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NUMERICAL RESULTS FOR LINEAR CAPUTO FRACTIONAL DIFFERENTIAL EQUATIONS WITH VARIABLE COEFFICIENTS AND APPLICATIONS

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ABSTRACT. Recently, we published results which includes the symbolic representation for the linear Caputo fractional differential equations in the journal "Mathematics." Also, we obtained numerical results by iterative methods. In this paper, we derive numerical results by direct numerical method using the symbolic representation we have obtained earlier. This direct numerical method is useful in developing monotone method and quasilinearization method for nonlinear problems. As an application of this result, we have obtained the numerical solution for a special Ricatti, type of differential equation which blows up in finite time.

AMS (MOS) Subject Classification. 34A08, 34A12.

1. Introduction

Dynamic systems with fractional derivative is known to be more useful, suitable and economical when compared to integer derivative, see [1–5] and the references there in for more details. In addition, the behavior of the solution of the fractional dynamic systems is global in nature compared with dynamic systems with integer derivatives. In the past three decades dynamic systems with fractional derivative has gained importance due to its advantage in applications. See [2–6, 9, 12, 14–20] for some applications. There are plenty of results available in literature for existence and uniqueness of solutions of nonlinear fractional differential equations, a vast majority of the results are via some kind of fixed point theorem methods. In order to compute the solution of the nonlinear fractional dynamical system, the method of upper and lower solution or the method of coupled lower and upper solution together with an iterative method is more appropriate. The advantage of such a method is that the interval of existence of the solution is guaranteed [6, 10]. In order to develop such an iterative method one of the important step is to compute the solution of the linear equation with either constant coefficients or variable coefficients. The solution of the Caputo linear fractional differential equation of order q, $0 < q < 1$, with constant coefficients will be in terms of the Mittag leffler function. The solution of the variable

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coefficients has been obtained symbolically in [8]. In addition, we have obtained the approximate solution of the Caputo linear fractional differential equation of order q, $0 < q < 1$, with variable coefficients by iterative method. In this paper we developed a direct numerical method for the Caputo linear equation with variable coefficients. Our direct numerical method is applicable when the coefficient term and the non homogeneous term are of the form $(t - t_0)^q$.

As an application of direct numerical method we have considered the example ${}^cD^qu = u^2,\ u(0) = 1$. Initially, we have developed Picards approximation as well as Quasilinearization approximation. Our direct numerical method encounter a hurdle. The hurdle is we do not have an explicit form of the iterate. Discrete values of the iterates are only known. We plan to improve on our direct numerical method when the initial iterates are known at discrete points. We have provided some numerical examples for linear Caputo fractional differential equations of order q, $0 < q < 1$, with variable coefficients by iterative method and direct numerical approximation method. The direct numerical approximations for $q \neq 1$ are developed in a way to provide the exact solution for $q = 1$.

2. **Preliminary Results**

In this section, initially we recall basic definitions, results of fractional derivatives and integrals. We also present symbolic representation for the linear Caputo fractional differential equation with variable coefficients see [8].

Definition 2.1. The Caputo (left-sided) fractional derivative of u(t) of order q, $n - 1 < q < n$, is given by equation

$$(2.1) \qquad {}^cD^qu(t) = \frac{1}{\Gamma(n-q)} \int_{t_0}^{t} (t-s)^{n-q-1} u^{(n)}(s)ds, \quad t \in [t_0, t_0 + T],$$

and (right-sided)

$$(2.2) \qquad {}^cD^qu(t) = \frac{(-1)^n}{\Gamma(n-q)} \int_{t}^{t_0+T} (s-t)^{n-q-1} u^{(n)}(s)ds, \quad t \in [t_0, t_0 + T],$$

where $u^{(n)}(t) = \frac{d^n(u)}{dt^n}$.

In particular, $q = n$, an integer, then ${}^cD^qu = u^{(n)}(t)$ and ${}^cD^qu = u'(t)$ if $q = 1$.

Definition 2.2. The Caputo (left-sided) fractional derivative of order q, where $0 < q < 1$ is given by equation

$$(2.3) \qquad {}^cD^qu(t) = \frac{1}{\Gamma(1-q)} \int_{t_0}^{t} (t-s)^{-q} u'(s)ds,$$

where $u'(t) = du/dt$.

Definition 2.3. The Riemann-Liouville (left-sided) fractional integral of order q is defined as,

$$D^{-q}u(t) = \frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}u(s)ds,\ t<T,\ 0<q<1. \tag{2.4}$$

One can also define the right-sided Riemann-Liouville fractional integral. In this work, we use only left sided integral. Note that $q=1$ in definitions (2.1) and (2.2) is the special case of the integer derivative. In order to compute the solutions, we introduce the two-parameter Mittag-Leffler functions.

Definition 2.4. The Mittag-Leffler function is given by

$$E_{\alpha,\beta}(\lambda(t-t_0)^{\alpha}) = \sum_{k=0}^{\infty}\frac{(\lambda(t-t_0)^{\alpha})^k}{\Gamma(\alpha k+\beta)}, \tag{2.5}$$

where $\alpha,\beta>0$ and λ is a constant. Furthermore, for $t_0=0$, $\alpha=q$ and $\beta=q$, it reduces to

$$E_{q,q}(\lambda t^q) = \sum_{k=0}^{\infty}\frac{(\lambda t^q)^k}{\Gamma(qk+q)}, \tag{2.6}$$

where $q>0$. If $\alpha=q$ and $\beta=1$, then

$$E_{q,1}(\lambda t^q) = \sum_{k=0}^{\infty}\frac{(\lambda t^q)^k}{\Gamma(qk+1)}, \tag{2.7}$$

where $q>0$.

If $q=1$, then $E_{1,1}(\lambda t)=e^{\lambda t}$. See [2, 4, 5, 8, 12] for more details. The work in [12] is exclusively for the study and application of the Mittag-Leffler function.

Consider the linear Caputo fractional differential equation

$${}^cD^qu(t) = p(t)u+f(t),\quad t_0<t<t_0+T,\quad T>0,\quad u(t_0)=u_0, \tag{2.8}$$

where $p(t)$ and $f(t)$ are continuous on $[t_0,t_0+T]$.

In particular, if $p(t)=\lambda$, the analytical solution of (2.8) is given by

$$u(t) = u_0E_{q,1}(\lambda(t-t_0)^q) + \int_{t_0}^{t}(t-s)^{q-1}E_{q,q}(\lambda(t-s)^q)f(s)ds. \tag{2.9}$$

For details see [2, 4].

If $p(t)$ is a function of t, $p(t)\in C([t_0,t_0+T],\mathbb{R})$, the solution of (2.8) has been obtained symbolically as follows,

$$u(t) = u_0e^{{}^cD^{-q}p(t)} + \frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}e^{{}^cD^{-q}p(\sigma)|_s^t}f(s)ds. \tag{2.10}$$

where

$${}^cD^{-q}p(\sigma)|_s^t = {}^cD^{-q}p(\sigma)|_0^t - {}^cD^{-q}p(\sigma)|_0^s \tag{2.11}$$

Note that e in (2.10) is not the usual exponential function and it does not hold the usual exponential properties.

For details see [8]. In our earlier work, we have obtained numerical solution for Caputo linear equation with variable coefficients by iterative method which is the same method adopted to obtain the symbolic representation. In this work, we have developed direct numerical methods to compute the solution of (2.8). The direct numerical approximations for $q \neq 1$ are developed in a way to provide the exact solution for $q = 1$. This is our next result.

3. Main Results

3.1. **Numerical results for linear Caputo fractional differential equations with variable coefficients.** Our main result consist of two parts. In section (3.1), we develop direct numerical approximation methods for the computation of solution of (2.8) for the special case when $p(t) = A+B(t-t_0)^q$ and $f(t) = \alpha(t-t_0)^q$. In section (3.2), we apply the direct numerical approximations to develop numerical methods to solve Ricatti type of Caputo fractional differential equation as an application of the results of section (3.1).

Consider the linear Caputo fractional differential equations with variable coefficients of order q.

$$ {}^cD^q u = p(t)u + f(t), \quad u(t_0) = u_0. \tag{3.1}$$

where $p(t)$ and $f(t)$ are continuous function on $[t_0, t_0 + T]$.

Here is the numerical examples for our explicit computation of solutions when $0 < q < 1$ by iterative method.

Example 1: If $p(t) = (t - t_0)^q$ *and* $f(t) = \alpha * (t - t_0)^q$, $\alpha = 1.5$, then (3.1) becomes

$$ {}^cD^q u(t) = (t - t_0)^q u + 1.5(t - t_0)^q, \quad u(t_0) = u_0 \ \ on \ \ J = [t_0, t_0 + T], \tag{3.2}$$

where $0 < q < 1$. The solution is obtained in the form

$$u(t) = u_0\{1 + \sum_{k=1}^{\infty}((t - t_0)^{2q})^k \prod_{r=1}^{k} \frac{\Gamma((2r-1)q+1)}{\Gamma(2rq+1)}\}$$
$$+ 1.5\sum_{k=1}^{\infty}((t - t_0)^{2q})^k \prod_{r=1}^{k} \frac{\Gamma((2r-1)q+1)}{\Gamma(2rq+1)}.$$

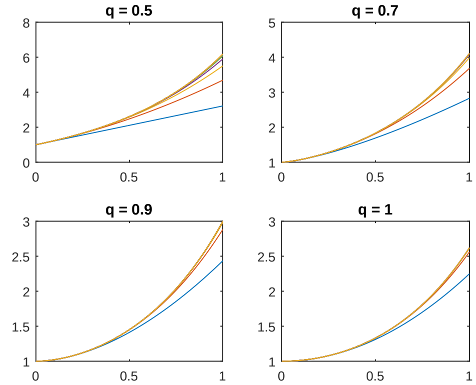

FIGURE 1. $p(t) = (t - t_0)^q$ and $f(t) = \alpha * (t - t_0)^q$ where $q = 0.5, 0.7, 0.9, 1.0$

Example 2: When $p(t) = -(t - t_0)^q$ and $f(t) = \alpha * (t - t_0)^q$, then (2.8) becomes

$$ {}^cD^q u(t) = -(t - t_0)^q u + 1.5(t - t_0)^q, \quad u(t_0) = u_0 \ \ on \ \ J = [t_0, t_0 + T], \tag{3.3} $$

where $0 < q < 1$.

The solution is obtained in the form

$$ u(t) = u_0\{1 + \sum_{k=1}^{\infty}((t - t_0)^{2q})^k(-1)^k \prod_{r=1}^{k} \frac{\Gamma((2r-1)q+1)}{\Gamma(2rq+1)}\} \\ - 1.5\sum_{k=1}^{\infty}((t - t_0)^{2q})^k(-1)^k \prod_{r=1}^{k} \frac{\Gamma((2r-1)q+1)}{\Gamma(2rq+1)}. $$

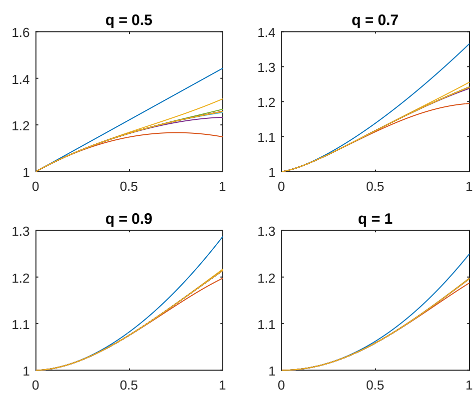

FIGURE 2. $p(t) = -(t - t_0)^q$ and $f(t) = \alpha * (t - t_0)^q$ where $q = 0.5, 0.7, 0.9, 1.0$

Example 3: When $p(t) = (t - t_0)^q$ and $f(t) = -\alpha * (t - t_0)^q$, then (2.8) becomes

$$ {}^cD^q u(t) = (t - t_0)^q u - 1.5(t - t_0)^q, \quad u(t_0) = u_0 \ \ on \ \ J = [t_0, t_0 + T], \tag{3.4} $$

where $0 < q < 1$.

The solution is obtained in the form

$$u(t) = u_0\{1 + \sum_{k=1}^{\infty}((t-t_0)^{2q})^k \prod_{r=1}^{k} \frac{\Gamma((2r-1)q+1)}{\Gamma(2rq+1)}\} - 1.5\sum_{k=1}^{\infty}((t-t_0)^{2q})^k \prod_{r=1}^{k} \frac{\Gamma((2r-1)q+1)}{\Gamma(2rq+1)}.$$

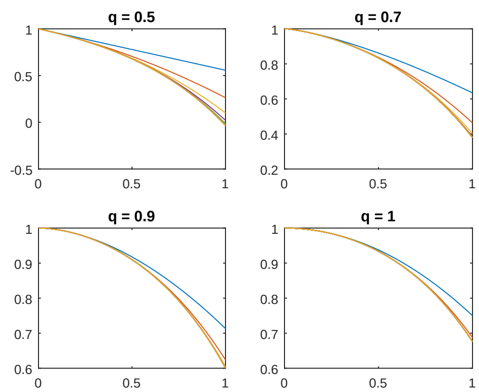

FIGURE 3. $p(t) = (t-t_0)^q$ and $f(t) = -\alpha * (t-t_0)^q$ where $q = 0.5, 0.7, 0.9, 1.0$

Example 4: When $p(t) = -(t-t_0)^q$ and $f(t) = -\alpha * (t-t_0)^q$, then (2.8) becomes

$$(3.5) \qquad {}^cD^q u(t) = -(t-t_0)^q u - 1.5(t-t_0)^q, \quad u(t_0) = u_0 \ \ on \ \ J = [t_0, t_0+T],$$

where $0 < q < 1$.

The solution is obtained in the form

$$u(t) = u_0\{1 + \sum_{k=1}^{\infty}((t-t_0)^{2q})^k(-1)^k \prod_{r=1}^{k} \frac{\Gamma((2r-1)q+1)}{\Gamma(2rq+1)}\} + 1.5\sum_{k=1}^{\infty}((t-t_0)^{2q})^k(-1)^k \prod_{r=1}^{k} \frac{\Gamma((2r-1)q+1)}{\Gamma(2rq+1)}.$$

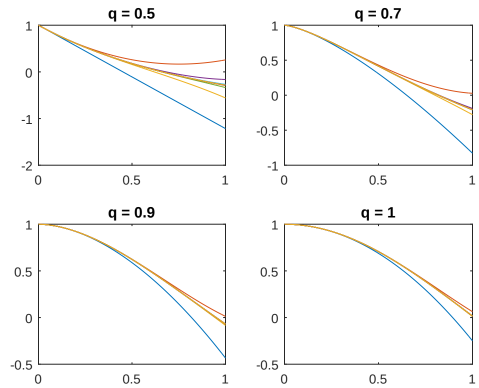

FIGURE 4. $p(t) = -(t-t_0)^q$ and $f(t) = -\alpha * (t-t_0)^q$ when $q = 0.5, 0.7, 0.9, 1.0$

Next we present the direct numerical approximation for symbolic representation form.

The symbolic representation of (3.1) is

$$u(t) = u_0 e^{{}^cD^{-q}p(t)} + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} e^{{}^cD^{-q}p(\sigma)|_s^t} f(s)ds, \tag{3.6}$$

where

$${}^cD^{-q}p(\sigma)|_s^t = {}^cD^{-q}p(\sigma)|_0^t - {}^cD^{-q}p(\sigma)|_0^s. \tag{3.7}$$

Our direct numerical approximation to solve (3.1) is by approximating the symbolic representation (3.6). The approximations are given by,

(3.8)
$$u(t_i) \approx u_0 E_{q,1}((\frac{p(t_i)+p(t_0)}{\Gamma(2q+1)})(\frac{(t(i)-t_0)^q}{\Gamma(q+1)})) + \alpha(E_{q,1}((\frac{p(t_i)+p(t_0)}{\Gamma(2q+1)})(\frac{(t(i)-t_0)^q}{\Gamma(q+1)})) - 1)$$

provided $\alpha \in \mathbb{R} \setminus \{-1\}$ and $p(t) > 0$.

(3.9)
$$u(t_i) \approx u_0 E_{q,1}((\frac{p(t_i)+p(t_0)}{\Gamma(2q+1)})(\frac{(t(i)-t_0)^q}{\Gamma(q+1)})) + \alpha(1 - E_{q,1}((\frac{p(t_i)+p(t_0)}{\Gamma(2q+1)})(\frac{(t(i)-t_0)^q}{\Gamma(q+1)})))$$

provided $\alpha \in \mathbb{R} \setminus \{1\}$ and $p(t) < 0$ where $p(t_i)$ is the value of $p(t)$ at each t_i and $f(t_i)$ is the value of $f(t)$ at each t_i.

Next we present the graph for our explicit computation of solutions when $0 < q < 1$ by direct numerical method. Here we assume $p(t)$ and $f(t)$ are known and continuous functions.

The red dotted lines in the following graph will represent the direct numerical graphs. Here we consider the same Example $(1, 2, 3, 4)$ for the direct numerical method. The graphs given below is the comparison of iterative and direct numerical method. When $q = 1$, our direct numerical graph matches with the integer result.

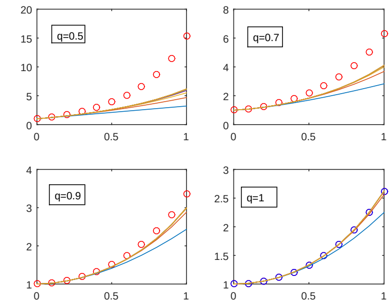

FIGURE 5. $p(t) = (t - t_0)^q$ and $f(t) = \alpha * (t - t_0)^q$ where $q = 0.5, 0.7, 0.9, 1.0$

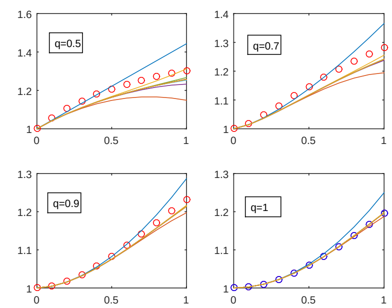

FIGURE 6. $p(t) = -(t - t_0)^q$ and $f(t) = \alpha * (t - t_0)^q$ where $q = 0.5, 0.7, 0.9, 1.0$

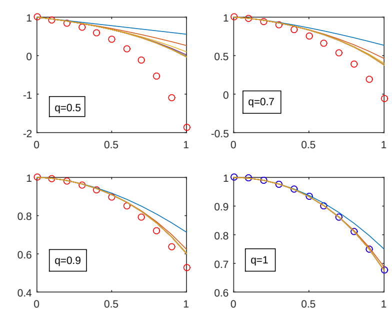

FIGURE 7. $p(t) = (t - t_0)^q$ and $f(t) = -\alpha * (t - t_0)^q$ where $q = 0.5, 0.7, 0.9, 1.0$

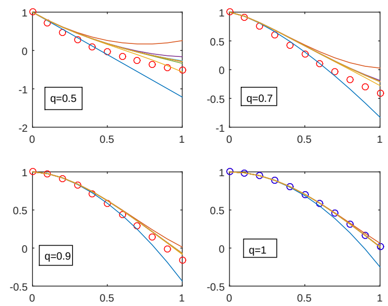

FIGURE 8. $p(t) = -(t - t_0)^q$ and $f(t) = -\alpha * (t - t_0)^q$ where $q = 0.5, 0.7, 0.9, 1.0$

We observe that our approximation (3.8) and (3.9) matches with the actual solution for $q = 1$, when

(i) $p(t) = A + B(t - t_0)^q$ and $f(t) \equiv 0$
(ii) $p(t) = (t - t_0)^q$ and $f(t) = \alpha(t - t_0)^q$, where α changes depends on the behavior of $f(t)$.

3.2. **Numerical results for nonlinear Caputo fractional differential equations.** Consider the nonlinear Caputo fractional differential equations with initial condition

$$ {}^cD^q u = f(t, u), \;\; u(t_0) = u_0, \tag{3.10} $$

where $f \in C([t_0, T] \times \mathbb{R}, \mathbb{R})$.

The Picard's approximation method to solve (3.10) is given by,

$$ u_n(t) = u_0 + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t - s)^{q-1} f(s, u_{n-1}(s)) ds, \tag{3.11} $$

where $n = 1, 2, 3, \ldots$

Here $f(t, u_{n-1}(t))$ are known function of t. Starting with $u_0 = u(t_0)$,

let us consider an example of nonlinear fractional differential equation with initial condition as follows,

Example 5:

$$ {}^cD^q u = u^2, \;\; u(t_0) = 1 = u_0. \tag{3.12} $$

The Picard's approximation method to solve (3.12) as follows,

$$ u_n(t) = u_0 + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t - s)^{q-1} f(s, u_{n-1}(s)) ds. \tag{3.13} $$

Our initial approximation is $u(t_0) = u_0$ and the first approximation is given by,

$$ u_1(t) = u_0 + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t - s)^{q-1} u_0 ds, \tag{3.14} $$

where $0 < q < 1$. Then the solution of (3.14) is given by

$$u_1(t) = 1 + \frac{(t-t_0)^q}{\Gamma(q+1)}. \tag{3.15}$$

Using the first approximation we develop the second approximation is as follows,

$$u_2(t) = 1 + \frac{(t-t_0)^q}{\Gamma(q+1)} + \frac{2(t-t_0)^{2q}}{\Gamma(2q+1)} + \frac{\Gamma(2q+1)}{(\Gamma(q+1))^2}\frac{(t-t_0)^{3q}}{\Gamma(3q+1)}. \tag{3.16}$$

Our third approximation will be as follows,

(3.17)

$$u_3(t) = 1 + \frac{(t-t_0)^q}{\Gamma(q+1)} + \frac{2(t-t_0)^{2q}}{\Gamma(2q+1)} + \left(4 + \frac{\Gamma(2q+1)}{(\Gamma(q+1))^2}\right)\frac{(t-t_0)^{3q}}{\Gamma(3q+1)}$$

(3.18)

$$+ \left(\frac{\Gamma(2q+1)}{\Gamma(q+1)} + \frac{4\Gamma(3q+1)}{\Gamma(q+1)\Gamma(2q+1)} + \frac{2\Gamma(2q+1)}{(\Gamma(q+1))^2}\right)\frac{(t-t_0)^{4q}}{\Gamma(4q+1)}$$

(3.19)

$$+ \left(\frac{4\Gamma(4q+1)}{(\Gamma(2q+1))^2} + \frac{\Gamma(2q+1)\Gamma(4q+1)}{(\Gamma(q+1))^2\Gamma(3q+1)}\right)\frac{(t-t_0)^{5q}}{\Gamma(5q+1)}$$

(3.20)

$$+ \left(\frac{4\Gamma(5q+1)}{\Gamma(3q+1)(\Gamma(q+1))^2}\right)\frac{(t-t_0)^{6q}}{\Gamma(6q+1)} + \left(\frac{\Gamma(6q+1)(\Gamma(2q+1))^2}{(\Gamma(3q+1))^2(\Gamma(q+1))^4}\right)\frac{(t-t_0)^{7q}}{\Gamma(7q+1)}.$$

The graph for the Example 5 is as follows

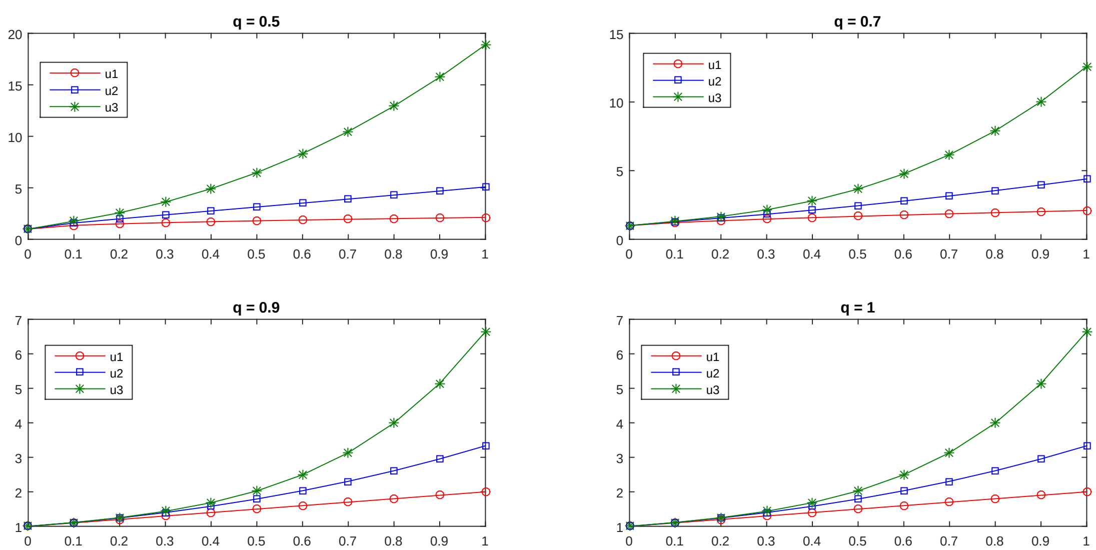

FIGURE 9. Picards Approximation method for $q = 0.5, 0.7, 0.9, 1.0$

However, the Picard's approximation method yields linear convergence whereas, the Quasilinearization method yields quadratic convergence.

Hence we consider the Quasilinear method to linearize the nonlinear fractional differential equations.

Let u_0 be the initial approximation namely, $u(t_0) = u_0$. Then we construct the sequence of iterates $u_n(t)$ as the solutions of the linear differential equations.

$$^cD^q u_n = f(t, u_{n-1}) + f_u(t, u_{n-1})(u_n - u_{n-1}), \;\; u_n(0) = u_0. \tag{3.21}$$

Now we consider the same example(3.12) for Quasilinearization method as follows,

$$^cD^q u = u^2, \;\; u(0) = 1 = u_0. \tag{3.22}$$

$$^cD^q u_n(t) = u_{n-1}^2 + 2u_{n-1}(u_n - u_{n-1}), \; u_0 = u(t_0) = 1, \tag{3.23}$$

where $n = 1, 2, 3, \ldots$

For $n = 1$, (3.23) reduces to

$$^cD^q u_1 = 1 + 2(u_1 - 1), \tag{3.24}$$

$$= 2u_1 - 1, \quad u_0 = 1. \tag{3.25}$$

The solution of the equation (3.25) is given by

$$u_1(t) = \frac{1 + E_{q,1}(2(t - t_0)^q)}{2}. \tag{3.26}$$

Using $u_1(t)$, we can solve $u_2(t)$ as follows,

$$^cD^q u_2 = 2u_1u_2 - u_1^2, \quad u_2(t_0) = 1. \tag{3.27}$$

$$^cD^q u_2 = 2(\frac{1 + E_{q,1}(2(t - t_0)^q)}{2})u_2 - (\frac{1 + E_{q,1}(2(t - t_0)^q)}{2})^2, \quad u_2(t_0) = 1. \tag{3.28}$$

Since we have Mittag Leffler function involved in our $u_1(t)$, it is difficult to find the solution of $u_2(t)$ by iterative method, so we use our direct numerical approximation method to solve $u2, u3, u4, \ldots$

From (3.8) and (3.9) we can compute $u_2(t)$ by taking

$$p(t) = 1 + E_{q,1}(2(t - t_0)^q) \text{ and } f(t) = \left(\frac{1 + E_{q,1}(2(t - t_0)^q)}{2}\right)^2.$$

Similarly we can compute $u3, u4, u5, \ldots$

Next, we present the graph for solution of (3.12) by Quasilinearization method.

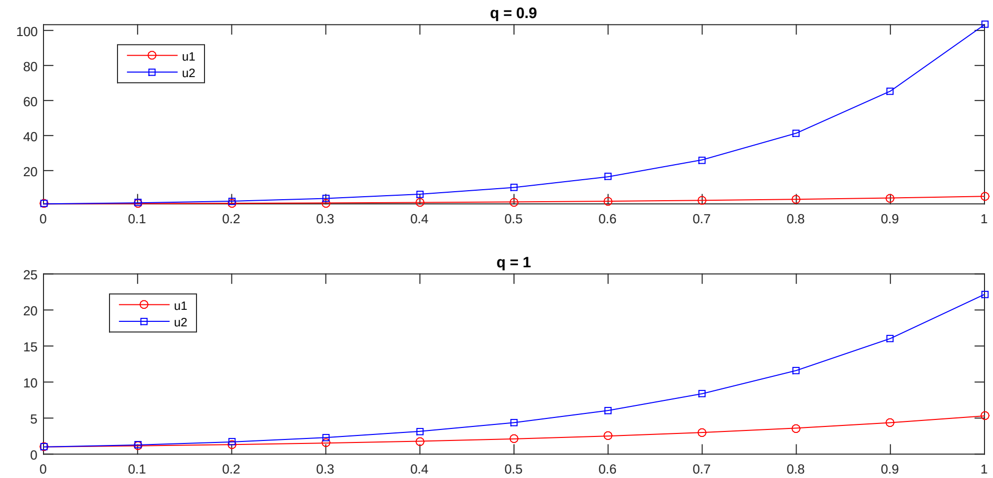

FIGURE 10. Quasilinearization method for u_1, u_2 when $q = 0.9, 1.0$

4. Conclusion

In this work we have developed direct numerical method for the computation of solution of the Caputo linear fractional differential equation of order q, $0 < q < 1$, with variable coefficients and with initial conditions. We were able to obtain this in the special case when the coefficient term and the non homogeneous term are of the form $(t - t_0)^q$ only. We plan to take up in future when the coefficients are of general polynomial form. In order to apply our direct numerical method we have considered the example

$$ {}^{c}D^{q}u = u^2, \ u(0) = 1. \tag{4.1}$$

Using the upper solution as $w = t^q$, we can find the interval of existence of solution of (4.1). The first iteration is relatively simple since the coefficient is constant whereas second iterate is relatively difficult to compute since it involves exponential property of Mittag Leffler function. The exponential property of the Mittag Leffler function is yet to be explored in future.

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