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# A STOCHASTIC DISCRETIZED VERSION OF A STAGE-STRUCTURED PREDATOR-PREY MODEL

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**ABSTRACT.** In this paper, a new stochastic discrete chaotic stage-structured predator-prey model is presented. The chaotic behavior of the proposed model is investigated. The existence and stability of the equilibria of the skeleton are studied. Numerical simulations are employed to show the model's complex dynamics by means of the largest Lyapunov exponents, bifurcations, time series diagrams and phase portraits. Time series diagrams are used to follow the dynamics of the model and discuss the marginal distribution of the state variables. The effects of noise intensity on its dynamics and the intermittency phenomenon are also discussed via simulation.

**AMS (MOS) Subject Classification.** 34K60, 34K23, 37M10.

## 1. INTRODUCTION

Stage-structured models in continuous time scale have been studied by many researchers. In stage-structured predator-prey models the immature predator has no direct effect on the prey or the mature predator, and the immature prey is not subject to predation. Aiello et al. [4] proposed a stage-structured model of population growth where the time to maturity is itself state dependent. They discussed the stability of the equilibria. The asymptotic behavior of a proposed predator-prey system with stage-structure for predator has been discussed by Wang and Chen [30]. Sufficient conditions under which positive equilibrium of the system is globally stable are also investigated. A predator-prey system, where the predator has two stages, a juvenile stage and a mature stage, has been constructed by Magnusson [26]. Hu and Huang [19] considered a predator-prey system of Lotka-Volterra type with time delays and stage structure for prey. The local stability of the equilibria and Hopf bifurcations occurring at the positive equilibrium under some conditions have been demonstrated. A stage-structured predator-prey system with time delays has been considered, where the time delays are regarded as bifurcation parameters, by Li et al. [25]. The local stability of a positive equilibrium has been investigated. They showed that Hopf bifurcations occurs when time delay crosses some critical values. Kar and Jana [21] studied the local and global stability for all possible non-negative equilibria

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of a predator-prey system with stage-structured for the predator population. However many authors used continuous time scale equations to describe stage-structured predator-prey systems, many authors [1, 2, 17, 27] have pointed out that the discrete time models governed by difference equations are more appropriate than the continuous ones when the populations have nonoverlapping generations. Discrete time models can also provide efficient computational models of continuous models for numerical simulations. Ecosystems are open systems and so they are often subject to environmental noises. Taking into account the complex behaviour of ecosystems it is more robust than the traditional methods to model observed time series by a stochastic chaotic models or non-linear chaotic models disturbed by dynamic noise. This type of models has been studied by many researchers, see [5, 6, 9-16, 18, 22, 23].

The main objectives of this paper are: to present a new chaotic stochastic discertized version of a stage structured predator-prey model for the predator population; to investigate parameters changes effects on the dynamics of the proposed model; to give a numerical evidence about the chaotic behaviour of stochastic systems; and to give a novel chaotic 3-dimensional discrete time scale system (functional coefficient nonlinear autoregressive model).

The organization of this paper is as follows. In section 2, the stochastic discrete stage structured predator-prey system is formulated. In section 3, equilibria are obtained and their asymptotic behaviour is discussed. The simulation is used in section 4, to discuss the analytical results, to show the effects of noise intensity on the dynamics of the system, and to give a numerical evidence about the chaotic behaviour of stochastic systems.

## 2. THE STOCHASTIC DISCRETE MODEL

Kar and Jana [21] proposed and analyzed a predator-prey system with stage-structured for the predator population. They set three different classes, namely, the prey, the juvenile predator and the matured predator. The class of prey biomass is denoted by $x_1$ and that of immature and matured predator populations by $x_2$ and $x_3$ respectively. The model has been formulated in the following form:

$$\left\{ \begin{array}{rcl} \dot{x_1} &=& rx_1(1-\frac{x_1}{k}) - \alpha x_1 x_3 \\ \dot{x_2} &=& sx_3 - \beta x_2 - \theta_1 x_2 \\ \dot{x_3} &=& mx_1x_3 + \beta x_2 - \theta_2 x_3 - \nu x_3^2, \end{array} \right. \tag{2.1}$$

where the matured predator population reproduces at a rate $s(>0)$, the transfer rate from the immature predator to the matured predator be $\beta(>0)$, the natural death rate for the immature predator be $\theta_1(>0)$, $m(>0)$ be the conversion factor from the prey population to the matured predator population, $\theta_2(>0)$ be the natural death rate of the matured predator population and $\nu(>0)$ be the intra-specific competition rate among the matured predator population. Applying the forward Euler scheme to model (2.1) and using suitable non dimensional conditions we obtain the following stochastic discrete stage structured predator prey model.

$$\begin{cases} x_t &= x_{t-1} + l(ax_{t-1} - x_{t-1}^2 - bx_{t-1}z_{t-1}) + k\varepsilon_t \\ y_t &= y_{t-1} + l(z_{t-1} - y_{t-1}) + k\eta_t \\ z_t &= z_{t-1} + l(-cz_{t-1} + dx_{t-1}z_{t-1} + fy_{t-1} - gz_{t-1}) + k\xi_t, \end{cases} \tag{2.2}$$

where $l$ is the step size $a, b, c, d, f$ and $g$ are the non dimentional version of the parameters defined in model (2.1) and $(\varepsilon_t, \eta_t, \xi_t)$ are assumed to be an i.i.d. white noise sequence conditional upon the history of the time series, which is denoted $\Omega_{t-1} = (x_{t-1}, y_{t-1}, z_{t-1})$ that is, $E[(\varepsilon_t, \eta_t, \xi_t)|\Omega_{t-1}] = 0$ and $E[\varepsilon_t^2|\Omega_{t-1}] = E[\eta_t^2|\Omega_{t-1}] = E[\xi_t^2|\Omega_{t-1}] = \sigma^2$, and $k$ is a scalar parameter of the noise intensity. All parameters are assumed to be positive.

## 3. THE SKELETON

The skeleton is defined by Tong [29], where $k = 0$. In this section we study the chaotic behaviour of the free noise system (2.2) caused by the change of time step. Where $k = 0$, the system (2.2) becomes:

$$\begin{cases} x_t &= x_{t-1} + l(ax_{t-1} - x_{t-1}^2 - bx_{t-1}z_{t-1}) \\ y_t &= y_{t-1} + l(z_{t-1} - y_{t-1}) \\ z_t &= z_{t-1} + l(-cz_{t-1} + dx_{t-1}z_{t-1} + fy_{t-1} - gz_{t-1}^2), \end{cases} \tag{3.1}$$

Equilibria of the system (3.1) are derived in the following.

**Lemma 3.1.** The equilibria of the system (3.1), are (i) $E_0 = (0, 0, 0)$, (ii) $E_1 = (0, \frac{f-c}{g}, \frac{f-c}{g})$, in the absence of the prey; (iii) $E_2 = (a, 0, 0)$, in the absence of the predator; (iv) $E_3 = (a - b\frac{ad+f-c}{bd+g}, \frac{ad+f-c}{bd+g}, \frac{ad+f-c}{bd+g})$, is a positive interior equilibrium point for $ag + bc > bf$, and $ad + f > c$.

**Proof.** The equilibria of the system (3.1) are obtained as the solution of the algebraic system:

$$\begin{cases} x &= x + l(ax - x^2 - bxz) \\ y &= y + l(z - y) \\ z &= z + l(-cz + dxz + fy - gz^2), \end{cases}$$

Which is obtained by setting $x_t = x_{t-1} = x$ , $y_t = y_{t-1} = y$ and $z_t = z_{t-1} = z$ in (3.1), it is easy to get the results.

Now, we study the stability of these equilibria of model (3.1). The local stability analysis of the system (3.1) can be studied by computing the variation matrix corresponding to each equilibrium. The variation matrix of the system at state variable is given by

$$J(x,y,z) = \begin{pmatrix} 1+l[a-2x-bz] & 0 & -blx \\ 0 & 1-l & l, \\ ldz & lf & 1+l[-c+dx-2gz] \end{pmatrix}. \tag{3.2}$$

For the local stability of $E_0$, we have the following result.

**Theorem 3.1.** If $1+c-\sqrt{(c-1)^2+4f} > 0$, the following conclusions hold.
(N1) If $(\frac{4}{(1+c+\sqrt{(c-1)^2+4f}}) < l < (\frac{4}{(1+c-\sqrt{(c-1)^2+4f}})$, then the equilibrium $E_0$ is a saddle point.
(N2) If $l > (\frac{4}{(1+c-\sqrt{(c-1)^2+4f}})$, then $E_0$ is a source.
(N3) If $l = (\frac{4}{(1+c+\sqrt{(c-1)^2+4f}})$ or $l = (\frac{4}{(1+c-\sqrt{(c-1)^2+4f}})$, then the equilibrium $E_0$ is non-hyperbolic.

**Proof.** In order to prove this result, we estimate the eigenvalues of Jacobian matrix at $E_0$. The Jacobian matrix for $E_0$ is

$$J(E_0) = \begin{pmatrix} 1+la & 0 & 0 \\ 0 & 1-l & l, \\ 0 & lf & 1-lc \end{pmatrix}.$$

Hence the eigenvalues of $J(E_0)$ are $\omega_1 = 1+la$, and $\omega_{2,3} = (1/2)[2-(1+c) \pm l\sqrt{(c-1)^2+4f}]$ and since all parameters are positive, we have $|\omega 1| > 1$. Let $1+c-\sqrt{(c-1)^2+4f} > 0$ and $(\frac{4}{(1+c+\sqrt{(c-1)^2+4f}}) < l < (\frac{4}{(1+c-\sqrt{(c-1)^2+4f}})$ then $|\omega 2| < 1$ and $|\omega 3| > 1$, which completes the proof of (N1). It is easy to get (N2) and (N3).

For the local stability of $E_1$, we have the following result.

**Theorem 3.2.** If $c-2f-1 < \sqrt{(1+2f-c)^2+4(c-f)} < 1+2f-c$ and $a < \frac{b(f-c)}{g}$, then
H1) $E_1$ is asymptotically stable if $0 < l < \min\{\alpha_1, \alpha_2, \alpha_3\}$,
H2) $E_1$ is a saddle if $\operatorname{mid}\{\alpha_1, \alpha_2, \alpha_3\} < l < \max\{\alpha_1, \alpha_2, \alpha_3\}$
H3) $E_1$ is unstable if $l > \max\{\alpha_1, \alpha_2, \alpha_3\}$ or $\min\{\alpha_1, \alpha_2, \alpha_3\} < l < \operatorname{mid}\{\alpha_1, \alpha_2, \alpha_3\}$,
H4) $E_1$ is non-hyperbolic if $l = \alpha_1$, or $\alpha_2$, or $\alpha_3$,
where $\alpha_1 = \frac{2g}{b(f-c)-ag}$, $\alpha_2 = \frac{4}{1+2f-c+\sqrt{(1+2f-c)^2+4(c-f)}}$, and $\alpha_3 = \frac{4}{1+2f-c-\sqrt{(1+2f-c)^2+4(c-f)}}$

**Proof.** The Jacobian matrix (3.2) at $E_1$ has the form:

$$J(E_1) = \begin{pmatrix} 1+l[a-\frac{b(f-c)}{g}] & 0 & 0 \\ 0 & 1-l & l, \\ \frac{ld(f-c)}{g} & lf & 1-l[2f-c] \end{pmatrix}.$$

The eigenvalues of $J(E_1)$ are $\omega_1 = 1+l[a-\frac{b(f-c)}{g}]$, and $\omega_{2,3} = (1/2)[2-l(1+2f-c \mp \sqrt{(1+2f-c)^2+4(c-f)})]$. After some calculations, we obtain the results.

For the local stability of $E_2$, we have the following result.

**Theorem 3.3.** If $ad - c - 1 < \sqrt{(ad-c-1)^2 + 4(ad-c+f)} < c + 1 - ad$, then

M1) $E_2$ is asymptotically stable if $0 < l < \min\{\beta_1, \beta_2, \beta_3\}$,

M2) $E_2$ is a saddle if $\mathrm{mid}\{\beta_1, \beta_2, \beta_3\} < l < \max\{\beta_1, \beta_2, \beta_3\}$

M3) $E_2$ is unstable if $l > \max\{\beta_1, \beta_2, \beta_3\}$ or $\min\{\beta_1, \beta_2, \beta_3\} < l < \mathrm{mid}\{\beta_1, \beta_2, \beta_3\}$,

M3) $E_2$ is non-hyperbolic if $l = \beta_1$, or $\beta_2$, or $\beta_3$,

where $\beta_1 = \frac{2}{a}$, $\beta_2 = \frac{4}{1+c-ad-\sqrt{(ad-c-1)^2+4(ad-c+f)}}$, and $\beta_3 = \frac{4}{1+c-ad+\sqrt{(ad-c-1)^2+4(ad-c+f)}}$.

**Proof.** The proof can be obtained by the same procedure in the previous theorem.

For local stability of $E_3$ we have the following result.

**Theorem 3.4.** The equilibrium point $E_3$ is asymptotically stable iff $1 + A + B + C > 0$,$1 - A + B - C > 0$,$1 - |C| > 0$, and $|1 - C| - |B - AC| > 0$, where $A = l - l\varphi - l\gamma - 3$, $B = (l\gamma + 1)(l\varphi - l + 2) - (l\varphi + 1)(l - 1) - fl^2 - bdl^2\alpha\varphi$, $C = (fl^2 + bdl^2\alpha\varphi)(l\varphi - l + 2) + (l\varphi + 1)(l\gamma + 1)(l - 1) + fl^2(l - 1) - bdl^2\alpha\varphi(l\varphi + 1)$,$\varphi = b\alpha - a$,$\alpha = \frac{ad+f-c}{bd+g}$, and $\gamma = -c - d\beta - 2g\alpha$.

**Proof.** The proof can be obtained by the same procedure in Theorem 3.2, and applying Jury criterion [28].

## 4. NUMERICAL SIMULATION

The main objective of this section is to introduce a numerical evidence about the chaotic behaviour of the system (2.2). Throughout this section, we consider the initial point $(x_0, y_0, z_0) = (0.3, 0.2, 0.1)$, and other parameters $a = 0.1$, $b = 7$, $c = 6$, $d = 0.01$, $f = 0.1$, and $g = 8$.

4.1. **Deterministic System.** In this subsection the qualitative behavior of the solution of the nonlinear system (3.1) is investigated. Various numerical results are presented here to show the chaoticity including its, bifurcation diagrams, Lyapunov exponents, and fractal dimension. Bifurcation diagrams of the system (3.1) are plotted on the interval $0.1 \le l \le 0.490$ in Fig. 1(a), (b), and (c). Its corresponding maximum Lyapunov exponent is given in Fig. 1(d). A positiveness of this exponent for $l > l^* \simeq 0.330$ confirms the chaotic character of attractors in this parametrical zone [8] ( here, the value $l^* \simeq 0.330$ is a tangent bifurcation point). To show the chaotic behaviour of the system, phase portraits are given for different values of $l$ In Fig. 3. For the given parameters the only positive equilibrium point is in the absence of the immature and matured predators, $E_2^* = (0.1, 0, 0)$ which is attracting point for $0.1 < l \le 0.330$ as we can see in Fig. 2(a). Fig. 2(b) shows the period-two orbit in the parameter zone $0.330 < l \le 0.408$. The period-four orbit in the parameter zone $0.408 < l \le 0.424$ is clear in Fig. 2(c). The chaotic attractor for $0.428 < l \le 0.490$ is clear in Fig. 2(d). Which means that the system (3.1) undergoes a discrete Hopf bifurcation. One of the commonly used characteristics for classifying and quantifying the chaoticity of a dynamical system is Lyapunov dimension [3, 20]. Via simulation we get three Lyapunov exponents $\lambda_1 = 0.947 > \lambda_2 = -0.432 > \lambda_3 = -0.622$ for $l = 0.470$, which means that $d_L \simeq 2.83$. Therefor the system (3.1) exhibits a fractal structure and its attractor has a fractal dimension which is chaotic behaviour.

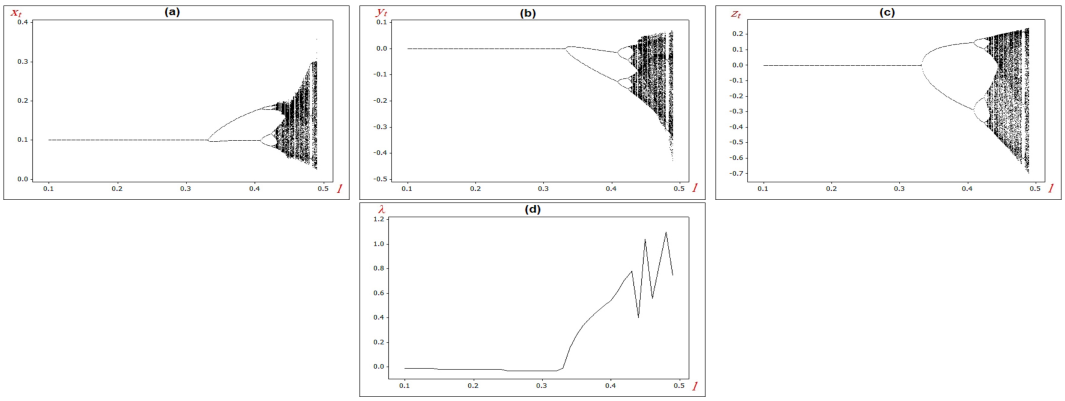


FIGURE 1. Bifurcation diagram of (3.1), for $0.1 \leq l \leq 0.490$ for initial point $(x_0, y_0, z_0) = (0.3, 0.2, 0.1)$ for: (a) $x_t$; (b) $y_t$; (c) $z_t$; (d) its corresponding Maximum Lyapunov exponents with $a = 0.1$, $b = 7$, $c = 6$, $d = 0.01$, $f = 0.1$, and $g = 8$.

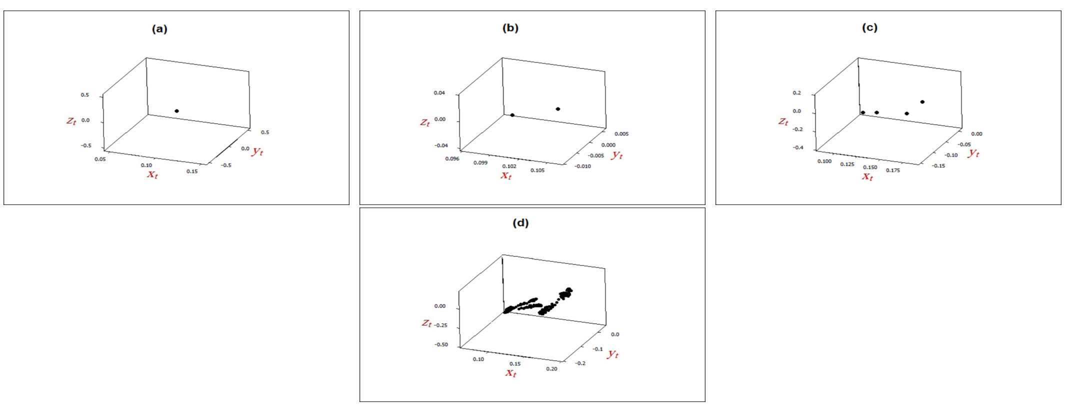


FIGURE 2. Phase portrait of the system (3.1) for: (a) $l = 0.330$; (b) $l = 0.334$; (c) $l = 0.42$; (d) $l = 0.44$ with $a = 0.1$, $b = 7$, $c = 6$, $d = 0.01$, $f = 0.1$, and $g = 8$.

4.2. **Stochastic System.** In this subsection, we consider a stochastic system forced by additive noise (2.2), where $\varepsilon_t$, $\eta_t$ and $\xi_t$ are uncorrelated Gaussian random processes with parameters $E\varepsilon_t = E\eta_t = E\xi_t = 0$, $E\varepsilon_t^2 = E\eta_t^2 = E\xi_t^2 = 1$ and $k$ is a scalar parameter of the noise intensity. Although the Frobenius-Perron integral equation [7, 24], is an exhaustive mathematical description of stochastic attractors, a direct using of these equations is very difficult technically even for the simplest cases. The main tool for the analysis of noise-induced phenomena is the direct numerical simulation of random trajectories. The main objectives of this subsection are: to give a numerical evidence about the interrelationship between the dynamics of the stochastic system and its skeleton; to discuss the phenomenon of noise-induced intermittency; and to investigate the changes of parameters and noise intensity effects on the asymptotic marginal distribution of the state variables, which may be regarded as an indicator of stochastic chaotic distribution of the system. We study a behavior of this stochastic system for different values of parameters $l$ and $k$. In Figs. 3(a)–3(i), bifurcations diagrams of the system (2.2) are plotted on the interval $0.1 \leq l \leq 0.490$ for three values of the noise intensity $k = 0.00008, 0.0001$, and 0.0008. Maximum Lyapunov exponents corresponding to each value of noise intensity are given in Figs. (3)(j), (k) and (l) respectively. By comparing Figs. 1 and 3(a)–3(i), we can see how noise deforms the deterministic attractor. The dynamical characteristics are also changed (compare Lyapunov exponents in Figs. 1(d) and 3(j)–3(l)). As noise intensity increases, a border between order and chaos moves to the left, see Figs. 3(j)–3(l). To follow the dynamics of the system (2.2), time series, phase portraits and stochastic attractors are plotted for different values of $l$ and $k$. For $l = 0.330$, with low noise $k = 0.0001$, the states $(x_t, y_t, z_t)$ oscillate with low amplitude around the stable deterministic equilibrium $E_2^*$. Random states are concentrated near the stable deterministic equilibrium, see Figs. 4(a)–4(c) and 6(a). With high noise $k = 0.0008$, we can observe the stochastic oscillations of large amplitude and the increase of the dispersion, see Figs. 4(d)–4(f), 6(b) and 7(a)–7(c). For $l = 0.334$, the stochastic oscillations of large amplitude around the deterministic period-two orbit $(0.09697, 0.00563, -0.03970)$ and $(0.10607, -0.00951, 0.03582)$, can be observed even though for low noise intensity $k = 0.0001$, see Figs. 4(g)–4(l). In this case, the stochastic system (2.2) exhibits a coexistence of two different dynamical regimes even if the system (3.1) has a stable equilibrium only. This type of dynamics of the system (2.2) can be determined as a noise-induced intermittency [5], see Figs. 4(g)–4(i), 6(c), 6(d) and 7(d)–7(f). For $l = 0.420$, the stochastic oscillations of large amplitude around the deterministic period-four orbit $(0.11394, -0.03902, -0.21811)$, $(0.18633, -0.11423, 0.16994)$, $(0.08648, 0.00512, -0.36002)$, and $(0.17850, -0.14824, 0.11181)$ can be observed even though for low noise intensity $k = 0.0001$, see Figs. 5(a)–5(f); 6(e), (f) and 7(g)–7(i). In this case, the stochastic system (2.2) exhibits a coexistence of four different dynamical regimes even if the system (3.1) has a stable equilibrium only. The stochastic chaotic behaviour can be observed for $l = 0.440$, even though for low noise intensity $k = 0.0001$, see Figs. 5(g)–5(l), 6(g), 6(h), and 7(j)–7(l). For stochastic dynamic characteristics, the dependence on noise level is illustrated in Fig. 8 for $l = 0.330$ and $l = 0.331$. To investigate the changes of parameters and noise intensity effects on the asymptotic marginal distribution of the state variables, we discuss it for different values of $l$ and $k$. For $l = 0.420$, and $0 \leq k \leq 0.0008$ the marginal distribution of the state variables is asymptotically normal, with mean $E_1^* = (0.1, 0, 0)$, which we may call it a stable distribution see Figs. 9(a)–9(c), the sample statistics are given in Table 1. As the control parameter $l$ increases the asymptotic distribution split into two regimes, see Figs. 9(d)–9(f) and Table 2, for $l = 0.334$, then into four regimes, see Figs. 9(g)–9(i) and Table 3, for $l = 0.420$, and so on until it reaches what we may call it chaotic distribution, see Figs. 9(j)–9(l), and Table 4, for $l = 0.440$. This phenomenon may be regarded as an indicator of stochastic chaotic behaviour.

## 5. CONCLUSION

The current paper has presented a new stochastic discrete chaotic stage-structured predator-prey model. The model shows rich and varied dynamics and chaos. Local stability of equilibria have been discussed. The results show that, for certain parametric restrictions there is a unique equilibrium point which is locally stable. When the time step parameter passes a critical value the attractor is a period-2 cycle. As the time step parameter increases, both branches split simultaneously, yielding a period-4 cycle. This splitting is the period-doubling bifurcation. A cascade of further period-doubling occurs as the time step parameter increases, yielding period-8, period-16, and so on until the map becomes chaotic and the attractor changes from a finite to an infinite set of points. The results referred to that there is a closed relationship between the time step parameter changes and the distribution of the states of the stochastic system. The remarkable feature of the dynamics of the model considered here is that small noises generate large-amplitude chaotic oscillation. The asymptotic distribution of the state variables is used as an indicator of the chaotic behaviour of stochastic systems, which still an open problem.

TABLE 1. Sample statistics of the variable state $(x_t, y_t, z_t)$ of the system (2.2), with initial values $(x_0, y_0, z_0) = (0.3, 0.2, 0.1)$, for $l = 0.330$ and $k \in [0.0, 0.0008]$, where $a = 0.1$, $b = 7$, $c = 6$, $d = 0.01$, $f = 0.1$, and $g = 8$.

| $l = 0.330$ | Mean | Median | St.D. | Skewness | Kurtosis |
|---|---|---|---|---|---|
| $x_t$ | 0.1000 | 0.1000 | 0.0019 | 0.03 | 2.25 |
| $y_t$ | 0.0000 | 0.0000 | 0.0008 | -0.11 | 2.57 |
| $z_t$ | 0.0000 | 0.0000 | 0.0027 | -0.05 | 2.35 |

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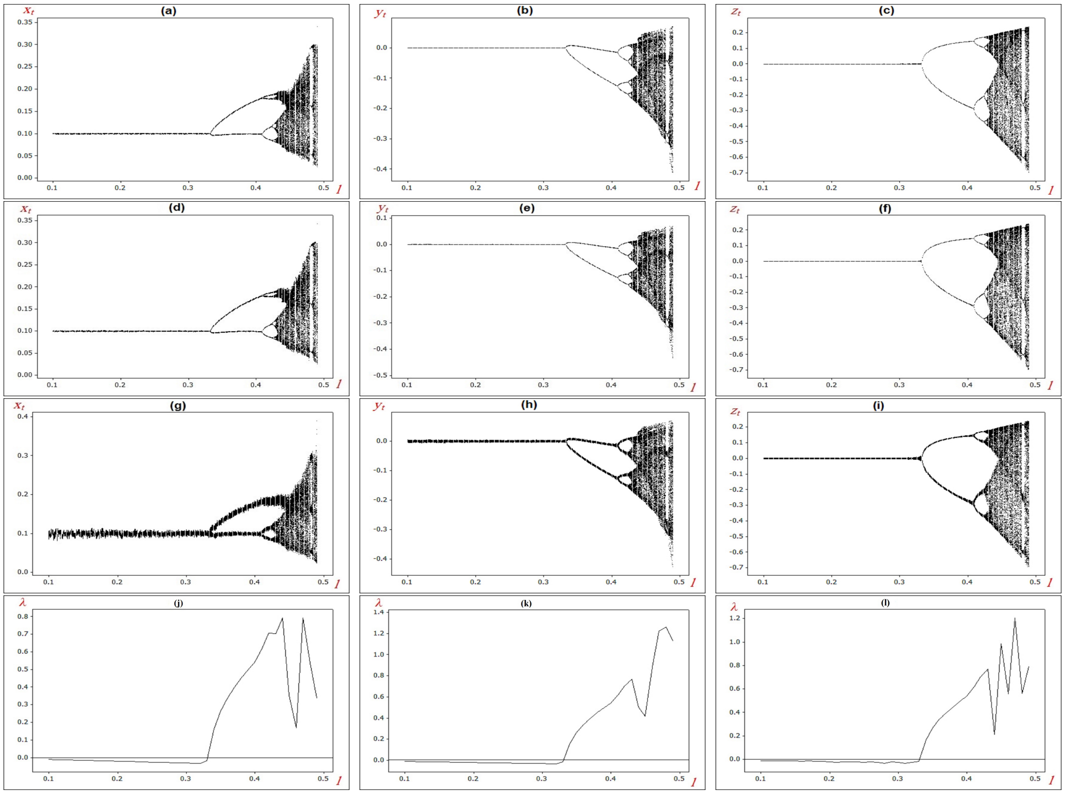


FIGURE 3. Bifurcation diagram of the stochastic model (2.2), with $0.1 \leq l \leq 0.490$ and the initial point $(x_0, y_0, z_0) = (0.3, 0.2, 0.1)$ for: $(a)x_t$; $(b)y_t$; $(c)z_t$, $(j)$ its corresponding Maximum Lyapunov exponents, with $k = 0.00008$; $(d)x_t$; $(e)y_t$; $(f)z_t$, $(k)$ its corresponding Maximum Lyapunov exponents, with $k = 0.0001$; $(g)x_t$; $(h)y_t$; $(i)z_t$, $(l)$ its corresponding Maximum Lyapunov exponents, with $k = 0.0008$; where $a = 0.1$, $b = 7$, $c = 6$, $d = 0.01$, $f = 0.1$, and $g = 8$.

TABLE 2. Sample statistics of the variable state $(x_t, y_t, z_t)$ of the system (2.2), with initial values $(x_0, y_0, z_0) = (0.3, 0.2, 0.1)$, for $l = 0.334$ and $k \in [0.0, 0.0008]$, where $a = 0.1$, $b = 7$, $c = 6$, $d = 0.01$, $f = 0.1$, and $g = 8$.

| $l = 0.334$ | Mean | Median | St.D. | Skewness | Kurtosis |
|---|---|---|---|---|---|
| $x_t$ | 0.1014 | 0.1009 | 0.0049 | 0.03 | -1.4 |
| $y_t$ | -0.0019 | -0.0014 | 0.0076 | -0.01 | -1.95 |
| $z_t$ | -0.0019 | 0.0009 | 0.0376 | -0.00 | -1.99 |

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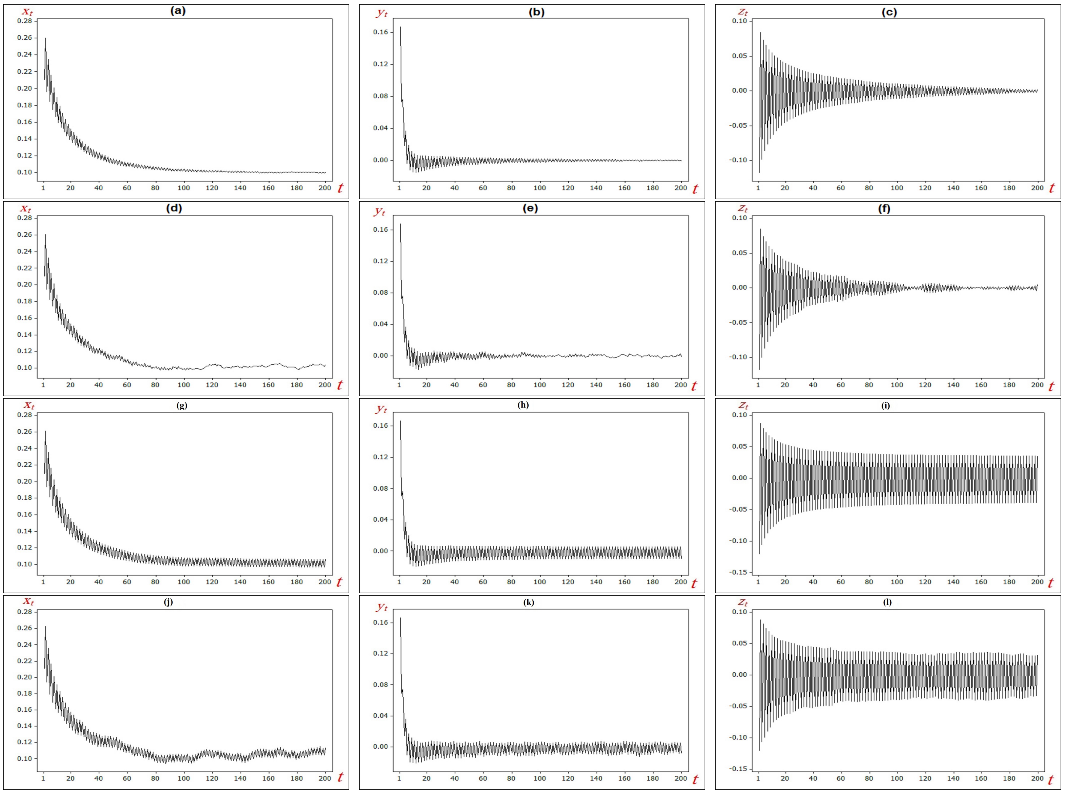

FIGURE 4. Time series of the stochastic model (2.2) with $l = 0.330$ and $k = 0.0001$ for: $(a)x_t$; $(b)y_t$; $(c)z_t$, with $l = 0.330$ and $k = 0.0008$ for: $(d)x_t$; $(e)y_t$; $(f)z_t$, with $l = 0.334$ and $k = 0.0001$ for: $(g)x_t$; $(h)y_t$; $(i)z_t$ and with $l = 0.334$ and $k = 0.0008$ for: $(j)x_t$; $(k)y_t$; $(l)z_t$, where $a = 0.1$, $b = 7$, $c = 6$, $d = 0.01$, $f = 0.1$, and $g = 8$.

TABLE 3. Sample statistics of the variable state $(x_t, y_t, z_t)$ of the system (2.2), with initial values $(x_0, y_0, z_0) = (0.3, 0.2, 0.1)$, for $l = 0.420$ and $k \in [0.0, 0.0008]$, where $a = 0.1$, $b = 7$, $c = 6$, $d = 0.01$, $f = 0.1$, and $g = 8$.

| $l = 0.420$ | Mean | Median | St.D. | Skewness | Kurtosis |
|---|---|---|---|---|---|
| $x_t$ | 0.1413 | 0.1442 | 0.0423 | -0.14 | -1.78 |
| $y_t$ | -0.0741 | -0.0770 | 0.0604 | 0.08 | -1.62 |
| $z_t$ | -0.0741 | 0.0477 | 0.2217 | -0.12 | -1.77 |

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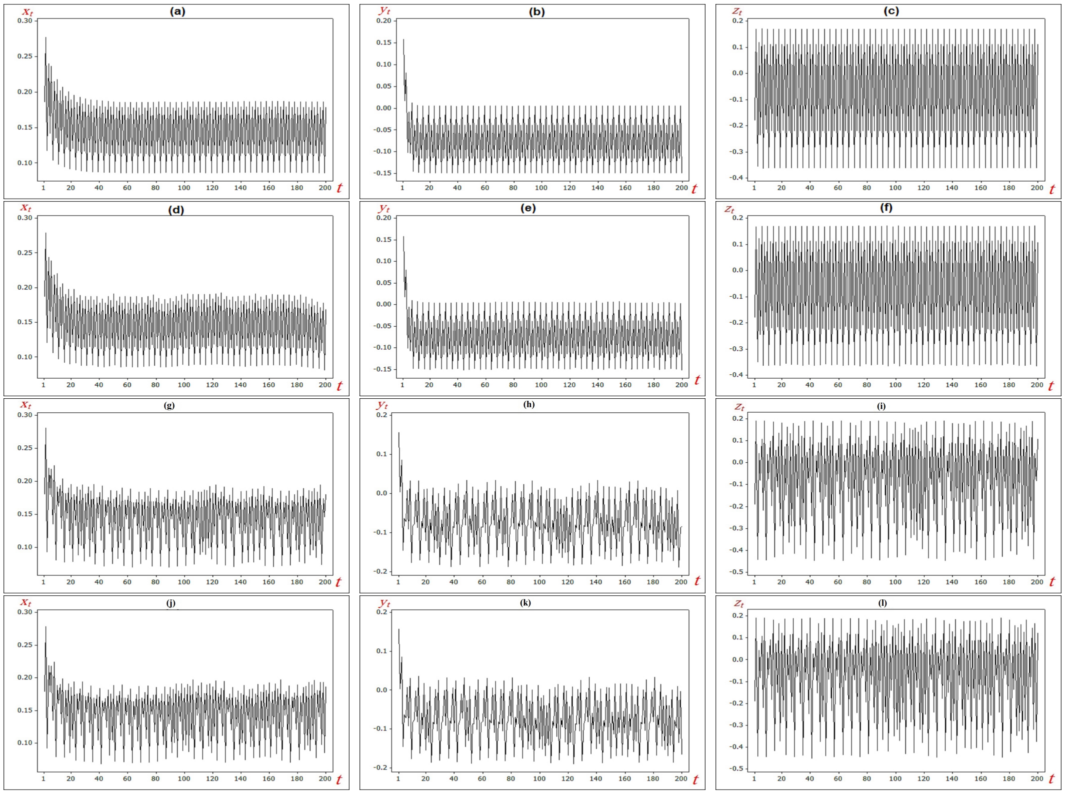

FIGURE 5. Time series of the stochastic model (2.2) with $l = 0.420$ and $k = 0.0001$ for: $(a)x_t$; $(b)y_t$; $(c)z_t$, with $l = 0.420$ and $k = 0.0008$ for: $(d)x_t$; $(e)y_t$; $(f)z_t$, with $l = 0.440$ and $k = 0.0001$ for: $(g)x_t$; $(h)y_t$; $(i)z_t$ and with $l = 0.440$ and $k = 0.0008$ for: $(j)x_t$; $(k)y_t$; $(l)z_t$, where $a = 0.1$, $b = 7$, $c = 6$, $d = 0.01$, $f = 0.1$, and $g = 8$.

TABLE 4. Sample statistics of the variable state $(x_t, y_t, z_t)$ of the system (2.2), with initial values $(x_0, y_0, z_0) = (0.3, 0.2, 0.1)$, for $l = 0.440$ and $k \in [0.0, 0.0008]$, where $a = 0.1$, $b = 7$, $c = 6$, $d = 0.01$, $f = 0.1$, and $g = 8$.

| $l = 0.440$ | Mean | Median | St.D. | Skewness | Kurtosis |
|---|---|---|---|---|---|
| $x_t$ | 0.1425 | 0.1564 | 0.0430 | -0.41 | -1.38 |
| $y_t$ | -0.0750 | -0.0823 | 0.0687 | -0.02 | -1.20 |
| $z_t$ | -0.0749 | 0.0112 | 0.2228 | -0.37 | -1.37 |

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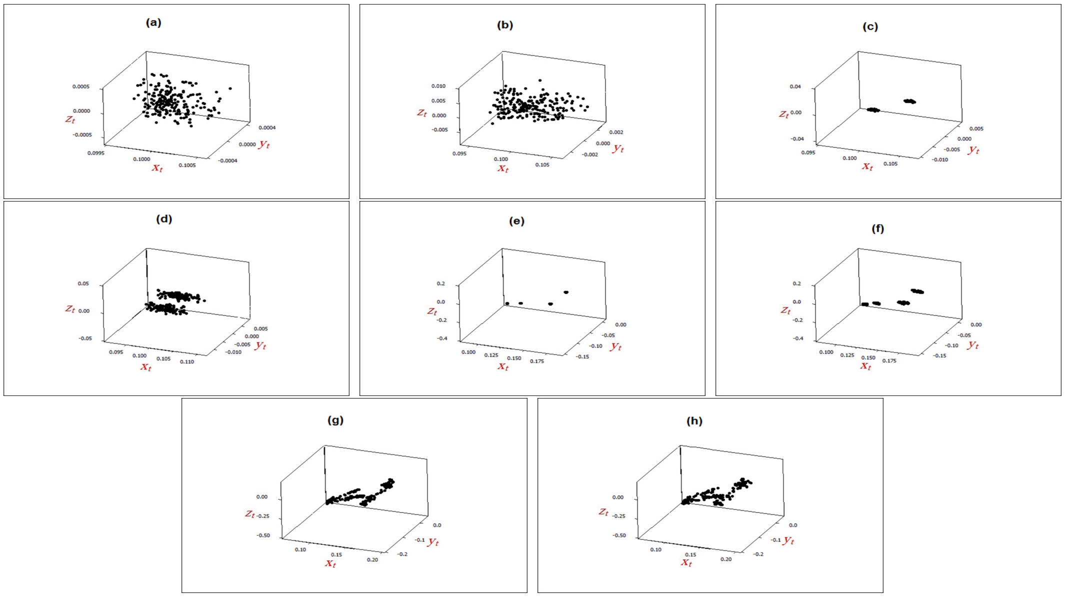


FIGURE 6. Phase portrait of the stochastic system (2.2) for: $(a)l = 0.330$ and $k = 0.0001$; $(b)l = 0.330$ and $k = 0.0008$; $(c)l = 0.334$ and $k = 0.0001$; $(d)l = 0.334$ and $k = 0.0008$; $(e)l = 0.420$ and $k = 0.0001$; $(f)l = 0.420$ and $k = 0.0008$; (g) $l = 0.440$ and $k = 0.0001$; $(h)l = 0.440$ and $k = 0.0008$, with $a = 0.1$, $b = 7$, $c = 6$, $d = 0.01$, $f = 0.1$, and $g = 8$.

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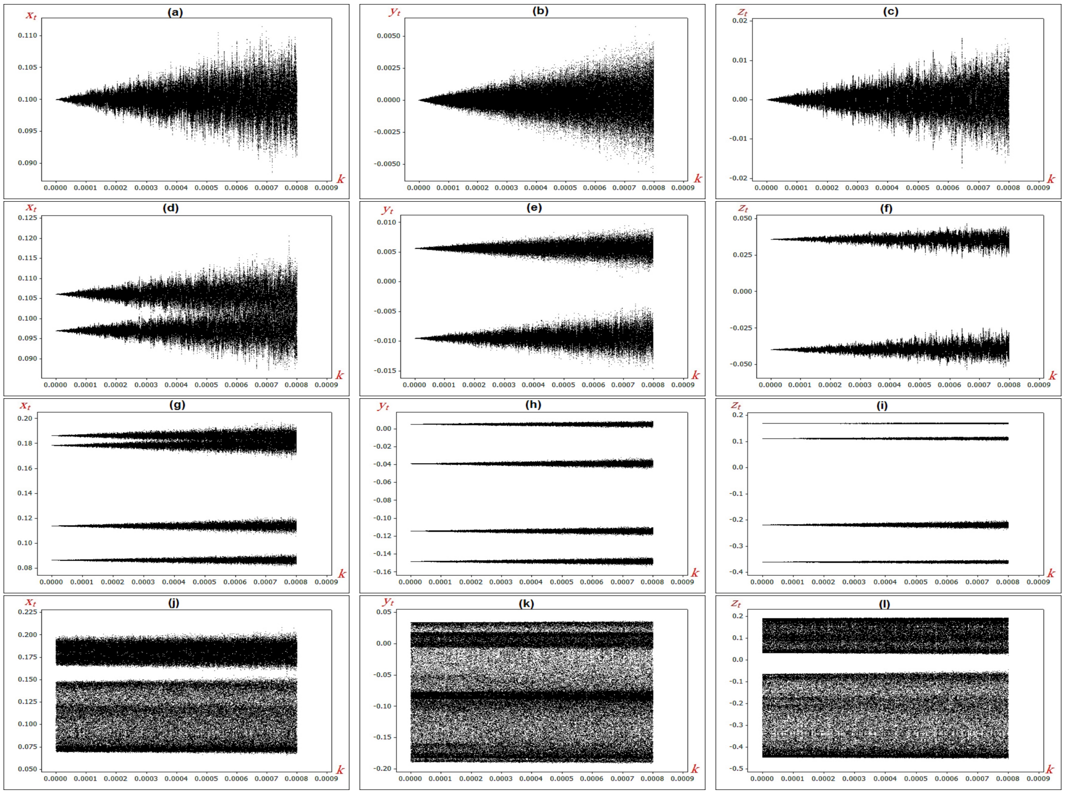

FIGURE 7. Attractors of the stochastic system (2.2) with $k \in [0.0, 0.0008]$,and the initial point $(x_0, y_0, z_0) = (0.3, 0.2, 0.1)$ for: $(a)x_t$; $(b)y_t$; $(c)z_t$, with $l = 0.330$; $(d)x_t$; $(e)y_t$; $(f)z_t$, with $l = 0.334$;$(g)x_t$; $(h)y_t$; $(i)z_t$, with $l = 0.420$;$(j)x_t$; $(k)y_t$; $(l)z_t$, with $l = 0.440$, where $a = 0.1$, $b = 7$, $c = 6$, $d = 0.01$, $f = 0.1$, and $g = 8$.

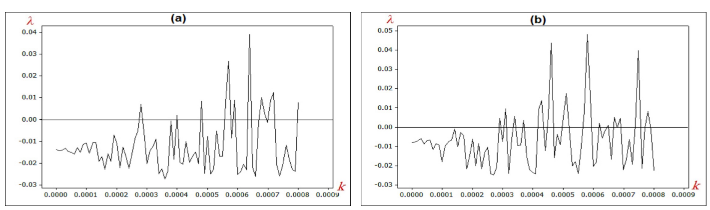

FIGURE 8. Lyapunov exponents of the stochastic system (2.2) with $k \in [0.0, 0.0008]$ and the initial point $(x_0, y_0, z_0) = (0.3, 0.2, 0.1)$ for: $(a)l = 0.330$; $(b)l = 0.331$, where $a = 0.1$, $b = 7$, $c = 6$, $d = 0.01$, $f = 0.1$, and $g = 8$.

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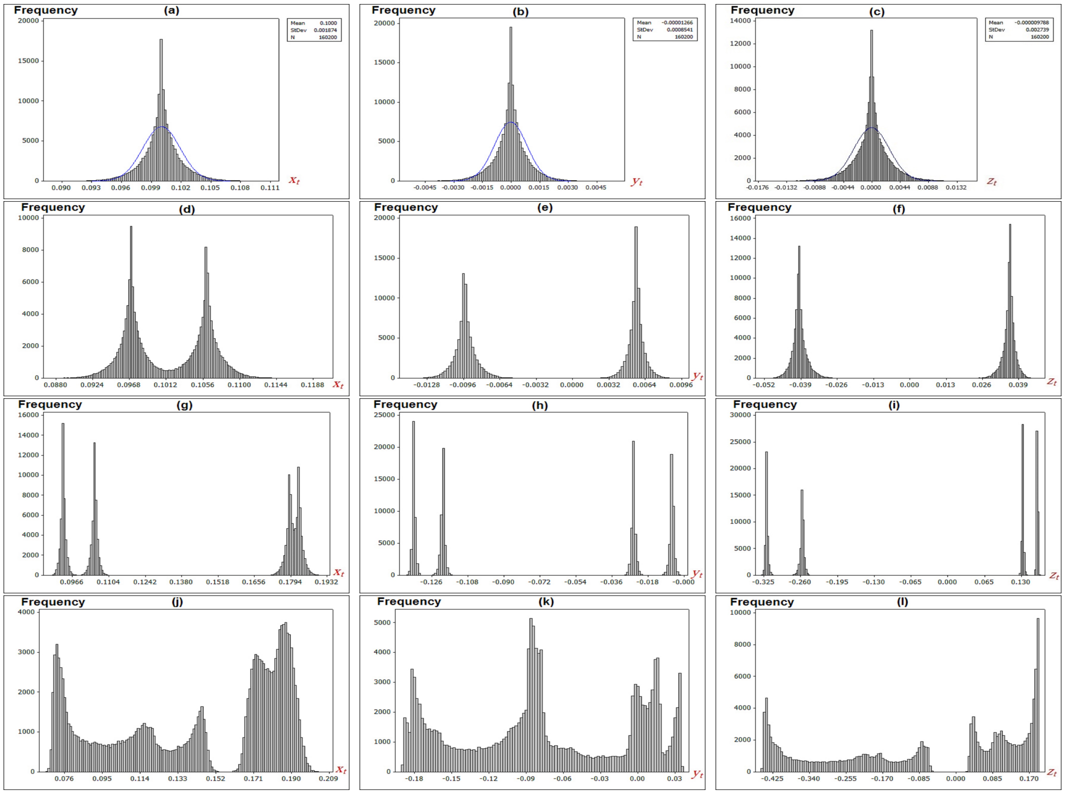

FIGURE 9. Asymptotic distributions of state variables $(x_t, y_t, z_t)$ of the system (2.2) with $k \in [0.0, 0.0008]$,and the initial point $(x_0, y_0, z_0) = (0.3, 0.2, 0.1)$ for: $(a)x_t$; $(b)y_t$; $(c)z_t$, with $l = 0.330$; $(d)x_t$; $(e)y_t$; $(f)z_t$, with $l = 0.334$;$(g)x_t$; $(h)y_t$; $(i)z_t$, with $l = 0.420$;$(j)x_t$; $(k)y_t$; $(l)z_t$, with $l = 0.440$, where $a = 0.1$, $b = 7$, $c = 6$, $d = 0.01$, $f = 0.1$, and $g = 8$.


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