{5}$; since $a < 0, b \leq 1$, and $u_0(x) > 0$, for $x \in (0,2)$, it is obvious that conditions (ii) and (iii) are satisfied, and if $L > \frac{\pi}{2}$, then the condition (i) is satisfied, it follows the Theorem 2.3 that a solution of the one-dimensional initial-boundary problem (4.1)–(4.3) quenches in finite time. Figure 2.1 shows the behavior of the solution $u(x,t)$, $x \in [0,2]$ before the quenching time; Figure 2.2 shows the solution at different times $t = 0.0$, 0.4, 0.8, 1.2, 1.6, 1.8 and 1.97. The numerical results indicate that the quenching time is approximately 1.971.

For the problem (4.1)–(4.3) with given $a = 1$, $b = 1$, $L = 2, f(s) = \frac{1}{1-s}$, and the initial condition $u_0(x) = -\frac{x(2-x)}{5}$, since $a < 0$, $b \leq 1$, and $u_0(x) > 0$, for $x \in (0,2)$, then conditions (ii) and (iii) are satisfied, and if $L > \frac{\pi}{2}$, then the condition (i) is satisfied, therefore, according to the Theorem 2.4, a solution of the one-dimensional initial-boundary problem (4.1)–(4.3) quenches in finite time. Figure 3.1 shows the behavior of the solution $u(x,t)$, $x \in [0,2]$ before the quenching time; Figure 3.2 shows the solution at different times $t = 0.0$, 0.4, 0.8, 1.2, 1.6, 2.0, 2.4 and 2.489. The numerical results indicate that the quenching time is approximately 2.490.

We investigate the effect of the length of the interval on the behavior of solutions of the problem (4.1)–(4.3). Let us consider the problem (4.1)–(4.3) with $a = -1$,

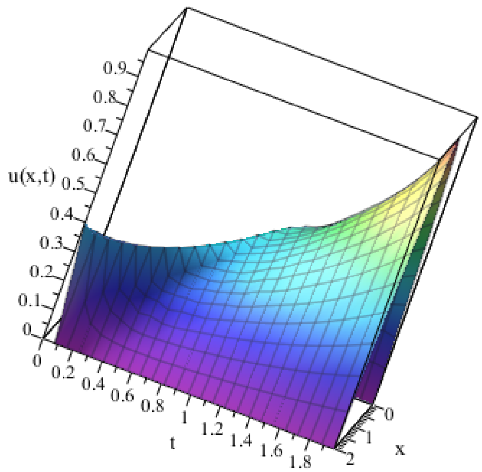


Figure 2.1. Solution u(x,t) in 3-dimensional space

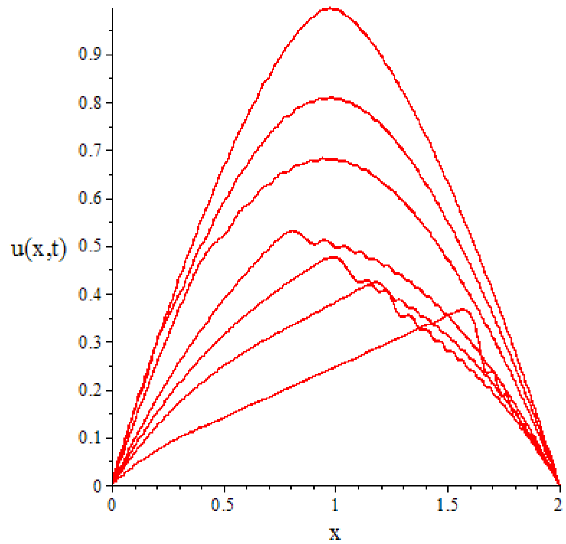


Figure 2.2. Solutions at t = 0.0, 0.4, 0.8, 1.2, 1.6, 1.8, 1.97

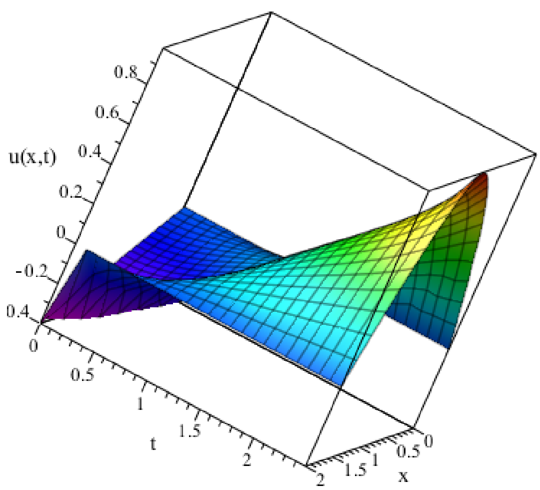


Figure 3.1. Solution u(x,t) in 3-dimensional space

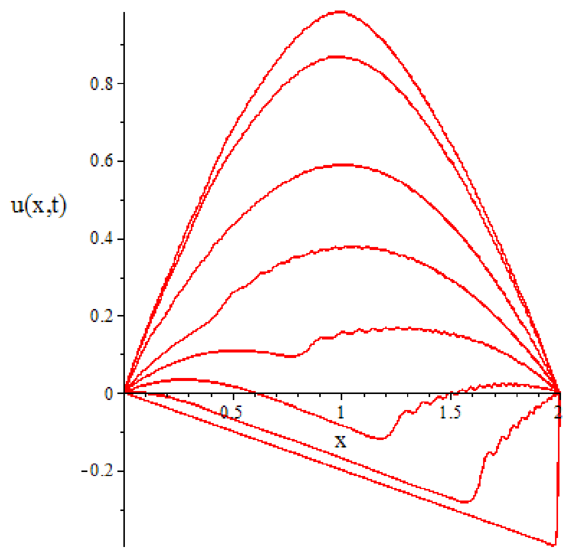


Figure 3.2. Solutions at t = 0.0, 0.4, 0.8, 1.2, 1.6, 2.0, 2,4, 2.489

$b = 0.5$, $f(s) = \frac{1}{1-s}$, and the initial condition $u_0(x) = 0$, also $L = 1.6 > \frac{\pi}{2}$. Figure 4.1 shows the behavior of the solution $u(x,t)$, $x \in [0,1]$ and $t \in [0, 1.45]$; Figure 4.2 shows the solution at different times $t = 0.0$, 0.2, 0.4, 0.6, 0.8, 1.0, 1.2, and 1.45. The numerical results indicate that the quenching time is approximately 1.451. Figures 1.1 and 1.2 and Figures 4.1 and 4.2 contrast the behavior of a solution of $u$ of (4.1)–(4.3) for different values of $L$, and the numerical results for $L = 2$ and $L = 1$ show that with the same $a, b$ and the initial condition $u_0(x)$, the length of the interval plays an important role in determining the quenching time.

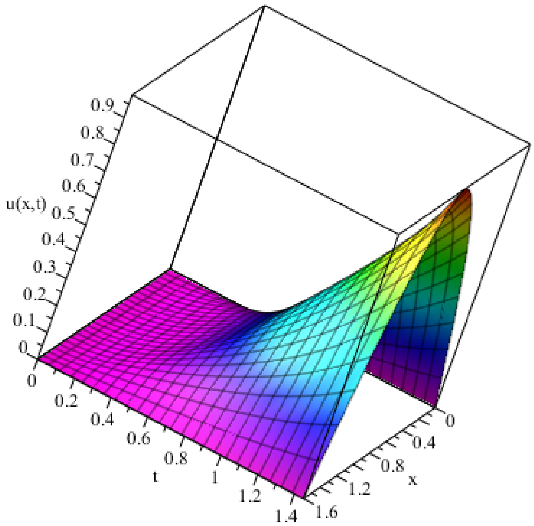


Figure 4.1. Solution u(x,t) in 3-dimensional space

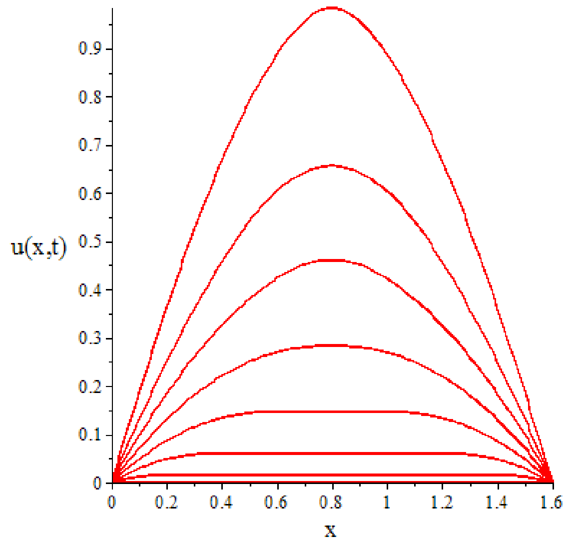


Figure 4.2. Solutions at t = 0.0, 0.2, 0.4, 0.6, 0.8, 1.0, 1.2, 1.45

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